非完整系统的共形不变性导致的新型守恒量*

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研究了非完整系统的共形不变性与新型守恒量. 提出了该系统共形不变性的概念; 得出了非完整系统的运动微分方程具有共形不变性并且是 Lie 对称性的充要条件. 利用规范函数满足的新型结构方程, 导出系统相应的新型守恒量. 最后给出应用算例.

关键词: 非完整系统, Lie 对称性, 共形不变性, 新型守恒量

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1 引 言

对称性在分析力学中也称不变性, 是自然界的普遍规律之一. 利用对称性寻求守恒量不仅能更直接认识世界的本质, 也使问题的解决更加简单. 自 1918 年 Noether[1] 发现对称性与守恒量之间的内在关系以来, 人们在对称性与守恒量方面的研究取得了一系列重要成果[2−7]. 近些年, 研究者主要通过 Lie 对称性, Noether 对称性, Mei 对称性和共形不变性来寻求守恒量[8−27], 由于得到的守恒量往往存在着计算复杂的不足, 许多学者在寻求计算更加简单的新型守恒量方面做了有益的探索, 文献 [28] 研究了 Appell 方程 Mei 对称性导致的一种新型守恒量, 文献 [29] 研究了非完整力学系统的 Lie 对称性直接导致的一种守恒量, 文献 [30] 研究了 Lagrange 系统 Lie 对称性导致的新型守恒量, 本文研究非完整系统的共形不变性导致的新型守恒量, 推导出非完整系统的运动微分方程具有共形不变性并且是 Lie 对称性的充分必要条件, 根据规范函数满足的新型结构方程, 求出系统相应的新型守恒量, 并给出应用算例.

2 系统运动的微分方程

设非完整约束力学系统的位形由 n 个广义坐标 q_s $(s=1,\cdots,n)$ 来确定, 它的运动受有 g 个理想双面 Chetaev 型非完整约束

$$f_\beta = f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0, \quad (\beta = 1,2,\cdots,g). \tag{1}$$

方程 (1) 加在虚位移 δq_s 上的限制为

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0, \ (\beta = 1,2,\cdots,g;\ s=1,\cdots,n). \tag{2}$$

则非完整约束力学系统的运动微分方程可表示为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s, \quad (s=1,\cdots,n). \tag{3}$$

式中 $L = L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为非完整约束力学系统的 Lagrange 函数, $Q_s = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为非势广义力; λ_β 为约束乘子, $\Lambda_s = \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为约束反力, 有

$$\Lambda_s = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}. \tag{4}$$

展开方程 (3), 有

$$F_s = A_{sk}\ddot{q}_k + B_s - Q_s - \Lambda_s = 0, \quad (s,k=1,\cdots,n), \tag{5}$$

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其中

$$A_{sk}=\frac{\partial^2 L}{\partial\dot q_s\partial\dot q_k},$$
$$B_s=\frac{\partial^2 L}{\partial\dot q_s\partial q_k}\dot q_k+\frac{\partial^2 L}{\partial\dot q_s\partial t}-\frac{\partial L}{\partial q_s}. \tag{6}$$

假设系统(3)非奇异, 即

$$\det\left(\frac{\partial^2 L}{\partial\dot q_s\partial\dot q_k}\right)\neq 0, \tag{7}$$

由方程(5)可以求得广义加速度

$$\ddot q_s=\alpha_s(t,\boldsymbol q,\dot{\boldsymbol q}),\quad (s=1,\cdots,n). \tag{8}$$

3 系统的共形不变性与共形因子

设ε为无限小参数, ξ_0, ξ_s为无限小变换生成元, 现引入时间和广义坐标的无限小群变换

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol q,\dot{\boldsymbol q}),\quad q_s^*(t^*)=q_s+\varepsilon\xi_s(t,\boldsymbol q,\dot{\boldsymbol q}),$$
$$(s=1,\cdots,n). \tag{9}$$

定义1 在无限小生成元$\xi_0(t,\boldsymbol q,\dot{\boldsymbol q})$, $\xi_s(t,\boldsymbol q,\dot{\boldsymbol q})$的变换下, 对于二阶微分方程$F_s$, 若非退化矩阵$\boldsymbol B_s^l$满足

$$X^{(2)}(F_s)=\boldsymbol B_s^l F_l,\quad (s,l=1,\cdots,n), \tag{10}$$

则称二阶微分方程F_s为共形不变的, (10)式是方程(5)共形不变的确定方程, $\boldsymbol B_s^l$称为共形因子.

对于与非完整系统(1)和(3)相应的完整系统(5), 如果无限小生成元$\xi_0(t,\boldsymbol q,\dot{\boldsymbol q})$, $\xi_s(t,\boldsymbol q,\dot{\boldsymbol q})$满足确定方程

$$X^{(2)}(F_s)|_{F_s=0}=0, \tag{11}$$

称这种对称性为系统的Lie对称性.

这里

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{12}$$
$$X^{(1)}=X^{(0)}+\left(\dot\xi_s-\dot q_s\dot\xi_0\right)\frac{\partial}{\partial\dot q_s}, \tag{13}$$
$$X^{(2)}=X^{(1)}+\left(\ddot\xi_s-2\ddot q_s\dot\xi_0-\dot q_s\ddot\xi_0\right)\frac{\partial}{\partial\ddot q_s}. \tag{14}$$

下面求共形因子$\boldsymbol B_s^l$. 因为

$$\begin{aligned}&X^{(2)}(F_s)\\ &=X^{(2)}\left(A_{sk}\ddot q_k+B_s-Q_s-\varLambda_s\right)\\ &=A_{sk}\left(\ddot\xi_k-2\ddot q_k\dot\xi_0-\dot q_k\ddot\xi_0\right)\\ &\quad+X^{(0)}(A_{sk})\ddot q_k+X^{(0)}\left(B_s-Q_s-\varLambda_s\right)\\ &\quad+\left(\dot\xi_k-\dot q_k\dot\xi_0\right)\times\frac{\partial\left(B_s-Q_s-\varLambda_s\right)}{\partial\dot q_k},\end{aligned} \tag{15}$$

又

$$\dot\xi_k=\frac{\partial\xi_k}{\partial t}+\frac{\partial\xi_k}{\partial q_r}\dot q_r+\frac{\partial\xi_k}{\partial\dot q_r}\ddot q_r,$$
$$(k=0,1,\cdots,n), \tag{16}$$

$$\begin{aligned}\ddot\xi_k=&\frac{\partial^2\xi_k}{\partial t^2}+2\frac{\partial^2\xi_k}{\partial q_r\partial t}\dot q_r+2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t}\ddot q_r\\ &+\frac{\partial^2\xi_k}{\partial q_r\partial q_j}\dot q_r\dot q_j+2\frac{\partial^2\xi_k}{\partial q_r\partial\dot q_j}\dot q_r\ddot q_j\\ &+\frac{\partial\xi_k}{\partial q_r}\ddot q_r+\frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}\ddot q_r\ddot q_j\\ &+\frac{\partial\xi_k}{\partial\dot q_r}\left(\frac{\partial\alpha_r}{\partial t}+\frac{\partial\alpha_r}{\partial q_j}\dot q_j+\frac{\partial\alpha_r}{\partial\dot q_j}\ddot q_j\right),\\ &(k=0,1,\cdots,n;\ r,j=1,\cdots,n),\end{aligned} \tag{17}$$

将(16), (17)式代入(15)式, 并考虑到$F_s=0$时, $\ddot q_s=\alpha_s(t,\boldsymbol q,\dot{\boldsymbol q})$得到

$$\begin{aligned}&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}\\ &=A_{sk}\left\{\left[2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial\xi_k}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\right]\right.\\ &\quad\times(\ddot q_r-\alpha_r)+\frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}(\ddot q_r\ddot q_j-\alpha_r\alpha_j)\\ &\quad-2(\ddot q_k-\alpha_k)\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot q_r\right)\\ &\quad-2\frac{\partial\xi_0}{\partial\dot q_r}(\ddot q_k\ddot q_r-\alpha_k\alpha_r)-\dot q_k\left(2\frac{\partial^2\xi_0}{\partial\dot q_r\partial t}\right.\\ &\quad\left.+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\right)(\ddot q_r-\alpha_r)\\ &\quad\left.-\dot q_k\frac{\partial^2\xi_0}{\partial\dot q_r\partial\dot q_j}(\ddot q_r\ddot q_j-\alpha_r\alpha_j)\right\}\\ &\quad+X^{(0)}(A_{sk})(\ddot q_k-\alpha_k)+\left(\frac{\partial\xi_k}{\partial\dot q_r}-\dot q_k\frac{\partial\xi_0}{\partial\dot q_r}\right)\\ &\quad\times(\ddot q_r-\alpha_r)\frac{\partial(B_s-Q_s-\varLambda_s)}{\partial\dot q_k},\\ &\quad(s,k,r,j=1,\cdots,n).\end{aligned} \tag{18}$$

由于

$$\begin{aligned}\ddot q_k-\alpha_k&=\ddot q_k+A^{kl}\left(B_l-Q_l-\varLambda_l\right)\\ &=A^{kl}\left(A_{lm}\ddot q_m+B_l-Q_l-\varLambda_l\right)\\ &=A^{kl}F_l,\\ \ddot q_k\ddot q_j&=\left(A^{kl}F_l+\alpha_k\right)\left(A^{jm}F_m+\alpha_j\right)\\ &=A^{kl}F_lA^{jm}F_m+\alpha_kA^{jm}F_m\\ &\quad+\alpha_jA^{kl}F_l+\alpha_k\alpha_j,\end{aligned}$$

$$(s,k,r,j,m,l=1,\cdots,n). \tag{19}$$

所以

$$\begin{aligned}
&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}\\
&=\Big\{A_{sk}\Big[2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial\xi_k}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\\
&\quad-\dot q_k\Big(2\frac{\partial^2\xi_0}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\Big)\Big]\\
&\quad\times A^{rl}-2\delta_s^l\Big(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot q_r\Big)+\Big(\frac{\partial\xi_k}{\partial\dot q_r}-\dot q_k\frac{\partial\xi_0}{\partial\dot q_r}\Big)\\
&\quad\times A^{rl}\frac{\partial[B_s-Q_s-\Lambda_s]}{\partial\dot q_k}+X^{(0)}(A_{sk})A^{kl}\\
&\quad+A_{sk}\frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}\left(\alpha_rA^{jl}+\alpha_jA^{rl}\right)\\
&\quad-2A_{sk}\frac{\partial\xi_0}{\partial\dot q_r}\left(\alpha_kA^{rl}+\alpha_rA^{kl}\right)\\
&\quad-A_{sk}\dot q_k\frac{\partial^2\xi_0}{\partial\dot q_r\partial\dot q_j}\left(\alpha_rA^{jl}+\alpha_jA^{rl}\right)\Big\}F_l.
\end{aligned}\tag{20}$$

令

$$\begin{aligned}
\boldsymbol M_s^l={}&A_{sk}\Big[2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial\xi_k}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\\
&-\dot q_k\Big(2\frac{\partial^2\xi_0}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_0}{\partial q_r}\\
&+\frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\Big)\Big]A^{rl}-2\delta_s^l\Big(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot q_r\Big)\\
&+\Big(\frac{\partial\xi_k}{\partial\dot q_r}-\dot q_k\frac{\partial\xi_0}{\partial q_r}\Big)A^{rl}\frac{\partial(B_s-Q_s-\Lambda_s)}{\partial\dot q_k}\\
&+X^{(0)}(A_{sk})A^{kl}+A_{sk}\frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}(\alpha_rA^{jl}\\
&+\alpha_jA^{rl})-2A_{sk}\frac{\partial\xi_0}{\partial\dot q_r}(\alpha_kA^{rl}+\alpha_rA^{kl})\\
&-A_{sk}\dot q_k\frac{\partial^2\xi_0}{\partial\dot q_r\partial\dot q_j}\left(\alpha_rA^{jl}+\alpha_jA^{rl}\right).\\
&(s,k,r,j,l=1,\cdots,n).
\end{aligned}\tag{21}$$

得

$$\begin{aligned}
&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}=\boldsymbol M_s^lF_l,\\
&(s,l=1,\cdots,n).
\end{aligned}\tag{22}$$

如系统具有共形不变性且是Lie对称性, 由(10), (11)和(22)式可得

$$\boldsymbol B_s^lF_l-\boldsymbol M_s^lF_l=X^{(2)}(F_s)|_{F_s=0}=0. \tag{23}$$

即

$$\boldsymbol B_s^l=\boldsymbol M_s^l\quad(s,l=1,\cdots,n). \tag{24}$$

命题1 对于与非完整系统(1)和(3)的相应完整系统(5), 其共形不变性且是Lie对称性的充分必要条件是生成元$\xi_0(t,\boldsymbol q,\dot{\boldsymbol q})$, $\xi_s(t,\boldsymbol q,\dot{\boldsymbol q})$ 满足

$$\begin{aligned}
&\boldsymbol B_s^l=\boldsymbol M_s^l\\
&=A_{sk}\Big[2\frac{\partial^2\xi_k}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot q_r}\dot q_j+\frac{\partial\xi_k}{\partial q_r}\\
&\quad+\frac{\partial\xi_k}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}-\dot q_k\Big(2\frac{\partial^2\xi_0}{\partial\dot q_r\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot q_r}\dot q_j\\
&\quad+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial\xi_0}{\partial\dot q_j}\frac{\partial\alpha_j}{\partial\dot q_r}\Big)\Big]A^{rl}-2\delta_s^l\Big(\frac{\partial\xi_0}{\partial t}\\
&\quad+\frac{\partial\xi_0}{\partial q_r}\dot q_r\Big)+\Big(\frac{\partial\xi_k}{\partial\dot q_r}-\dot q_k\frac{\partial\xi_0}{\partial q_r}\Big)A^{rl}\\
&\quad\times\frac{\partial(B_s-Q_s-\Lambda_s)}{\partial\dot q_k}+X^{(0)}(A_{sk})A^{kl}\\
&\quad+A_{sk}\frac{\partial^2\xi_k}{\partial\dot q_r\partial\dot q_j}\left(\alpha_rA^{jl}+\alpha_jA^{rl}\right)\\
&\quad-2A_{sk}\frac{\partial\xi_0}{\partial\dot q_r}\left(\alpha_kA^{rl}+\alpha_rA^{kl}\right)\\
&\quad-A_{sk}\dot q_k\frac{\partial^2\xi_0}{\partial\dot q_r\partial\dot q_j}\left(\alpha_rA^{jl}+\alpha_jA^{rl}\right).\\
&\quad(s,k,r,j,l=1,\cdots,n).
\end{aligned}\tag{25}$$

4 共形不变性与新型守恒量

由非完整系统的共形不变性, 通过Lie对称性可导出相应的新型守恒量, 有如下结论.

命题2 对于非完整系统, 如果共形不变性的无限小生成元$\xi_0(t,\boldsymbol q,\dot{\boldsymbol q})$, $\xi_s(t,\boldsymbol q,\dot{\boldsymbol q})$和规范函数$G$满足如下Lie对称性的结构方程:

$$\begin{aligned}
&X^{(1)}(L)-\dot L\xi_0+(Q_s+\Lambda_s)(\xi_s-\dot q_s\xi_0)\\
&+\dot G=0,
\end{aligned}\tag{26}$$

则相应于非完整系统(1)和(3)的完整系统(5)的共形不变性存在守恒量

$$I=\frac{\partial L}{\partial\dot q_s}(\xi_s-\dot q_s\xi_0)+G=\mathrm{const.} \tag{27}$$

证明 将(27)式对时间t求导并将(26)式代入可得

$$\begin{aligned}
\frac{\mathrm dI}{\mathrm dt}={}&\frac{\partial L}{\partial\dot q_s}(\dot\xi_s-\ddot q_s\xi_0-\dot q_s\dot\xi_0)+\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial\dot q_s}(\xi_s-\dot q_s\xi_0)\\
&-\dot L\xi_0-X^{(1)}(L)-(Q_s+\Lambda_s)(\xi_s-\dot q_s\xi_0)\\
={}&(\xi_s-\dot q_s\xi_0)\left[\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial\dot q_s}-\frac{\partial L}{\partial q_s}-(Q_s+\Lambda_s)\right]\\
={}&0.
\end{aligned}$$

在无限小变换 (9) 下, 非完整约束方程 (1) 的不变性可表示为如下限制方程:

$$X^{(1)} f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, 2, \cdots, g). \tag{28}$$

考虑到非完整约束 (2) 对无限小生成元 $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 的限制, 附加限制方程为

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) = 0. \tag{29}$$

定义 2 对于非完整系统, 如果无限小生成元 $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 满足确定方程 (11) 和限制方程 (28), 则这种对称性为系统的弱 Lie 对称性, 相应的共形不变性是弱 Lie 对称性的共形不变性. 如果无限小生成元 $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 满足确定方程 (11)、限制方程 (28) 及附加限制方程 (29), 则称这种对称性为系统的强 Lie 对称性, 相应的共形不变性是强 Lie 对称性的共形不变性.

由命题 2 可得如下结论:

命题 3 如果 $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 是非完整系统 (1) 和 (3) 的弱 (强)Lie 对称性生成元, 且存在规范函数 G 满足结构方程 (26), 则非完整系统 (1) 和 (3) 的弱 (强)Lie 对称性的共形不变性导致守恒量 (27).

5 算 例

设一非完整系统为

$$L = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} k m q_2^2, \tag{30}$$

$$Q_1 = Q_2 = 0, \tag{31}$$

$$f = q_2 - \dot{q}_1 = 0. \tag{32}$$

其中 m, k 为常数, 试研究系统的共形不变性与守恒量.

由方程 (3) 可得

$$m\ddot{q}_1 = -\lambda, \tag{33}$$

$$m\ddot{q}_2 - k m q_2 = 0. \tag{34}$$

由 (33) 式, 并对 (32) 式求导, 得到

$$\lambda = -m\ddot{q}_1 = -m\dot{q}_2, \tag{35}$$

根据 (4) 式得到

$$\Lambda_1 = -\lambda = m\dot{q}_2, \quad \Lambda_2 = 0. \tag{36}$$

系统的微分方程为

$$\ddot{q}_1 = \dot{q}_2 = \alpha_1,$$

$$\ddot{q}_2 = k q_2 = \alpha_2. \tag{37}$$

或

$$F_1 = \ddot{q}_1 - \dot{q}_2 = 0,$$

$$F_2 = \ddot{q}_2 - k q_2 = 0. \tag{38}$$

取无限小变换生成元为

$$\xi_0 = 0, \quad \xi_1 = q_1 + 1, \quad \xi_2 = q_2. \tag{39}$$

则

$$\begin{aligned} X^{(2)}(\boldsymbol{F}) &= \left(\xi_s \frac{\partial}{\partial q_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \ddot{\xi}_s \frac{\partial}{\partial \ddot{q}_s}\right) \\ &\quad \times \begin{pmatrix} \ddot{q}_1 - \dot{q}_2 \\ \ddot{q}_2 - k q_2 \end{pmatrix} \\ &= \begin{pmatrix} -\dot{q}_2 + \ddot{q}_1 \\ -k q_2 + \ddot{q}_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \end{aligned}$$

因此, 共形因子为

$$\boldsymbol{B}_s^l = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{40}$$

也可从 (25) 式求出共形因子

$$\boldsymbol{B}_s^l = \boldsymbol{M}_s^l = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

该结果与 (40) 式的一样, 共形不变性的确定方程可表示为

$$X^{(2)}(\boldsymbol{F}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{41}$$

这时, 非完整系统 (1) 和 (3) 既是共形不变性的又是 Lie 对称性的.

相应的对称性是系统的弱 Lie 对称性, 因为生成元满足限制方程 (28), 但不满足附加限制方程 (29).

将 (30), (31), (32), (36) 和 (39) 式代入 (26) 式并积分可得

$$\begin{aligned} G = & -\frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2}(k-1) m q_2^2 \\ & - m(q_1 + 1)\dot{q}_1 - m q_2 \dot{q}_2, \end{aligned} \tag{42}$$

将 (42) 式代入 (27) 式, 可得守恒量

$$I = \frac{1}{2}m\dot{q}_2^2 - \frac{1}{2}(k-1)mq_2^2 - \frac{1}{2}m\dot{q}_1^2 = \text{const.} \quad (43)$$

6 结 论

本文给出了非完整系统共形不变性导致的新型守恒量, 其表达式 (27) 比原来的守恒量表达式 [7] 更为简洁, 其计算将更加简单. 此外, 本文的结果更具一般性, 当系统的运动不受 g 个理想双面 Chetaev 型非完整约束, 且 $Q_s = Q_s(t, q, \dot{q}) = 0$ 时, 本文的结果与文献 [30] 所得新型守恒量完全一样.

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A new type of conserved quantity deduced from conformal invariance in nonholonomic mechanical system*

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Abstract

Conformal invariance and a new type of conserved quantity in nonholonomic mechanical system are studied. The definition and determining equation of conformal invariance for the nonholonomic mechanical system are provided; and the necessary and sufficient conditions that the conformal invariance for a nonholonomic mechanical system should be of Lie symmetry are deduced. With the aid of a new structure equation that the gauge function satisfies, the system's corresponding new conserved quantity is obtained. Finally an example is given to illustrate the application of the results.

Keywords: nonholonomic system, Lie symmetry, conformal invariance, new conserved quantity

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