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含非线性微扰项的二阶动力学系统的一阶近似守恒量的一种新求法*

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从一阶近似守恒量的性质出发, 把受微扰系统视为未受微扰系统与微扰项的迭加, 提出一种分三步求得一阶近似守恒量的新方法: 先选择合适的方法求得未受微扰系统的守恒量 I_0, 再考虑微扰项对守恒量 I_0 的影响, 最后利用一阶近似守恒量的性质求得一阶近似守恒量. 用该方法研究了一实际的受非线性微扰作用的两自由度动力学系统, 得到 4 个稳定的一阶近似守恒量. 用坐标变换法和微扰法得到系统一阶近似解的表达式, 并讨论 4 种特殊情况下的一阶近似解.

关键词: 非线性微扰, 二阶动力学系统, 一阶近似守恒量, 坐标变换法

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1 引 言

许多实际力学系统的某些参数常常会随着位移、速度和时间发生微小的变化, 其运动微分方程一般是含非线性微扰项的二阶微分方程, 此类系统的近似守恒量和近似对称性研究对于研究力学系统的特性以及得到方程的近似解至关重要. 近年来关于常微分方程、偏微分方程近似对称性和近似守恒量的研究已取得不少的成果[1−10]. 目前研究近似对称性和近似守恒量主要采用近似 Lie 对称性理论[1] 和近似 Noether 对称性理论[2], 引进近似的群无限小变换, 微分方程在此变换下近似保持不变则为近似 Lie 对称性; 哈密顿作用量在此变换下近似保持不变则为近似 Noether 对称性, 所得的守恒量为近似守恒量. 用近似对称性理论求近似守恒量要用到近似的群无限小变换, 并需解出近似的无限小生成元、规范函数, 计算较繁复且易遗漏, 理论性强又比较抽象. 本文从一阶近似守恒量的性质出发, 把受微扰系统视为未受微扰系统与微扰项的迭加, 提出一种分三步求得一阶近似守恒量的新方法, 先选择合适的方法求得未受微扰系统的守恒量 I_0, 再考虑微扰项对守恒量 I_0 的影响, 最后利用一阶近似守恒量的性质求得守恒量. 用新方法研究了一实际的受非线性微扰作用的两自由度动力学系统, 得到了 4 个稳定的一阶近似守恒量. 先通过坐标变换对微分方程解耦, 再用微扰法解得新坐标系下的一阶近似解, 通过坐标反变换得到原坐标系下的一阶近似解, 并对 4 种特殊情况进行了讨论.

未受微扰系统的守恒量 I_0 有多种求法, 如: 对称性法[11−16] (即 Lie 对称性法, Noether 对称性法和 Mei 对称性法)、Ermakov 方法[17−19]、Poisson 括号方法[20−23]、直接积分法[24,25]、扩展 Prelle-Singer (P-S) 法[26−28]、坐标变换法[29] (改变坐标标度与旋转坐标轴法), 每种方法各有特点和适用条件. 用对称性法求守恒量要已知系统的 Lagrange 函数 (或 Hamilton 函数), 用到群的无限小变换, 理论性强且比较抽象, 但能同时研究相应的对称性. Ermakov 方法只能求可表示成 Ermakov 形式系统的守恒量, 适用范围较小. Poisson 括号法要已知系统的 Hamilton 函数, 且只能求线性耦合系统的守

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恒量, 但只需通过Poisson括号的计算就能得到守恒量, 理论简单, 数学计算较方便. 直接积分法适用求线性耦合系统的守恒量, 数学计算简单, 不需要已知系统的Lagrange函数. 扩展P-S法求守恒量, 要用到较多的积分乘子, 数学计算较繁复, 但不需要已知系统的Lagrange函数. 坐标变换法(改变坐标标度与旋转坐标轴法)适合求线性耦合系统的守恒量, 方法简单, 物理意义明确. 因此, 可以根据未受微扰系统的运动微分方程的形式选择合适的方法, 甚至于可采用几种方法联合求得未受微扰系统的守恒量I_0, 方法灵活.

2 求二阶动力学系统一阶近似守恒量的基本理论

实际的含非线性微扰项的二阶动力学系统的运动微分方程一般不显含时间和广义速度, 可表示成

$$\ddot{x}_1 = g_1(x_1, x_2, \varepsilon) = g_1(\varepsilon^0) + \varepsilon g_1(\varepsilon^1), \tag{1a}$$

$$\ddot{x}_2 = g_2(x_1, x_2, \varepsilon) = g_2(\varepsilon^0) + \varepsilon g_2(\varepsilon^1), \tag{1b}$$

其中, $g_1(x_1, x_2, \varepsilon)$, $g_2(x_1, x_2, \varepsilon)$为广义加速度, 可表示成未受微扰作用时的广义加速度$g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$和因微扰作用产生的一阶微扰项$\varepsilon g_1(\varepsilon^1)$, $\varepsilon g_2(\varepsilon^1)$之和; $g_1(\varepsilon^1)$, $g_2(\varepsilon^1)$表示一阶微扰项的系数(下文表示类同). 与系统(1) 相应的未受微扰作用系统的运动微分方程可表示成

$$\ddot{x}_1 = g_1(x_1, x_2) = g_1(\varepsilon^0), \tag{2a}$$

$$\ddot{x}_2 = g_2(x_1, x_2) = g_2(\varepsilon^0). \tag{2b}$$

系统(1)可视为系统(2a)与一阶微扰项$\varepsilon g_1(\varepsilon^1)$, $\varepsilon g_2(\varepsilon^1)$的迭加. 因此, 求系统(1)的一阶近似守恒量的步骤可归纳如下: 首先根据方程(2a) 的形式, 选择一种较合适的方法求得其守恒量I_0, 由此求得的守恒量中不含微扰项; 其次, 考虑微扰项$\varepsilon g_1(\varepsilon^1)$, $\varepsilon g_2(\varepsilon^1)$对守恒量$I_0$ 的影响, 计算守恒量I_0对时间一阶导数的一阶微扰项大小$\dfrac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1)$, 即$\dfrac{\mathrm{d}I_0}{\mathrm{d}t}$中的$\dot{x}_1, \dot{x}_2$用(1)式替代; 最后, 根据一阶近似守恒量的性质及$g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$的形式求得I_1.

一阶近似守恒量可表示成[3]

$$I = I_0 + \varepsilon I_1, \tag{3}$$

一阶近似守恒量的性质为[3]

$$\frac{\mathrm{d}I}{\mathrm{d}t} = O(\varepsilon^2). \tag{4}$$

将(3)式代入(4)式并展开, 令$\varepsilon^0, \varepsilon^1$的系数分别等于0, 忽略$\varepsilon^2$以上项, 可得

$$\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^0) = 0, \tag{5a}$$

$$\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1) + \frac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0) = 0, \tag{5b}$$

从系统(2a)求得的I_0一定满足(5a)式, 将$\dfrac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1)$代入(5b) 式可得$\dfrac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0)$, 同时考虑$g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$的形式就可求得I_1. I_0和I_1均不为0, 称为稳定的一阶近似守恒量.

综上所述, 一阶近似守恒量 (3)式中的第一部分I_0的形式是由未受微扰系统(2a) 决定的, 而第二部分I_1的形式是由未受微扰系统和微扰项共同决定的, 通过微扰项$\varepsilon g_1(\varepsilon^1)$, $\varepsilon g_2(\varepsilon^1)$对守恒量$I_0$产生影响, 此影响体现在$\dfrac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1)$中 (也即体现在$\dfrac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0)$中), 而由$\dfrac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0)$确定$I_1$时又要考虑未受微扰项$g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$的形式, 实现$g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$项对I_1的影响.

3 两自由度动力学系统的一阶近似守恒量

设受非线性微扰作用的两自由度动力学系统的运动微分方程可表示成

$$\begin{aligned}\ddot{x}_1 &= -\frac{5}{2}x_1 + \frac{3}{2}x_2 + \varepsilon(x_1^2 + x_2^2)\\ &= g_1 = g_1(\varepsilon^0) + \varepsilon g_1(\varepsilon^1),\end{aligned} \tag{6a}$$

$$\begin{aligned}\ddot{x}_2 &= -\frac{5}{2}x_2 + \frac{3}{2}x_1 + \varepsilon(2x_1x_2)\\ &= g_2 = g_2(\varepsilon^0) + \varepsilon g_2(\varepsilon^1),\end{aligned} \tag{6b}$$

其中ε为非线性耦合系数, 且$0 < \varepsilon \ll 1$. 与系统(6) 相应的未受微扰系统为

$$\ddot{x}_1 = -\frac{5}{2}x_1 + \frac{3}{2}x_2, \tag{7a}$$

$$\ddot{x}_2 = -\frac{5}{2}x_2 + \frac{3}{2}x_1, \tag{7b}$$

系统(7)是一线性耦合系统, 其守恒量可用坐标变换法和扩展P-S法(或Lie对称性法)联合求得. 设

$$u_1 = \frac{\sqrt{2}}{2}(x_1 - x_2), \quad u_2 = \frac{\sqrt{2}}{2}(x_1 + x_2), \tag{8}$$

则 (7) 式变为

$$\ddot{u}_1 = -4u_1, \tag{9a}$$

$$\ddot{u}_2 = -u_2, \tag{9b}$$

(9) 式表示频率比为 2 : 1 的两维各向异性谐振子系统, 存在 4 个守恒量 [29]

$$I_0^1 = \frac{1}{2}\dot{u}_1^2 + 2u_1^2, \tag{10a}$$

$$I_0^2 = \frac{1}{2}\dot{u}_2^2 + \frac{1}{2}u_2^2, \tag{10b}$$

$$I_0^3 = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) + \frac{1}{2}(4u_1^2 + u_2^2), \tag{10c}$$

$$I_0^4 = u_1 u_2^2 - u_1\dot{u}_2^2 + u_2\dot{u}_1\dot{u}_2, \tag{10d}$$

(10) 式中的前三个守恒量可以根据 (9) 式直接积分得到; I_0^1, I_0^2 分别表示两分振子的能量; I_0^3 表示总能量; 它们不相互独立, 存在 $I_0^3 = I_0^1 + I_0^2$ 关系. 第 4 个守恒量可用扩展 P-S 法或 Lie 对称性法等多种方法得到, 文献 [29] 是用扩展 P-S 法求得的.

利用 (8) 式进行坐标反变换, 可得 4 个守恒量在原坐标下的表示

$$I_0^1 = \frac{1}{4}(\dot{x}_1 - \dot{x}_2)^2 + (x_1 - x_2)^2, \tag{11a}$$

$$I_0^2 = \frac{1}{4}(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{4}(x_1 + x_2)^2, \tag{11b}$$

$$I_0^3 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{4}(5x_1^2 + 5x_2^2 - 6x_1x_2), \tag{11c}$$

$$\begin{aligned} I_0^4 = {} & (x_1 - x_2)(x_1 + x_2)^2 - (x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ & + (x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2). \end{aligned} \tag{11d}$$

下面计算 $\frac{\mathrm{d}I_0^\alpha}{\mathrm{d}t}(\varepsilon^1)(\alpha = 1, 2, 3, 4)$. 将 (11) 式分别代入 $\frac{\mathrm{d}I_0^\alpha}{\mathrm{d}t}$, 并考虑 (6) 式中的 $g_1(\varepsilon^1)$, $g_2(\varepsilon^1)$ 项, 得

$$\begin{aligned} \frac{\mathrm{d}I_0^1}{\mathrm{d}t}(\varepsilon^1) &= \frac{1}{2}(\dot{x}_1 - \dot{x}_2)[g_1(\varepsilon^1) - g_2(\varepsilon^1)] \\ &= \frac{1}{2}(x_1 - x_2)^2(\dot{x}_1 - \dot{x}_2), \end{aligned} \tag{12a}$$

$$\begin{aligned} \frac{\mathrm{d}I_0^2}{\mathrm{d}t}(\varepsilon^1) &= \frac{1}{2}(\dot{x}_1 + \dot{x}_2)\left[g_1(\varepsilon^1) + g_2(\varepsilon^1)\right] \\ &= \frac{1}{2}(x_1 + x_2)^2(\dot{x}_1 + \dot{x}_2), \end{aligned} \tag{12b}$$

$$\begin{aligned} \frac{\mathrm{d}I_0^3}{\mathrm{d}t}(\varepsilon^1) &= \dot{x}_1 g_1(\varepsilon^1) + \dot{x}_2 g_2(\varepsilon^1) \\ &= (x_1^2 + x_2^2)\dot{x}_1 + 2x_1x_2\dot{x}_2, \end{aligned} \tag{12c}$$

$$\begin{aligned} \frac{\mathrm{d}I_0^4}{\mathrm{d}t}(\varepsilon^1) = {} & -2(x_1 - x_2)(\dot{x}_1 + \dot{x}_2)(g_1(\varepsilon^1) + g_2(\varepsilon^1)) \\ & + 2(x_1 + x_2)(\dot{x}_1 g_1(\varepsilon^1) - \dot{x}_2 g_2(\varepsilon^1)) \\ = {} & -2(x_1 - x_2)(x_1 + x_2)^2(\dot{x}_1 + \dot{x}_2) \\ & + (x_1 + x_2)(x_1 - x_2)^2(\dot{x}_1 + \dot{x}_2) \\ & + (x_1 + x_2)^3(\dot{x}_1 - \dot{x}_2). \end{aligned} \tag{12d}$$

根据 (5b) 式和 (12) 式, 并同时考虑 (6) 式中 $g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$ 项的形式, 可求得如下 I_1^α

$$I_1^1 = -\frac{1}{6}(x_1 - x_2)^3, \tag{13a}$$

$$I_1^2 = -\frac{1}{6}(x_1 + x_2)^3, \tag{13b}$$

$$I_1^3 = -\frac{1}{3}x_1^3 - x_1x_2^2, \tag{13c}$$

$$\begin{aligned} I_1^4 = \frac{1}{12}\Big[& 2(x_1^2 - x_2^2)(\dot{x}_1^2 - \dot{x}_2^2) \\ & - (x_1 + x_2)^2(\dot{x}_1 - \dot{x}_2)^2 \\ & - (x_1 - x_2)^2(\dot{x}_1 + \dot{x}_2)^2 \\ & - 3(x_1^2 - x_2^2)^2 - 8(x_1 - x_2)(x_1 + x_2)^3 \\ & - 32(x_1^2 - x_2^2)(\dot{x}_1 + \dot{x}_2)^2 \\ & + 4(x_1 + x_2)^2(\dot{x}_1^2 - \dot{x}_2^2) \\ & - 8(\dot{x}_1 - \dot{x}_2)(\dot{x}_1 + \dot{x}_2)^3\Big]. \end{aligned} \tag{13d}$$

将 (11) 式和 (13) 式代入 (3) 式, 可得 4 个稳定的一阶近似守恒量:

$$\begin{aligned} I^1 = I_0^1 + \varepsilon I_1^1 = {} & \frac{1}{4}(\dot{x}_1 - \dot{x}_2)^2 + (x_1 - x_2)^2 \\ & - \frac{1}{6}\varepsilon(x_1 - x_2)^3, \end{aligned} \tag{14a}$$

$$\begin{aligned} I^2 = I_0^2 + \varepsilon I_1^2 = {} & \frac{1}{4}(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{4}(x_1 + x_2)^2 \\ & - \frac{1}{6}\varepsilon(x_1 + x_2)^3, \end{aligned} \tag{14b}$$

$$\begin{aligned} I^3 = I_0^3 + \varepsilon I_1^3 = {} & \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) \\ & + \frac{1}{4}(5x_1^2 + 5x_2^2 - 6x_1x_2) \\ & - \varepsilon\left(\frac{1}{3}x_1^3 + x_1x_2^2\right), \end{aligned} \tag{14c}$$

$$\begin{aligned} I^4 = I_0^4 + \varepsilon I_1^4 = {} & (x_1 - x_2)(x_1 + x_2)^2 \\ & - (x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ & + (x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2) \\ & + \frac{1}{12}\varepsilon[2(x_1^2 - x_2^2)(\dot{x}_1^2 - \dot{x}_2^2) \\ & - (x_1 + x_2)^2(\dot{x}_1 - \dot{x}_2)^2 \\ & - (x_1 - x_2)^2(\dot{x}_1 + \dot{x}_2)^2 \\ & - 3(x_1^2 - x_2^2)^2 - 8(x_1 - x_2)(x_1 + x_2)^3 \\ & - 32(x_1^2 - x_2^2)(\dot{x}_1 + \dot{x}_2)^2 \\ & + 4(x_1 + x_2)^2(\dot{x}_1^2 - \dot{x}_2^2) \end{aligned}$$

$$-8(\dot{x}_1-\dot{x}_2)(\dot{x}_1+\dot{x}_2)^3\big]. \tag{14d}$$

4 两自由度动力学系统的一阶近似解

先对 (6) 式进行坐标变换, 将 (8) 式代入 (6) 式可得

$$\ddot{u}_1=-4u_1+\varepsilon u_1^2, \tag{15a}$$
$$\ddot{u}_2=-u_2+\varepsilon u_2^2, \tag{15b}$$

(15) 式为解耦的含平方微扰项的二阶微分方程. 根据微扰法理论 [30], 其一阶近似解可以表示成

$$\begin{aligned} u_1&=u_1^0(\varepsilon^0)+\varepsilon u_1^1(\varepsilon^1),\\ u_2&=u_2^0(\varepsilon^0)+\varepsilon u_2^1(\varepsilon^1). \end{aligned} \tag{16}$$

将 (16) 式代入 (15) 式并比较等式两边 ε^0, ε^1 项的系数, 可得如下 4 个微分方程:

$$\ddot{u}_1^0=-4u_1^0, \tag{17a}$$
$$\ddot{u}_1^1=-4u_1^1+(u_1^0)^2, \tag{17b}$$
$$\ddot{u}_2^0=-u_2^0, \tag{17c}$$
$$\ddot{u}_2^1=-u_2^1+(u_2^0)^2. \tag{17d}$$

由 (17a) 和 (17c) 式可解得

$$u_1^0=A_{10}\cos(2t)+B_{10}\sin(2t), \tag{18a}$$
$$u_2^0=A_{20}\cos t+B_{20}\sin t, \tag{18b}$$

其中 $A_{10}, B_{10}, A_{20}, B_{20}$ 为积分常数, 由初始条件确定. 将 (18a) 和 (18b) 式分别代入 (17b) 和 (17d) 式, 得

$$\begin{aligned} \ddot{u}_1^1=&-4u_1^1+\frac{1}{2}(A_{10}^2+B_{10}^2)\\ &+\frac{1}{2}(A_{10}^2-B_{10}^2)\cos(4t)\\ &+A_{10}B_{10}\sin(4t), \end{aligned} \tag{19a}$$

$$\begin{aligned} \ddot{u}_2^1=&-u_2^1+\frac{1}{2}(A_{20}^2+B_{20}^2)\\ &+\frac{1}{2}(A_{20}^2-B_{20}^2)\cos(2t)\\ &+A_{20}B_{20}\sin(2t). \end{aligned} \tag{19b}$$

(19) 式的解为

$$\begin{aligned} u_1^1=&A_{11}\cos(2t)+B_{11}\sin(2t)\\ &+\frac{1}{8}(A_{10}^2+B_{10}^2)-\frac{1}{24}(A_{10}^2-B_{10}^2)\cos(4t)\\ &-\frac{1}{12}A_{10}B_{10}\sin(4t), \end{aligned} \tag{20a}$$

$$\begin{aligned} u_2^1=&A_{21}\cos t+B_{21}\sin t+\frac{1}{2}(A_{20}^2+B_{20}^2)\\ &-\frac{1}{6}(A_{20}^2-B_{20}^2)\cos(2t)\\ &-\frac{1}{3}A_{20}B_{20}\sin(2t), \end{aligned} \tag{20b}$$

其中 $A_{11}, B_{11}, A_{21}, B_{21}$ 为积分常数, 由初始条件确定. 将 (18) 式和 (20) 式代入 (16) 式得系统 (15) 的一阶近似解

$$\begin{aligned} u_1=&A_{10}\cos(2t)+B_{10}\sin(2t)\\ &+\varepsilon\Big(A_{11}\cos(2t)+B_{11}\sin(2t)\\ &+\frac{1}{8}(A_{10}^2+B_{10}^2)-\frac{1}{24}(A_{10}^2-B_{10}^2)\cos(4t)\\ &-\frac{1}{12}A_{10}B_{10}\sin(4t)\Big), \end{aligned} \tag{21a}$$

$$\begin{aligned} u_2=&A_{20}\cos t+B_{20}\sin t\\ &+\varepsilon\Big(A_{21}\cos t+B_{21}\sin t\\ &+\frac{1}{2}(A_{20}^2+B_{20}^2)-\frac{1}{6}(A_{20}^2-B_{20}^2)\cos(2t)\\ &-\frac{1}{3}A_{20}B_{20}\sin(2t)\Big). \end{aligned} \tag{21b}$$

利用 (8) 式对 (21) 式进行坐标反变换, 得系统 (6) 的一阶近似解

$$\begin{aligned} x_1=&\frac{\sqrt{2}}{2}\Big[\big(A_{10}\cos(2t)+B_{10}\sin(2t)\\ &+A_{20}\cos t+B_{20}\sin t\big)+\varepsilon\Big(A_{11}\cos(2t)\\ &+B_{11}\sin(2t)+A_{21}\cos t+B_{21}\sin t\\ &+\frac{1}{8}(A_{10}^2+B_{10}^2)-\frac{1}{24}(A_{10}^2-B_{10}^2)\cos(4t)\\ &-\frac{1}{12}A_{10}B_{10}\sin(4t)+\frac{1}{2}(A_{20}^2+B_{20}^2)\\ &-\frac{1}{6}(A_{20}^2-B_{20}^2)\cos(2t)\\ &-\frac{1}{3}A_{20}B_{20}\sin(2t)\Big)\Big], \end{aligned} \tag{22a}$$

$$\begin{aligned} x_2=&\frac{\sqrt{2}}{2}\Big[\big(A_{20}\cos t+B_{20}\sin t-A_{10}\cos(2t)\\ &-B_{10}\sin(2t)\big)+\varepsilon\Big(A_{21}\cos t+B_{21}\sin t\\ &-A_{11}\cos(2t)-B_{11}\sin(2t)+\frac{1}{2}(A_{20}^2+B_{20}^2)\\ &-\frac{1}{6}(A_{20}^2-B_{20}^2)\cos(2t)-\frac{1}{3}A_{20}B_{20}\sin(2t)\\ &-\frac{1}{8}(A_{10}^2+B_{10}^2)+\frac{1}{24}(A_{10}^2-B_{10}^2)\cos(4t)\\ &+\frac{1}{12}A_{10}B_{10}\sin(4t)\Big)\Big]. \end{aligned} \tag{22b}$$

由初始条件 $x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}$ 可得

$$\begin{aligned}
A_{10} &= \frac{\sqrt{2}}{2}(x_{10} - x_{20}),\\
B_{10} &= \frac{\sqrt{2}}{4}(\dot{x}_{10} - \dot{x}_{20}),\\
A_{20} &= \frac{\sqrt{2}}{2}(x_{10} + x_{20}),\\
B_{20} &= \frac{\sqrt{2}}{2}(\dot{x}_{10} + \dot{x}_{20}),\\
A_{11} &= -\frac{1}{24}(x_{10} - x_{20})^2 - \frac{1}{48}(\dot{x}_{10} - \dot{x}_{20})^2,\\
B_{11} &= \frac{1}{24}(x_{10} - x_{20})(\dot{x}_{10} - \dot{x}_{20}),\\
A_{21} &= -\frac{1}{6}(x_{10} + x_{20})^2 - \frac{1}{3}(\dot{x}_{10} + \dot{x}_{20})^2,\\
B_{21} &= \frac{1}{3}(x_{10} + x_{20})(\dot{x}_{10} + \dot{x}_{20}).
\end{aligned} \tag{23}$$

下面分四种情况讨论.

1) $\dot{x}_{10} = \dot{x}_{20} = 0$, 且 $x_{10} = x_{20} = x_0$, 则

$$\begin{aligned}
&B_{10} = B_{20} = B_{11} = B_{21} = 0, \quad A_{10} = A_{11} = 0,\\
&A_{20} = \sqrt{2}x_0, \quad A_{21} = -\frac{2}{3}x_0^2,
\end{aligned} \tag{24}$$

将 (24) 式代入 (22) 式, 得

$$\begin{aligned}
x_1 = x_2 = x_0\cos t + \varepsilon\bigg(&\frac{\sqrt{2}}{2}x_0^2 - \frac{\sqrt{2}}{3}x_0^2\cos t\\
&- \frac{\sqrt{2}}{6}x_0^2\cos(2t)\bigg),
\end{aligned} \tag{25}$$

此时, 两质点做同相运动.

2) $\dot{x}_{10} = \dot{x}_{20} = 0$, 且 $x_{10} = -x_{20} = x_0$, 则

$$\begin{aligned}
&B_{10} = B_{20} = B_{11} = B_{21} = 0, \quad A_{20} = A_{21} = 0,\\
&A_{10} = \sqrt{2}x_0, \quad A_{11} = -\frac{1}{6}x_0^2,
\end{aligned} \tag{26}$$

将 (26) 式代入 (22) 式, 得

$$\begin{aligned}
x_1 = x_0\cos(2t) + \varepsilon\bigg(&\frac{\sqrt{2}}{8}x_0^2 - \frac{\sqrt{2}}{12}x_0^2\cos(2t)\\
&- \frac{\sqrt{2}}{24}x_0^2\cos(4t)\bigg),
\end{aligned} \tag{27a}$$

$$\begin{aligned}
x_2 = -x_0\cos(2t) + \varepsilon\bigg(&-\frac{\sqrt{2}}{8}x_0^2 + \frac{\sqrt{2}}{12}x_0^2\cos(2t)\\
&+ \frac{\sqrt{2}}{24}x_0^2\cos(4t)\bigg),
\end{aligned} \tag{27b}$$

此时, 两质点做反相运动.

3) $x_{10} = x_{20} = 0$, 且 $\dot{x}_{10} = \dot{x}_{20} = \dot{x}_0$, 则

$$\begin{aligned}
&A_{10} = A_{20} = B_{10} = 0, \quad B_{11} = B_{21} = A_{11} = 0,\\
&B_{20} = \sqrt{2}\dot{x}_0, \quad A_{21} = -\frac{4}{3}\dot{x}_0^2,
\end{aligned} \tag{28}$$

将 (28) 式代入 (22) 式, 得

$$\begin{aligned}
x_1 = x_2 = \dot{x}_0\sin t + \varepsilon\bigg(&\frac{\sqrt{2}}{2}\dot{x}_0^2 - \frac{2\sqrt{2}}{3}\dot{x}_0^2\cos t\\
&+ \frac{\sqrt{2}}{6}\dot{x}_0^2\cos(2t)\bigg),
\end{aligned} \tag{29}$$

此时, 两质点做同相运动.

4) $x_{10} = x_{20} = 0$, 且 $\dot{x}_{10} = -\dot{x}_{20} = \dot{x}_0$, 则

$$\begin{aligned}
&A_{10} = A_{20} = B_{20} = 0, \quad B_{11} = B_{21} = A_{21} = 0,\\
&B_{10} = \frac{\sqrt{2}}{2}\dot{x}_0, \quad A_{11} = -\frac{1}{12}\dot{x}_0^2,
\end{aligned} \tag{30}$$

将 (30) 式代入 (22) 式, 得

$$\begin{aligned}
x_1 = \frac{1}{2}\dot{x}_0\sin(2t) + \varepsilon\bigg(&\frac{\sqrt{2}}{32}\dot{x}_0^2 - \frac{\sqrt{2}}{24}\dot{x}_0^2\cos(2t)\\
&+ \frac{\sqrt{2}}{96}\dot{x}_0^2\cos(4t)\bigg),
\end{aligned} \tag{31a}$$

$$\begin{aligned}
x_2 = -\frac{1}{2}\dot{x}_0\sin(2t) + \varepsilon\bigg(&-\frac{\sqrt{2}}{32}\dot{x}_0^2 + \frac{\sqrt{2}}{24}\dot{x}_0^2\cos(2t)\\
&- \frac{\sqrt{2}}{96}\dot{x}_0^2\cos(4t)\bigg),
\end{aligned} \tag{31b}$$

此时, 两质点做反相运动.

5 结 论

本文从一阶近似守恒量的性质出发, 将受微扰作用的运动微分方程视为未受微扰作用的微分方程上迭加了微扰项, 提出了一种求一阶近似守恒量的新方法: 先根据未受微扰作用微分方程的形式选择合适的方法求得未受微扰系统的守恒量 I_0, 再计算微扰项对 I_0 的影响, 即 $\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1)$, 最后根据一阶近似守恒量的性质求得 $\frac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0)$ 并考虑 $g_1(\varepsilon^0)$, $g_2(\varepsilon^0)$ 的形式得到 I_1, 从而求得一阶近似守恒量. 由于 I_0 是未受微扰作用系统的守恒量, 因此 $\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^0)$ 必为 0, 则 εI_0 一定是受微扰作用系统的平凡的一阶近似守恒量, 平凡的一阶近似守恒量比较简单, 其研究意义不大, 文中未做详细研究. 未受微扰系统的守恒量 I_0 可用多种求法求得, 甚至也可以采用多种方法联合求解, 方法灵活又简单方便, 且不易遗漏. 求得一阶近似守恒量后可以进一步讨论相应的近似对称性. 本方法可以推广应用于研究其他实际力学系统, 是求近似守恒量的一种有效方法. 文中用新方法研究了受非线性微扰作用的两自由度动力学系统, 到得了 4 个稳定的一阶近似守恒

量. 先通过坐标变换对微分方程解耦, 再用微扰法解得新坐标系下的一阶近似解, 通过坐标反变换得到原坐标系下的一阶近似解, 并对4种特殊情况进行了讨论. 用坐标变换法对微分方程解耦后再求一阶近似解, 降低了求解的难度.

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A new method to obtain first order approximate conserved quantities of second-ordinary dynamics system containing nonlinear perturbation terms*

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Abstract

We consider the perturbed system as the combination of unperturbed system and perturbed term according to the characteristic of the first order approximate conserved quantities, and we suggest a new method to obtain the first order approximate conserved quantities by three steps: first, we select a suitable method to obtain the conserved quantity I_0 of unperturbed system, second, we calculate the influence of perturbed terms on conserved quantity I_0, and finally we obtain the first order approximate conserved quantities of the system by using the characteristic of the first order approximate conserved quantities. An actual two-dimensional nonlinear dynamics perturbed system is studied in this paper, and four stable first order approximate conserved quantities are obtained by using this new method. The expressions of first order approximate solution of the system are also obtained by transforming coordinates and using the perturbation method, and four special cases are discussed in this paper.

Keywords: nonlinear perturbed, second-ordinary dynamics systems, first order approximate conserved quantities, transformation of coordinates

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