

## 黏弹性问题的改进的复变量无单元 Galerkin 方法\*

彭妙娟† 刘茜

(上海大学土木工程系, 上海 200072)

(2014年4月29日收到; 2014年5月20日收到修改稿)

基于改进的复变量移动最小二乘法, 提出了二维黏弹性问题的改进的复变量无单元 Galerkin 方法. 采用改进的复变量移动最小二乘法建立形函数, 根据 Galerkin 积分弱形式建立求解方程, 并用罚函数法施加本质边界条件, 推导了二维黏弹性问题的改进的复变量无单元 Galerkin 方法的计算公式. 最后, 通过实际算例, 将计算结果与复变量无单元 Galerkin 方法及有限元法的结果进行了对比, 说明了本文方法具有更高的计算精度和计算效率.

**关键词:** 无网格方法, 复变量移动最小二乘法, 改进的复变量无单元 Galerkin 方法, 黏弹性问题

**PACS:** 02.60.Cb, 02.60.Lj, 46.35.+z

**DOI:** 10.7498/aps.63.180203

## 1 引言

无网格方法 (meshless method) 是近年来兴起的一种新的数值方法. 不同于以往基于网格的方法, 无网格方法在构造近似函数时采用基于点的近似思想, 不像有限元法或边界元法那样进行初始网格划分和网格重构, 能够克服有限元法和边界元法对单元或网格的依赖性<sup>[1,2]</sup>.

目前已发展的无网格方法有无单元 Galerkin 法<sup>[3-6]</sup>、重构核粒子法<sup>[7]</sup>、光滑粒子法<sup>[8]</sup>、有限点法<sup>[9-11]</sup>、无网格 Petrov-Galerkin 方法<sup>[12]</sup>、单位分解法<sup>[13,14]</sup>、复变量无网格方法<sup>[15,16]</sup>和边界积分方程的无网格方法<sup>[17-23]</sup>等.

黏弹性问题由于本构关系比较复杂, 在求解其数学模型的精确解时难度较大, 所以数值方法被广泛用于求解黏弹性问题内部应力和应变的分布规律. 随着无网格方法的发展, 许多学者将该方法用于黏弹性问题. Yang 和 Liu<sup>[24]</sup>将时域精细算法与 EFG 方法相结合, 求解黏弹性力学问题; Sladek 等<sup>[25,26]</sup>运用无网格局部 Petrov-Galerkin 方法对各向异性材料进行应力分析, 并求解了二维非均质

线性黏弹性材料的准静态和瞬态响应问题; Cheng 等<sup>[27]</sup>将复变量无单元 Galerkin 法应用于黏弹性问题.

为了减少基于移动最小二乘法的无网格方法的计算量, 程玉民等<sup>[28-31]</sup>提出了一种新的构造无网格方法形函数的方法——复变量移动最小二乘法. 复变量移动最小二乘法的优点是其试函数中所含的待定常数减少了, 这样其形成的二维问题的无网格方法可取较少的节点. 对任一场点来说, 其影响域中所含的最小节点数就大大减少了, 进而在整个求解域中所需选取的节点数目也可以大大减少. 这样, 基于复变量移动最小二乘法的无网格方法的计算量可以大幅度减小. 基于复变量移动最小二乘法, Cheng 等建立了各种复变量无网格方法, 包括复变量无单元 Galerkin 方法 (complex variable element-free Galerkin method, CVEFG)<sup>[32,33]</sup>、复变量边界无单元方法<sup>[34]</sup>、复变量无网格流形方法<sup>[35,36]</sup>和改进的复变量无单元 Galerkin 方法 (improved complex variable element-free Galerkin method, ICVEFG)<sup>[37-42]</sup>. 彭妙娟等<sup>[43-47]</sup>提出了势问题、弹性力学问题、弹塑性力学、弹性大变形和黏弹性问题的复变量无单元 Galerkin 方法. Li

\* 国家自然科学基金 (批准号: 11171208) 资助的课题.

† 通讯作者. E-mail: [mjpeng@shu.edu.cn](mailto:mjpeng@shu.edu.cn)

等 [48] 讨论了弹塑性大变形问题的复变量无单元 Galerkin 方法. 文献 [49—53] 基于复变量理论提出了复变量重构核粒子法. Ren 等 [54] 建立了复变量插值型移动最小二乘法.

无网格方法目前还存在着一些问题, 如求解时间常常比有限元法等用时更长, 需要更合理地选择数值积分方法以及影响域大小的比例参数等, 以保证较高的计算精度和计算速度. 复变量移动最小二乘法建立了向量函数的逼近格式, 而改进的复变量移动最小二乘法引入了共轭基函数, 使计算过程简化, 在节点相同的情况下, 其计算精度更高、计算时间更短.

本文将 ICVEFG 方法应用于二维黏弹性问题. 基于改进的复变量移动最小二乘法建立形函数, 根据 Galerkin 积分弱形式建立求解方程, 采用罚函数法施加本质边界条件, 提出了蠕变情况下的二维黏弹性问题的 ICVEFG 方法. 最后通过数值算例验证了本文方法比 CVEFG 方法具有更高的计算精度和计算效率.

## 2 改进的复变量移动最小二乘法

在改进的复变量移动最小二乘法中, 设函数  $u(z)$  的逼近函数为  $u^h(z)$ , 则

$$\begin{aligned} u^h(z) &= u_1^h(z) + iu_2^h(z) \\ &= \sum_{i=1}^m \bar{p}_i(z) a_i(z) \\ &= \bar{\mathbf{p}}^T(z) \mathbf{a}(z) \quad z = x_1 + ix_2 \in \Omega, \end{aligned} \quad (1)$$

其中  $\bar{\mathbf{p}}^T = (\bar{p}_1(z), \bar{p}_2(z), \dots, \bar{p}_m(z))$  是基向量为  $\mathbf{p}^T = (p_1(z), p_2(z), \dots, p_m(z))$  的共轭向量,  $\mathbf{a}^T = (a_1(z), a_2(z), \dots, a_m(z))$  是相应的系数向量,  $m$  是基函数的个数,  $\Omega$  是问题所在的域.

相对于整体逼近, 在  $z$  影响域内的局部逼近定义为

$$u^h(z, \hat{z}) = \sum_{i=1}^m \bar{p}_i(\hat{z}) \cdot a_i(z) = \bar{\mathbf{p}}^T(\hat{z}) \cdot \mathbf{a}(z). \quad (2)$$

定义泛函

$$\begin{aligned} J &= \sum_{I=1}^n w(z - z_I) |u^h(z_I, z) - u(z_I)|^2 \\ &= (\bar{\mathbf{P}}\mathbf{a}(z) - \mathbf{u}^*(z))^T \mathbf{W}(z) \\ &\quad \times \overline{(\bar{\mathbf{P}}\mathbf{a}(z) - \mathbf{u}^*(z))}, \end{aligned} \quad (3)$$

其中

$$\bar{\mathbf{P}} = \begin{bmatrix} \bar{p}_1(z_1) & \bar{p}_2(z_1) & \cdots & \bar{p}_m(z_1) \\ \bar{p}_1(z_2) & \bar{p}_2(z_2) & \cdots & \bar{p}_m(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{p}_1(z_n) & \bar{p}_2(z_n) & \cdots & \bar{p}_m(z_n) \end{bmatrix}, \quad (4)$$

$$\mathbf{W}(z) = \begin{bmatrix} w(z - z_1) & 0 & \cdots & 0 \\ 0 & w(z - z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(z - z_n) \end{bmatrix}, \quad (5)$$

$$\mathbf{u}^* = (u(z_1), u(z_2), \dots, u(z_n))^T = \mathbf{Q}\mathbf{u}, \quad (6)$$

$$\mathbf{u} = (u_1(z_1), u_2(z_1), u_1(z_2), u_2(z_2), \dots, u_1(z_n), u_2(z_n))^T, \quad (7)$$

$$\mathbf{Q} = \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & i \end{bmatrix}_{n \times 2n}. \quad (8)$$

对  $J$  取极小值,

$$\frac{\partial J}{\partial \mathbf{a}} = 0, \quad (9)$$

可得

$$\mathbf{a}(z) = \mathbf{C}^{-1}(z) \mathbf{B}(z) \mathbf{u}^*, \quad (10)$$

其中

$$\mathbf{B}(z) = \mathbf{P}^T \mathbf{W}(z), \quad (11)$$

$$\mathbf{C}(z) = \mathbf{P}^T \mathbf{W}(z) \bar{\mathbf{P}}. \quad (12)$$

于是, 局部逼近  $u^h(z, \hat{z})$  的表达式可以写成

$$u^h(z, \hat{z}) = \Phi(z) \mathbf{u}^* = \sum_{I=1}^n \Phi_I(z) u(z_I), \quad (13)$$

这里  $\Phi(z)$  为逼近函数的形函数,

$$\begin{aligned} \Phi(z) &= (\Phi_1(z), \Phi_2(z), \dots, \Phi_n(z)) \\ &= \bar{\mathbf{p}}^T(z) \mathbf{C}^{-1}(z) \mathbf{B}(z). \end{aligned} \quad (14)$$

由 (13) 式可得

$$\begin{aligned} u_1^h(z) &= \text{Re}[\Phi(z) \mathbf{u}^*] \\ &= \text{Re} \left[ \sum_{I=1}^n \Phi_I(z) u(z_I) \right], \end{aligned} \quad (15)$$

$$u_2^h(z) = \text{Im}[\Phi(z)u^*] = \text{Im}\left[\sum_{I=1}^n \Phi_I(z)u(z_I)\right]. \quad (16)$$

改进的复变量移动最小二乘法在复变量移动最小二乘法的基础上引入了共轭基函数, 其泛函具有明确的物理意义, 可以提高计算精度和计算效率 [37].

### 3 黏弹性问题的ICVEFG方法

黏弹性体具有弹性性质和黏性性质, 由于变形随时间而变化, 所以不能使用弹性力学的方法来研究黏弹性体. 表征黏弹性体特性, 通常使用服从胡克定律的弹性元件和牛顿黏性定律的黏性元件, 通过这两种元件的不同组合方式, 可以反映多种复杂的黏弹性体的应力应变关系. Maxwell 模型和 Kelvin 模型是黏弹性材料的两种典型的本构模型 [55].

应力、应变与时间的关系在线性黏弹性中可以表示为

$$\varepsilon(t) = \sigma_0 J(t), \quad (17)$$

其中  $J(t)$  是蠕变柔量.

如果给定求解域  $\Omega$  内的体力  $\mathbf{b}$ , 在面力边界  $\Gamma_t$  上的面力  $\bar{\mathbf{t}}$ , 在位移边界  $\Gamma_u$  上的位移分布  $\bar{\mathbf{u}}$  (边界条件与时间无关).

考虑二维黏弹性问题, 其平衡方程为

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = 0 \quad (\text{在 } \Omega \text{ 域}), \quad (18)$$

其中  $\mathbf{L}$  为微分算子矩阵,

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \quad (19)$$

几何方程为

$$\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u} \quad (\text{在 } \Omega \text{ 域}), \quad (20)$$

物理方程为

$$\boldsymbol{\sigma}(z) = \mathbf{M}_1 \begin{Bmatrix} e \\ e \\ 0 \end{Bmatrix} + \mathbf{M}_2 \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{Bmatrix}, \quad (21)$$

其中

$$\mathbf{M}_1 = \begin{bmatrix} 3K & 0 & 0 \\ 0 & 3K & 0 \\ 0 & 0 & 3K \end{bmatrix}, \quad (22)$$

$$\mathbf{M}_2 = \begin{bmatrix} \frac{1}{J_1(t)} & 0 & 0 \\ 0 & \frac{1}{J_1(t)} & 0 \\ 0 & 0 & \frac{1}{J_1(t)} \end{bmatrix}. \quad (23)$$

边界条件为

$$\mathbf{u} = \bar{\mathbf{u}} \quad (\text{在边界 } \Gamma_u \text{ 上}), \quad (24)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad (\text{在边界 } \Gamma_t \text{ 上}), \quad (25)$$

其中

$$\mathbf{n} = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}, \quad (26)$$

$n_1$  和  $n_2$  为边界点的外法线方向余弦.

由改进的复变量移动最小二乘法的试函数表达式 (14), 域内任意场点  $z$  的位移  $u^*(z)$  表示为

$$u^*(z) = u_1(z) + iu_2(z) = \Phi(z)u^* = \sum_{I=1}^n \Phi_I(z)u^*(z_I), \quad (27)$$

其中

$$u^*(z_I) = u_1(z_I) + iu_2(z_I). \quad (28)$$

结合 (27) 和 (28) 式可以得到

$$\begin{aligned} \mathbf{u}(z) &= \begin{Bmatrix} u_1(z) \\ u_2(z) \end{Bmatrix} \\ &= \sum_I^n \begin{bmatrix} \text{Re}[\Phi_I(z)] & -\text{Im}[\Phi_I(z)] \\ \text{Im}[\Phi_I(z)] & \text{Re}[\Phi_I(z)] \end{bmatrix} \\ &\quad \times \begin{Bmatrix} u_1(z_I) \\ u_2(z_I) \end{Bmatrix} \\ &= \sum_I^n \tilde{\Phi}_I(z) \cdot \mathbf{u}_I = \tilde{\Phi}(z) \cdot \mathbf{U}, \end{aligned} \quad (29)$$

其中  $\mathbf{U}$  是节点位移向量,  $\tilde{\Phi}(z)$  是形函数矩阵,  $n$  是场点  $z$  影响域内的节点总数,

$$\tilde{\Phi}(z) = (\tilde{\Phi}_1(z), \tilde{\Phi}_2(z), \dots, \tilde{\Phi}_n(z)), \quad (30)$$

$$\tilde{\Phi}_I(z) = \begin{bmatrix} \text{Re}[\Phi_I(z)] & -\text{Im}[\Phi_I(z)] \\ \text{Im}[\Phi_I(z)] & \text{Re}[\Phi_I(z)] \end{bmatrix}, \quad (31)$$

$$\mathbf{U}^T = (\mathbf{u}^T(z_1), \mathbf{u}^T(z_2), \dots, \mathbf{u}^T(z_n)), \quad (32) \quad \text{其中}$$

$$\mathbf{u}(z_I) = \begin{Bmatrix} u_1(z_I) \\ u_2(z_I) \end{Bmatrix}. \quad (33)$$

$$\begin{Bmatrix} e \\ e \\ 0 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ 0 \end{Bmatrix} = \mathbf{B}^{(1)}(z) \cdot \mathbf{u}, \quad (35)$$

黏弹性平面问题中, 应力可表示为

$$\boldsymbol{\sigma}(z) = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ \varepsilon_{22} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ \frac{1}{2}\varepsilon_{12} \end{Bmatrix} = \mathbf{B}^{(2)}(z) \cdot \mathbf{u}, \quad (36)$$

$$= \begin{Bmatrix} \sigma_v \\ \sigma_v \\ 0 \end{Bmatrix} + \begin{Bmatrix} \sigma_{11} - \sigma_v \\ \sigma_{22} - \sigma_v \\ \sigma_{12} \end{Bmatrix}$$

$$= \mathbf{M}_1 \begin{Bmatrix} e \\ e \\ 0 \end{Bmatrix} + \mathbf{M}_2 \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{Bmatrix}, \quad (34)$$

$\mathbf{B}^{(1)}(z)$  是正应变矩阵

$$\mathbf{B}^{(1)}(z) = (\mathbf{B}_1^{(1)}(z), \mathbf{B}_2^{(1)}(z), \dots, \mathbf{B}_n^{(1)}(z)), \quad (37)$$

$\mathbf{B}^{(2)}(z)$  是偏应变矩阵

$$\mathbf{B}^{(2)}(z) = (\mathbf{B}_1^{(2)}(z), \mathbf{B}_2^{(2)}(z), \dots, \mathbf{B}_n^{(2)}(z)). \quad (38)$$

对平面应力情况, 即  $\sigma_{33} = 0$ , 可得

$$\mathbf{B}_I^{(1)}(z) = \frac{1}{3KJ_1(t) + 2} \cdot \begin{bmatrix} \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ 0 & 0 \end{bmatrix}, \quad (39)$$

$$\mathbf{B}_I^{(2)}(z) = \begin{bmatrix} \beta_1 \text{Re}[\Phi_{I,1}(z)] - \beta_2 \text{Im}[\Phi_{I,2}(z)] & -\beta_1 \text{Im}[\Phi_{I,1}(z)] - \beta_2 \text{Re}[\Phi_{I,2}(z)] \\ -\beta_2 \text{Re}[\Phi_{I,1}(z)] + \beta_1 \text{Im}[\Phi_{I,2}(z)] & \beta_2 \text{Im}[\Phi_{I,1}(z)] + \beta_1 \text{Re}[\Phi_{I,2}(z)] \\ \frac{1}{2}\{\text{Re}[\Phi_{I,2}(z)] + \text{Im}[\Phi_{I,1}(z)]\} & \frac{1}{2}\{-\text{Im}[\Phi_{I,2}(z)] + \text{Re}[\Phi_{I,1}(z)]\} \end{bmatrix}, \quad (40)$$

其中  $\beta_1$  和  $\beta_2$  是常数,

$$\beta_1 = \frac{3KJ_1(t) + 1}{3KJ_1(t) + 2}, \quad (41)$$

$$\beta_2 = \frac{1}{3KJ_1(t) + 2}. \quad (42)$$

令

$$\mathbf{B}_I^{(1*)}(z) = \begin{bmatrix} \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ 0 & 0 \end{bmatrix}, \quad (43)$$

则 (39) 式可写为

$$\mathbf{B}_I^{(1)}(z) = \frac{1}{3KJ_1(t) + 2} \cdot \mathbf{B}_I^{(1*)}(z). \quad (44)$$

对平面应变情况, 即  $\varepsilon_{33} = 0$ , 可得

$$\mathbf{B}_I^{(1)}(z) = \frac{1}{3} \cdot \begin{bmatrix} \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ \text{Re}[\Phi_{I,1}(z)] + \text{Im}[\Phi_{I,2}(z)] & -\text{Im}[\Phi_{I,1}(z)] + \text{Re}[\Phi_{I,2}(z)] \\ 0 & 0 \end{bmatrix}, \quad (45)$$

$$\mathbf{B}_I^{(2)}(z) = \begin{bmatrix} \frac{2}{3}\text{Re}[\Phi_{I,1}(z)] - \frac{1}{3}\text{Im}[\Phi_{I,2}(z)] & -\frac{2}{3}\text{Im}[\Phi_{I,1}(z)] - \frac{1}{3}\text{Re}[\Phi_{I,2}(z)] \\ -\frac{2}{3}\text{Re}[\Phi_{I,1}(z)] + \frac{1}{3}\text{Im}[\Phi_{I,2}(z)] & \frac{2}{3}\text{Im}[\Phi_{I,1}(z)] + \frac{1}{3}\text{Re}[\Phi_{I,2}(z)] \\ \frac{1}{2}\{\text{Re}[\Phi_{I,2}(z)] + \text{Im}[\Phi_{I,1}(z)]\} & \frac{1}{2}\{-\text{Im}[\Phi_{I,2}(z)] + \text{Re}[\Phi_{I,1}(z)]\} \end{bmatrix}. \quad (46)$$

黏弹性平面问题中, 应变可表示为

$$\begin{aligned} \boldsymbol{\varepsilon}(z) &= \begin{Bmatrix} \varepsilon_{11}(z) \\ \varepsilon_{22}(z) \\ \varepsilon_{12}(z) \end{Bmatrix} = \mathbf{L}\mathbf{u}^*(z) = \mathbf{L}(\tilde{\boldsymbol{\Phi}}(z)\mathbf{u}^*) = \mathbf{L}(\tilde{\boldsymbol{\Phi}}(z)\mathbf{Q}\mathbf{u}) \\ &= \begin{bmatrix} \left(\text{Re}\left[\sum_{I=1}^n \Phi_I(z)u^*(z_I)\right]\right)_{,1} \\ \left(\text{Im}\left[\sum_{I=1}^n \Phi_I(z)u^*(z_I)\right]\right)_{,2} \\ \left(\text{Re}\left[\sum_{I=1}^n \Phi_I(z)u^*(z_I)\right]\right)_{,2} + \left(\text{Im}\left[\sum_{I=1}^n \Phi_I(z)u^*(z_I)\right]\right)_{,1} \end{bmatrix} = \mathbf{B}(z) \cdot \mathbf{u}, \end{aligned} \quad (47)$$

其中

$$\mathbf{B}(z) = (\mathbf{B}_1(z), \mathbf{B}_2(z), \dots, \mathbf{B}_n(z)), \quad (48)$$

$$\mathbf{B}_I(z) = \begin{bmatrix} \text{Re}[\Phi_{I,1}(z)] & -\text{Im}[\Phi_{I,1}(z)] \\ \text{Im}[\Phi_{I,2}(z)] & \text{Re}[\Phi_{I,2}(z)] \\ \text{Re}[\Phi_{I,2}(z)] + \text{Im}[\Phi_{I,1}(z)] & -\text{Im}[\Phi_{I,2}(z)] + \text{Re}[\Phi_{I,1}(z)] \end{bmatrix}. \quad (49)$$

采用罚函数法引入位移边界条件, 得到黏弹性问题的 Galerkin 积分弱形式为

$$\begin{aligned} &\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega \\ &- \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma_t + \alpha \int_{\Gamma_u} \delta \mathbf{u}^T \mathbf{S}(\mathbf{u} - \mathbf{u}^*) d\Gamma_u \\ &= 0. \end{aligned} \quad (50)$$

将 (34) 和 (47) 式代入 (50) 式得到

$$\begin{aligned} &\int_{\Omega} \delta \mathbf{U}^T (\mathbf{B}^T \mathbf{M}_1 \mathbf{B}^{(1)}) \mathbf{U} d\Omega \\ &+ \int_{\Omega} \delta \mathbf{U}^T (\mathbf{B}^T \mathbf{M}_2 \mathbf{B}^{(2)}) \mathbf{U} d\Omega \\ &- \int_{\Omega} \delta \mathbf{U}^T \tilde{\boldsymbol{\Phi}}^T \mathbf{b} d\Omega \\ &- \int_{\Gamma_t} \delta \mathbf{U}^T \tilde{\boldsymbol{\Phi}}^T \bar{\mathbf{t}} d\Gamma_t \\ &+ \alpha \int_{\Gamma_u} \delta \mathbf{U}^T (\tilde{\boldsymbol{\Phi}}^T \mathbf{S} \tilde{\boldsymbol{\Phi}}) \mathbf{U} d\Gamma_u \\ &- \alpha \int_{\Gamma_u} \delta \mathbf{U}^T (\tilde{\boldsymbol{\Phi}}^T \mathbf{S} \mathbf{u}^*) d\Gamma_u = 0. \end{aligned} \quad (51)$$

离散系统方程为

$$(\mathbf{K} + \mathbf{K}^\alpha) \cdot \mathbf{U} = \mathbf{F} + \mathbf{F}^\alpha, \quad (52)$$

其中

$$\begin{aligned} \mathbf{K} &= \frac{1}{3KJ_1(t) + 2} \cdot \int_{\Omega} \mathbf{B}^T \cdot \mathbf{M}_1 \cdot \mathbf{B}^{(1*)} d\Omega \\ &+ \int_{\Omega} \mathbf{B}^T \cdot \mathbf{M}_2 \cdot \mathbf{B}^{(2)} d\Omega, \end{aligned} \quad (53)$$

$$\mathbf{F} = \int_{\Omega} \tilde{\boldsymbol{\Phi}}^T(z) \cdot \mathbf{b} d\Omega + \int_{\Gamma_t} \tilde{\boldsymbol{\Phi}}^T(z) \cdot \bar{\mathbf{t}} d\Gamma_t, \quad (54)$$

$$\mathbf{K}^\alpha = \alpha \int_{\Gamma_u} \tilde{\boldsymbol{\Phi}}^T(z) \cdot \mathbf{S} \cdot \tilde{\boldsymbol{\Phi}}(z) d\Gamma_u, \quad (55)$$

$$\mathbf{F}^\alpha = \alpha \int_{\Gamma_u} \tilde{\boldsymbol{\Phi}}^T(z) \cdot \mathbf{S} \cdot \mathbf{u}^* d\Gamma_u, \quad (56)$$

$$\mathbf{S} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}, \quad (57)$$

其中, 当  $x_1$  (或  $x_2$ ) 有位移约束时, 相应的  $S_x$  (或  $S_y$ ) 等于 1, 否则为 0.

上述即为二维黏弹性问题的 ICVEFG 方法.

#### 4 数值算例

本文基于上述理论, 对两个典型的黏弹性问题算例进行了计算, 并将计算的精度与速度分别与

CVEFG方法和有限元方法(FEM)进行了比较. 算例中, 影响域为矩形区域, 构造逼近函数时采用线性基函数, 权函数采用三次样条权函数, 背景积分单元上数值积分采用 $4 \times 4$ 阶 Gauss 积分.

**算例1 受均布荷载的悬臂梁**

图1所示为受均布荷载的悬臂梁, 梁长 $L = 8$  m, 梁高 $H = 2$  m, 取单位厚度. 材料的弹性模量 $E = 1.0 \times 10^8$  Pa, Poisson 比 $\nu = 0.25$ , 剪切变形的流变性质满足 Kelvin 黏弹性模型, 参数为 $G = 2.0 \times 10^8$  Pa. 均布荷载集度 $q = 3.0 \times 10^4$  Pa, 不计自重, 按平面应力问题计算.

图2所示为求解域内的节点布置情况. 其中 $d_{\max} = 3.8$ 为决定影响域大小的比例参数取值, 罚因子 $\alpha = (1.0 \times 10^3) \times E$ .

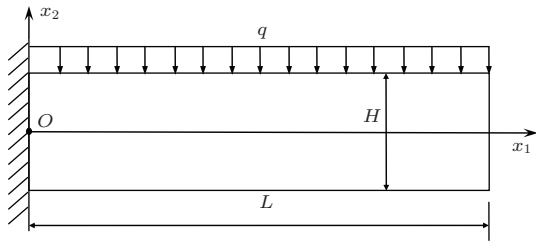


图1 受均布荷载的悬臂梁

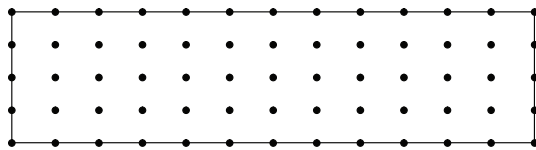


图2 节点布置

图3和图4分别给出了 $t = 20$  s时 $x_1 = 1$  m截面上节点正应力和梁右端中点(8, 0)的挠度随时间的变化. 从图3和图4计算结果可以看出, ICVEFG方法比CVEFG方法更接近于有限元解, 计算精度更高.

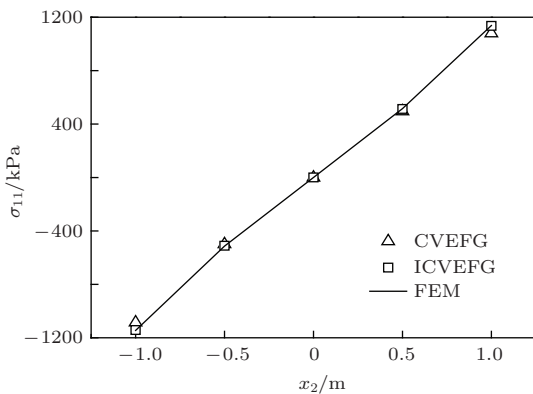


图3  $t = 20$  s时 $x_1 = 1$  m截面上节点正应力

**算例2 受纯弯曲的梁**

图5为受纯弯曲的梁, 梁长 $L = 5$  m, 梁高 $H = 2$  m, 取单位厚度. 材料的弹性模量 $E = 1.0 \times 10^6$  Pa, Poisson 比 $\nu = 0.3$ , 剪切变形的流变性质满足三参数模型, 其参数为 $G_1 = 5.0 \times 10^5$  Pa,  $G_2 = 1.0 \times 10^6$  Pa和 $\eta = 2.0 \times 10^6$  Pa·s. 其所受三角形分布荷载大小如图5所示, 不计体力, 按平面应力问题计算.

由于梁具有对称性, 则取四分之一梁进行计算. 在 $x_1$ 轴和 $x_2$ 轴上, 位移 $u_1$ 均为零, 令原点处 $u_2$ 为零, 消除结构在 $x_2$ 方向上刚体位移. 图6所示为节点布置情况. 决定影响域大小的比例参数取 $d_{\max} = 4.0$ . 罚因子 $\alpha = (1.0 \times 10^2) \times E$ .

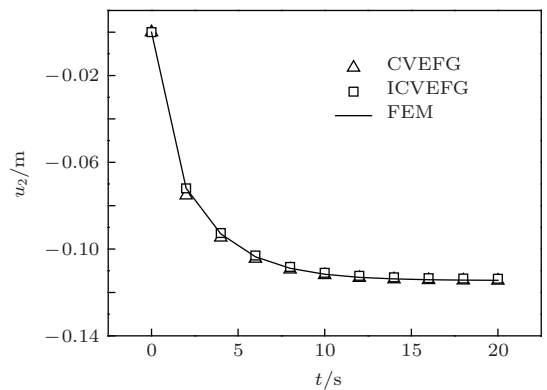


图4 梁右端中点(8, 0)的挠度随时间的变化

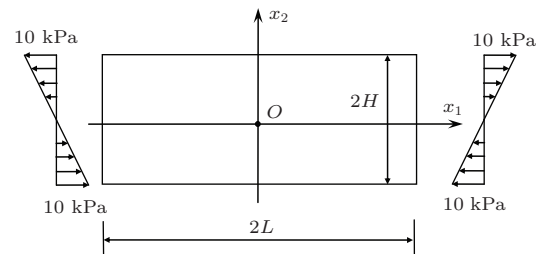


图5 受纯弯曲的梁

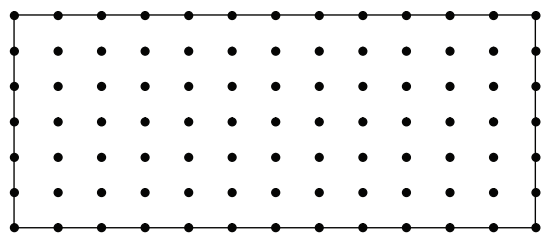


图6 四分之一梁的节点布置

图7为 $t = 30$  s时梁右端边界上节点水平位移, 图8为 $t = 30$  s时 $x_1 = L/2$ 截面上节点正应力. 当梁中的节点布置相同时, 采用ICVEFG方法所得的计算结果与CVEFG方法和FEM 计算结果进行

比较. 从图7和图8可知, ICVEFG方法所得到的计算结果与FEM算出的结果相差非常小, 而且此方法比CVEFG方法更接近于有限元解, 计算精度更高.

图9为  $t = 30$  s时梁中性轴上节点挠度, 当精度相似时, ICVEFG方法比CVEFG方法所使用的计算时间更短. 可以看出, ICVEFG方法能够更加节省计算时间.

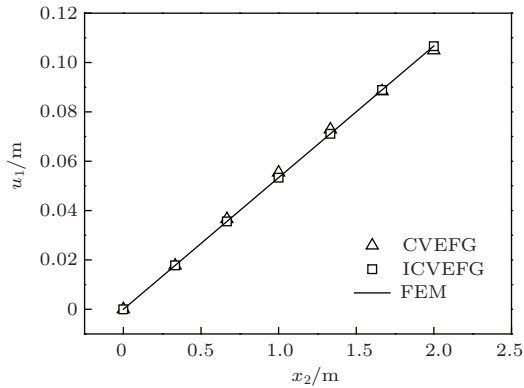


图7  $t = 30$  s时梁右端边界上节点水平位移

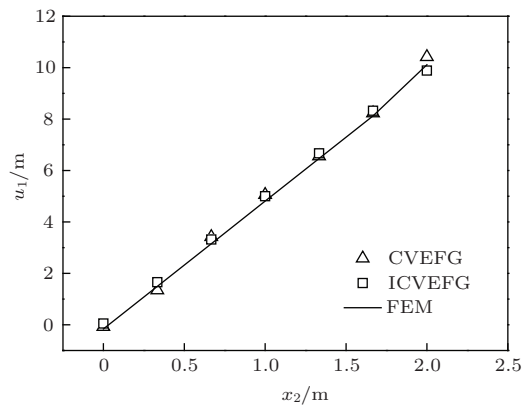


图8  $t = 30$  s时  $x_1 = L/2$  截面上节点正应力

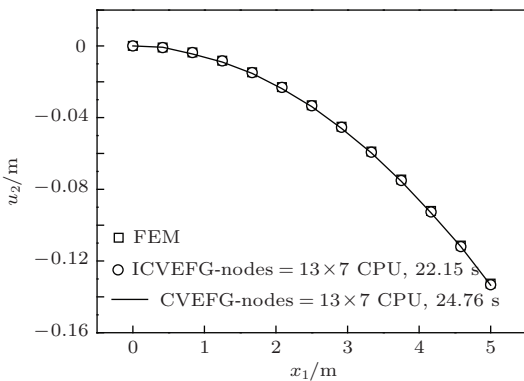


图9  $t = 30$  s时梁中性轴上节点挠度

## 5 结 论

本文基于改进的复变量移动最小二乘法, 将改进的复变量无单元Galerkin方法应用于黏弹性问题, 通过算例验证了该方法相对于复变量无单元Galerkin方法计算精度和计算速度更高, 说明了该方法的有效性.

## 参考文献

- [1] Belytschko T, Krongauz Y, Organ D 1996 *Comput. Methods Appl. Mech. Engineer.* **139** 3
- [2] Cheng Y M, Ji X, He P F 2004 *Acta Mech. Sin.* **36** 43 (in Chinese) [程玉民, 嵇醒, 贺鹏飞 2004 *力学学报* **36** 43]
- [3] Belytschko T, Lu Y Y, Gu L 1994 *Int. J. Numer. Methods Engineer.* **37** 229
- [4] Zhang Z, Li D M, Cheng Y M, Liew K M 2012 *Acta Mech. Sin.* **28** 808
- [5] Zhang Z, Wang J F, Cheng Y M, Liew K M 2013 *Sci. China G: Phys. Mech. Astron.* **56** 1568
- [6] Zhang Z, Hao S Y, Liew K M, Cheng Y M 2013 *Engineering Analysis with Boundary Elements* **37** 1576
- [7] Chen L, Cheng Y M 2008 *Acta Phys. Sin.* **57** 1 (in Chinese) [陈丽, 程玉民 2008 *物理学报* **57** 1]
- [8] Monaghan J J 1998 *Comput. Phys. Commun.* **48** 89
- [9] Cheng R J, Cheng Y M 2008 *Appl. Numer. Math.* **58** 884
- [10] Cheng R J, Cheng Y M 2007 *Acta Phys. Sin.* **56** 5569 (in Chinese) [程荣军, 程玉民 2007 *物理学报* **56** 5569]
- [11] Cheng R J, Cheng Y M 2007 *Acta Mech. Sin.* **39** 843 (in Chinese) [程荣军, 程玉民 2007 *力学学报* **39** 843]
- [12] Dai B D, Cheng J, Zheng B J 2013 *Int. J. Appl. Mech.* **5** 1350011
- [13] Li S C, Cheng Y M 2004 *Acta Mech. Sin.* **36** 496 (in Chinese) [李树忱, 程玉民 2004 *力学学报* **36** 496]
- [14] Li S C, Li S C, Cheng Y M 2005 *Theoret. Appl. Fract. Mech.* **44** 234
- [15] Chen L, Cheng Y M 2008 *Acta Phys. Sin.* **57** 6047 (in Chinese) [陈丽, 程玉民 2008 *物理学报* **57** 6047]
- [16] Chen L, Cheng Y M 2010 *Sci. China G: Phys. Mech. Astron.* **40** 242 (in Chinese) [陈丽, 程玉民 2010 *中国科学 G 辑 物理学 力学 天文学* **40** 242]
- [17] Cheng Y M, Chen M J 2003 *Acta Mech. Sin.* **35** 181 (in Chinese) [程玉民, 陈美娟 2003 *力学学报* **35** 181]
- [18] Cheng Y M, Peng M J 2005 *Sci. China G: Phys. Mech. Astron.* **35** 435 (in Chinese) [程玉民, 彭妙娟 2005 *中国科学 G 辑 物理学 力学 天文学* **35** 435]
- [19] Cheng Y M, Peng M J 2005 *Sci. China G: Phys. Mech. Astron.* **48** 641
- [20] Qin Y X, Cheng Y M 2006 *Acta Phys. Sin.* **55** 3215 (in Chinese) [秦义校, 程玉民 2006 *物理学报* **55** 3215]
- [21] Dai B D, Cheng Y M 2007 *Acta Phys. Sin.* **56** 597 (in Chinese) [戴保东, 程玉民 2007 *物理学报* **56** 597]

- [22] Peng M J, Cheng Y M 2009 *Engineering Analysis with Boundary Elements* **33** 77
- [23] Cheng Y M, Liew K M, Kitipornchai S 2009 *Int. J. Numer. Methods Engineer.* **78** 1258
- [24] Yang H T, Liu Y 2003 *Int. J. Solids Struct.* **40** 701
- [25] Sladek J, Sladek V, Zhang C H 2005 *Engineering Analysis with Boundary Elements* **29** 597
- [26] Sladek J, Sladek V, Zhang C H, Schanz M 2006 *Computat. Mech.* **37** 279
- [27] Cheng Y M, Li R X, Peng M J 2012 *Chin. Phys. B* **21** 090205
- [28] Cheng Y M, Peng M J, Li J H 2005 *Acta Mech. Sin.* **37** 719 (in Chinese) [程玉民, 彭妙娟, 李九红 2005 力学学报 **37** 719]
- [29] Cheng Y M, Li J H 2005 *Sci. China G: Phys. Mech. Astron.* **35** 548 (in Chinese) [程玉民, 李九红 2005 中国科学 G 辑 物理学 力学 天文学 **35** 548]
- [30] Cheng Y M, Li J H 2005 *Acta Phys. Sin.* **54** 4463 (in Chinese) [程玉民, 李九红 2005 物理学报 **54** 4463]
- [31] Cheng Y M, Li J H 2006 *Sci. China G: Phys. Mech. Astron.* **49** 46
- [32] Liew K M, Feng C, Cheng Y M, Kitipornchai S 2007 *Int. J. Numer. Methods Engineer* **70** 46
- [33] Cheng Y M, Wang J F, Li R X 2012 *Int. J. Appl. Mech.* **4** 1250042
- [34] Liew K M, Cheng Y 2009 *Comput. Methods Appl. Mech. Engineer.* **198** 3925
- [35] Gao H, Cheng Y 2010 *Int. J. Computat. Methods* **7** 55
- [36] Gao H F, Cheng Y M 2009 *Acta Mech. Sin.* **41** 480 (in Chinese) [高洪芬, 程玉民 2009 力学学报 **41** 480]
- [37] Bai F N, Li D M, Wang J F, Cheng Y M 2012 *Chin. Phys. B* **21** 020204
- [38] Cheng Y M, Wang J F, Bai F N 2012 *Chin. Phys. B* **21** 090203
- [39] Wang J F, Cheng Y M 2012 *Chin. Phys. B* **21** 120206
- [40] Wang J F, Cheng Y M 2013 *Chin. Phys. B* **22** 030208
- [41] Li D M, Bai F N, Cheng Y M, Liew K M 2012 *Comput. Methods Appl. Mech. Engineer.* **233–236** 1
- [42] Li D M, Liew K M, Cheng Y M 2014 *Comput. Methods Appl. Mech. Engineer.* **269** 72
- [43] Peng M, Liu P, Cheng Y 2009 *Int. J. Appl. Mech.* **1** 367
- [44] Liu P, Peng M J, Cheng Y M 2009 *Comput. Aided Engineer.* **18** 10 (in Chinese) [刘沛, 彭妙娟, 程玉民 2009 计算机辅助工程 **18** 10]
- [45] Peng M J, Li D M, Cheng Y M 2011 *Engineer. Struct.* **33** 127
- [46] Li D M, Peng M J, Cheng Y M 2011 *Sci. China G: Phys. Mech. Astron.* **41** 1003 (in Chinese) [李冬明, 彭妙娟, 程玉民 2011 中国科学 G 辑 物理学 力学 天文学 **41** 1003]
- [47] Peng M J, Li R X, Cheng Y M 2014 *Engineering Analysis with Boundary Elements* **40** 104
- [48] Li D M, Liew K M, Cheng Y M 2014 *Computat. Mech.* **53** 1149
- [49] Chen L, Cheng Y 2010 *Sci. China G: Phys. Mech. Astron.* **53** 954
- [50] Chen L, Cheng Y M 2010 *Chin. Phys. B* **19** 090204
- [51] Chen L, Ma H P, Cheng Y M 2013 *Chin. Phys. B* **22** 050202
- [52] Weng Y J, Cheng Y M 2013 *Chin. Phys. B* **22** 090204
- [53] Weng Y J, Zhang Z, Cheng Y M 2014 *Engineering Analysis with Boundary Elements* **44** 36
- [54] Ren H, Cheng J, Huang A 2012 *Appl. Math. Computat.* **219** 1724
- [55] Flügge W 1975 *Viscoelasticity* (2nd Ed) (New York: Springer-Verlag)



# Improved complex variable element-free Galerkin method for viscoelasticity problems\*

Peng Miao-Juan<sup>†</sup> Liu Qian

(Department of Civil Engineering, Shanghai University, Shanghai 200072, China)

( Received 29 April 2014; revised manuscript received 20 May 2014 )

## Abstract

In this paper, based on the improved complex variable least-square (ICVMLS) approximation, the improved complex variable element-free Galerkin (ICVEFG) method for two-dimensional viscoelasticity problems is proposed. The ICVMLS approximation is used to form the shape function, the Galerkin weak form is used to obtain the system equations, and the penalty method is used to impose the essential boundary conditions, then the corresponding formulae of the ICVEFG method for two-dimensional viscoelasticity problems are obtained. Finally, some numerical examples are given, and the numerical results from the ICVEFG method are compared with those from the CVEFG method and finite element method, and the results show that the ICVEFG method in this paper has the high computational precision and efficiency.

**Keywords:** meshless method, complex variable moving least-squares approximation, improved complex variable element-free Galerkin method, viscoelasticity problem

**PACS:** 02.60.Cb, 02.60.Lj, 46.35.+z

**DOI:** [10.7498/aps.63.180203](https://doi.org/10.7498/aps.63.180203)

---

\* Project supported by the National Natural Science Foundation of China (Grant No. 11171208)

<sup>†</sup> Corresponding author. E-mail: [mjpeng@shu.edu.cn](mailto:mjpeng@shu.edu.cn)