薛定谔扰动耦合系统孤波的行波近似解法*

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研究了一类薛定谔非线性耦合系统. 利用精确解与近似解相关联的特殊技巧, 首先讨论了对应的无扰动耦合系统, 利用投射法得到了精确的孤波解. 再利用泛函映射方法得到了薛定谔非线性扰动耦合系统的行波近似解.

关键词: 薛定谔系统, 孤波, 渐近解

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1 引 言

孤波理论在凝聚态物理、量子物理、光学、流体力学等研究领域中是一个十分关注的对象. 研究方法不断地在改进. 许多学者在孤波的求解方法上作了许多工作 [1−5]. 目前, 使用渐近方法来求解孤波就是一种新的方法. 它改变了以往单纯用数值模拟来讨论孤波的性态, 而是通过解析理论得到孤波的近似表达式. 其优点在于可以通过近似表达式进一步用解析运算工具来对孤波性态作更深入的研究 [6,7]. 近来, 许多渐近方法不断地在发展, 包括合成展开法, 平均法, 边界层法, 匹配法和多重尺度法等 [8−11]. 作者等也应用渐近方法讨论了一类非线性问题 [12−24]. 由于光传播时具有波、粒二象性, 其中 Schrödinger 方程是重要的光学模型之一. 特别是非线性 Schrödinger 方程已经广泛应用于现代光通信技术, 而光通信的理论基础就是光孤波传播时的状态. 本文就是利用一种新改进的渐近方法讨论了一类广义 Schrödinger 扰动耦合系统, 并得到了对应孤波的近似行波解.

考虑如下一类广义 Schrödinger 非线性扰动耦合系统:

$$a_1\frac{\partial^2 u_1}{\partial x^2} - a_2u_1 + a_3u_1u_2 = F_1\left(\frac{\partial u_1}{\partial t}, \varepsilon\right), \tag{1}$$

$$b_1\frac{\partial u_2}{\partial t} - b_2\frac{\partial u_1}{\partial x} = F_2\left(\frac{\partial u_2}{\partial t}, \varepsilon\right), \tag{2}$$

其中 ε 为正的小参数, $u_1(x,t)$, $u_2(x,t)$ 为对应系统的物理场函数, a_i, b_j $(i = 1, 2, 3, j = 1, 2)$ 为对应物理量的加权参数, F_i $(i = 1, 2)$ 为物理场函数的扰动项, 它是在相应的变化范围内的充分光滑的函数. 本系统代表了一类光导纤维中光孤子传播的并受到场函数的速率非线性扰动和参数 ε 的广义孤波传播系统的模型. 马松华等 [25] 研究了该系统孤波脉冲、飞秒孤波和时间孤子的激励, 以及讨论了孤波间的弹性相互作用. 其详细物理背景参见文献 [25, 26]. 现用一个简单的解析方法来求得 Schrödinger 扰动耦合系统 (1), (2) 的渐近解.

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2 无扰动耦合系统

首先讨论如下简单的Schrödinger无扰动耦合系统

$$a_1\frac{\partial^2 u_1}{\partial x^2}-a_2u_1+a_3u_1u_2=0, \tag{3}$$

$$b_1\frac{\partial u_2}{\partial t}-b_2\frac{\partial u_1}{\partial x}=0. \tag{4}$$

我们引入行波变换 $s=x+ct$. 这时系统(3), (4)为

$$a_1\frac{\partial^2 u_1}{\partial s^2}-a_2u_1+a_3u_1u_2=0, \tag{5}$$

$$b_1c\frac{\partial u_2}{\partial s}-b_2\frac{\partial u_1}{\partial s}=0. \tag{6}$$

由(5),(6)式有

$$a_1\frac{\partial^2 u_1}{\partial s^2}-a_2u_1+\frac{a_3b_2}{b_1c}u_1^2=0, \tag{7}$$

$$u_2=\frac{b_2}{b_1c}u_1. \tag{8}$$

在方程(8)中, 不妨选取出现的任意常数为零. 利用投射理论[25], 我们设方程(7) 有如下形式的孤波解:

$$u_1(s)=k_0+k_1v+k_2v^2, \tag{9}$$

其中 k_i $(i=0,1,2)$ 为待定常数, 而 $v(s)$ 满足方程

$$\frac{\mathrm{d}v}{\mathrm{d}s}=v^2-\sigma^2. \tag{10}$$

不难得到方程(10)具有如下孤波解:

$$v(s)=-\sigma\tanh(\sigma s), \quad \sigma>0. \tag{11}$$

作为特殊情形, 由(11)式, 当 $k_0=k_2=\sigma=1$, $k_1=0.1$ 和 $k_0=k_2=\sigma=1$, $k_1=-0.1$ 时, 形如(9)式的孤波曲线 $u(s)$ 由图1和图2 所示.

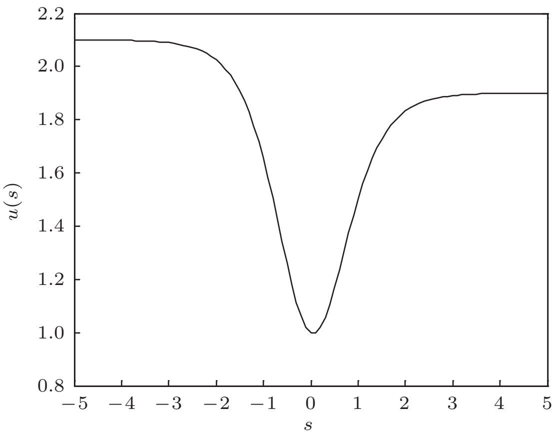

图1 $u(s)$ 的孤波曲线 $(k_0=k_2=\sigma=1,\ k_1=0.1)$

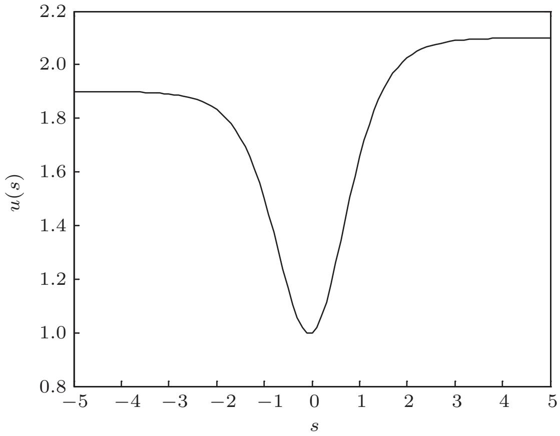

图2 $u(s)$ 的孤波曲线 $(k_0=k_2=\sigma=1,\ k_1=-0.1)$

由(9)和(10)式, 我们有

$$\frac{\partial u_1}{\partial s}=-k_1\sigma^2-2k_2\sigma^2v+k_1v^2+2k_2v^3, \tag{12}$$

$$\frac{\partial^2 u_1}{\partial s^2}=2k_2\sigma^4-2k_1\sigma^2v-6k_2\sigma^2v^2+2k_1v^3+6k_2v^4. \tag{13}$$

将(9), (12)和(13)式代入方程(7), 得到

$$\begin{aligned}&a_1b_1c\big(2k_2\sigma^4-2k_1\sigma^2v-6k_2\sigma^2v^2+2k_1v^3+6k_2v^4\big)\\&+a_2b_1c\left(k_1\sigma^2+2k_2\sigma^2v-k_1v^2-2k_2v^3\right)\\&+a_3b_2\big(k_0^2+2k_0k_1v+\left(2k_0k_2+k_1^2\right)v^2\\&+2k_1k_2v^3+k_2^2v^4\big)=0.\end{aligned}$$

即

$$\begin{aligned}&\left(2a_1b_1ck_2\sigma^4+a_2b_1ck_1\sigma^2+a_3b_2k_0^2\right)\\&+\left(-2a_1b_1ck_1\sigma^2+2a_2b_1ck_2\sigma^2+2a_3b_2k_0k_1\right)v\\&+\left(-6a_1b_1ck_2\sigma^2-a_2b_1ck_1+2a_3b_2k_0k_2+a_3b_2k_1^2\right)v^2\\&+2(a_1b_1ck_1-a_2b_1k_2+a_3b_2k_1k_2)v^3\\&+\left(6a_1b_1ck_2+a_3b_2k_2^2\right)v^4=0.\end{aligned}$$

令上式 v^i $(i=0,1,2,3,4)$ 的系数为零. 并可解得

$$\begin{aligned}\bar{k}_0=&\frac{a_3b_1^2b_2\left(6a_1^2c^2\sigma^2+3a_2^2\right)}{a_3^2b_2^3(a_1c-3a_2a_3b_1)\sigma^2-a_2a_3b_1b_2}\\&-\frac{a_1a_2\left(c\sigma^2-a_3b_2\right)c\sigma^2}{a_3^2b_2^3(a_1c-3a_2a_3b_1)\sigma^2-a_2a_3b_1b_2},\end{aligned} \tag{14}$$

$$\begin{aligned}\bar{k}_1=&-6\bigg[\frac{a_2b_1}{a_3b_2}+\frac{a_3b_1^2b_2\left(6a_1^2c^2\sigma^2+3a_2^2\right)}{a_3b_2^2(a_1c-3a_2a_3b_1)\sigma^2-a_2b_1}\\&-\frac{a_1a_2\left(c\sigma^2-a_3b_2\right)c\sigma^2}{a_3b_2^2(a_1c-3a_2a_3b_1)\sigma^2-a_2b_1}\bigg],\end{aligned} \tag{15}$$

$$\bar{k}_2=-\frac{6a_1b_1c}{a_3b_2}, \tag{16}$$

而 c, σ 为如下方程的解 $\bar{c}$, $\bar{\sigma}$(其构造从略):

$$2a_1b_1c\bar{k}_2\sigma^4 + a_2b_1k_1\sigma^2 + a_3b_2\bar{k}_0^2 = 0, \tag{17}$$

$$6a_1b_1c\bar{k}_2\sigma^2 + a_2b_1c\bar{k}_1 - 2a_3b_2\bar{k}_0\bar{k}_2 + a_3b_2\bar{k}_1^2 = 0. \tag{18}$$

将 (14)—(18) 式得到的结果代入 (9) 式并注意到 (11) 式, 可得方程 (7) 的一个孤波解

$$U_1(s) = \bar{k}_0 - \bar{k}_1\bar{\sigma}\tanh(\bar{\sigma}s) - \bar{k}_3\bar{\sigma}^2\tanh^2(\bar{\sigma}s). \tag{19}$$

再由 (8) 式, 我们有

$$\begin{aligned} U_2(s) &= \frac{b_2}{b_1\bar{c}}U_1 \\ &= \frac{b_2}{b_1\bar{c}}\big(\bar{k}_0 - \bar{k}_1\bar{\sigma}\tanh(\bar{\sigma}s)\big) \\ &\quad - \bar{k}_3\bar{\sigma}^2\tanh^2(\bar{\sigma}s). \end{aligned} \tag{20}$$

由变换 $s = x\bar{c}t$, 我们得到了 Schrödinger 无扰动耦合系统 (3), (4) 的一组孤波的行波精确解

$$\begin{aligned} U_1(x,t) &= \bar{k}_0 - \bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right) \\ &\quad - \bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right), \end{aligned} \tag{21}$$

$$\begin{aligned} U_2(x,t) &= \frac{b_2}{b_1\bar{c}}\left(\bar{k}_0 - \bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right) \\ &\quad - \bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right), \end{aligned} \tag{22}$$

其中 $\bar{k}_i$ $(i = 0, 2, 3)$, $\bar{\sigma}$, $\bar{c}$ 由 (14)—(18) 式决定.

3 Schrödinger 扰动耦合系统的近似解

由于非线性耦合系统 (1), (2) 一般不能得到有限项初等函数形式的精确解. 因此我们需要求出 Schrödinger 非线性扰动耦合系统 (1), (2) 孤波解的近似表示式.

设 Schrödinger 非线性扰动耦合系统的一个孤波解为

$$\begin{aligned} u_1 &= \sum_{i=0}^{\infty} u_{1i}(s)r^i, \\ u_2 &= \sum_{i=0}^{\infty} u_{2i}(s)r^i, \end{aligned} \tag{23}$$

其中 r 为一个人工参数 [6,7].

今引入同伦映射 $H_i(u_1, u_2, r)$, $i = 1, 2$, $(R^2 \times I \to R)$:

$$\begin{aligned} H_i(u_1,u_2,r) &= L_i(u_1,u_2) - L_i(U_1,U_2) \\ &\quad + r\bigg[L_i(u_1,u_2) + N_1(u_1,u_2) \\ &\quad - F_i\left(\frac{\partial u_i}{\partial t},\varepsilon\right)\bigg], \end{aligned} \tag{24}$$

其中 $I = [0, 1]$, 算子 L_i, N_i 为

$$\begin{aligned} L_1(u_1,u_2) &= a_1\frac{\partial^2 u_1}{\partial s^2} - a_2u_1, \\ L_2(u_1,u_2) &= b_1\frac{\partial u_2}{\partial s} - b_2\frac{\partial u_1}{\partial s}, \\ N_1(u_1,u_2) &= a_3u_1u_2, \\ N_2(u_1,u_2) &= 0. \end{aligned}$$

由映射 (24), 将 (19) 式代入

$$H_i(u_1,u_2,r) = 0 \quad (i = 1,2),$$

按 r 展开非线性项, 合并 r 的同次幂项的系数, 并令其为零. 我们可依次得到 $u_{1i}(s)$, $u_{2i}(s)$ $(i = 0, 1, 2, \cdots)$ 的方程, 并可求出对应的解.

将得到的 u_{1i}, u_{2i}, 并将变换 $s = x + \bar{c}t$ 代入 (19) 式, 可依次得到了系统 (1), (2) 的各次近似解. 在 F_i $(i = 1, 2)$ 的假设下, 利用不动点原理 [6,27,28] 可知, 在选择初始近似

$$u_{10}(s) \equiv u_{10}\left(x+\bar{c}t\right), \quad u_{20}(s) \equiv u_{20}\left(x+\bar{c}t\right)$$

后, 这时 (19) 式就是非线性系统

$$L_i(u_1,u_2) + N_i(u_1,u_2) = F_i, \quad i = 1,2. \tag{25}$$

对应的

$$u_1 = \sum_{i=0}^{\infty} u_{1i}(x,t)s^i, \quad u_2 = \sum_{i=0}^{\infty} u_{2i}(x,ut)s^i,$$

在 $s \in [0, 1]$ 上为一致收敛的解.

显然, 由映射 (24), 这时

$$H_i(u_1,u_2,1) = 0 \quad (i = 1,2)$$

当 $s = x + \bar{c}t$ 时, 它和系统 (1), (2) 相同. 于是系统 (1), (2) 的解 u_1, u_2 就是

$$H_i(u_1,u_2,r) = 0 \quad (i = 1,2)$$

当 $s = x + \bar{c}t$ 时, 在 $r \to 1$ 时的解.

下面来求出 (19) 式的各次系数. 比较

$$H_i(u_1,u_2,r) = 0 \quad (i = 1,2)$$

关于 r 的同次幂的系数. 由

$$H_i(u_1,u_2,r) = 0 \quad (i = 1,2)$$

关于 r 的零次幂的系数, 有

$$L_i(u_{10},u_{20}) = L_i(U_1,U_2), \quad i = 1,2. \tag{26}$$

于是由 (21),(22) 式, 有

$$\begin{aligned} u_{10}(s) &= \bar{k}_0 - \bar{k}_1\bar{\sigma}\tanh(\bar{\sigma}s) \\ &\quad - \bar{k}_3\bar{\sigma}^2\tanh^2(\bar{\sigma}s), \end{aligned} \tag{27}$$

$$\begin{aligned}u_{20}(s) &= \frac{b_2}{b_1\bar{c}}u_{10}(s)\\ &= \frac{b_2}{b_1\bar{c}}\left(\bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}s\right)\right)\\ &\quad -\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}s\right).\end{aligned}\tag{28}$$

比较

$$H_i(u_1,u_2,r)=0\quad(i=1,2)$$

关于 s 的同次幂的系数. 由

$$H_i(u_1,u_2,r)=0\quad(i=1,2)$$

关于 r 的一次幂的系数, 有

$$\begin{aligned}&a_1\frac{\partial^2u_{11}}{\partial s^2}-a_2u_{11}+a_3u_{10}u_{20}\\ &-F_1\left(\frac{\partial u_{10}}{\partial s},\varepsilon\right)=0,\end{aligned}\tag{29}$$

$$b_1\bar{c}\frac{\partial u_{21}}{\partial s}-b_2\frac{\partial u_{11}}{\partial s}-F_2\left(\frac{\partial u_{20}}{\partial s},\varepsilon\right)=0.\tag{30}$$

不难得到系统 (29), (30) 当 $u_{11}(0)=u_{21}(0)=0$ 时的解 (u_{11},u_{21}) 为

$$\begin{aligned}u_{11}(s) =& -\int_0^s\left[a_3u_{10}u_{20}-F_1\left(\frac{\partial u_{10}}{\partial s},\varepsilon\right)\right]\\ &\times\left[\exp\sqrt{\frac{a_2}{a_1}}(s-\eta)\right.\\ &\left.+\exp\left(-\sqrt{\frac{a_2}{a_1}}(s-\eta)\right)\right]\mathrm{d}\eta,\end{aligned}\tag{31}$$

$$u_{21}(s)=\frac{1}{b_1\bar{c}}\int_0^s\left[b_2\frac{\partial u_{11}}{\partial\tau}+F_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\right]\mathrm{d}\tau.\tag{32}$$

其中 u_{10},u_{20},u_{11} 分别由 (27), (28), (31) 式决定.

比较

$$H_i(u_1,u_2,r)=0\quad(i=1,2)$$

关于 r^2 的同次幂的系数. 由

$$H_i(u_1,u_2,s)=0\quad(i=1,2)$$

关于 r 的二次幂的系数, 有

$$\begin{aligned}&a_1\frac{\partial^2u_{12}}{\partial s^2}-a_2u_{12}+a_3(u_{10}u_{21}+u_{11}u_{20})\\ &-\bar{F}_1\left(\frac{\partial u_{10}}{\partial s},\varepsilon\right)\frac{\partial u_{11}}{\partial s}=0,\end{aligned}\tag{33}$$

$$\begin{aligned}&b_1\bar{c}\frac{\partial u_{22}}{\partial s}-b_2\frac{\partial u_{12}}{\partial s}+b_1\frac{\partial u_{21}}{\partial s}-b_2\frac{\partial u_{11}}{\partial s}\\ &-\bar{F}_2\left(\frac{\partial u_{20}}{\partial s},\varepsilon\right)\frac{\partial u_{21}}{\partial s}=0,\end{aligned}\tag{34}$$

其中

$$\bar{F}_i(y,\varepsilon)=\frac{\partial F_i(y,\varepsilon)}{\partial y}\quad(i=1,2).$$

系统 (33), (34) 当 $u_{12}(0)=u_{22}(0)=0$ 时的解 (u_{12},u_{22}) 为

$$\begin{aligned}u_{12}(s) =& -\int_0^s\left[a_3(u_{10}u_{21}+u_{11}u_{20})\right.\\ &\left.-\bar{F}_1\left(\frac{\partial u_{10}}{\partial\eta},\varepsilon\right)\frac{\partial u_{11}}{\partial\eta}\right]\\ &\times\left[\exp\sqrt{\frac{a_2}{a_1}}(s-\eta)\right.\\ &\left.+\exp\left(-\sqrt{\frac{a_2}{a_1}}(s-\eta)\right)\right]\mathrm{d}\eta,\end{aligned}\tag{35}$$

$$\begin{aligned}u_{22}(s) =& \frac{1}{\bar{c}}\int_0^s\left[\frac{b_2}{b_1}\frac{\partial u_{12}}{\partial\tau}-\frac{\partial u_{21}}{\partial\tau}+\frac{b_2}{b_1}\frac{\partial u_{11}}{\partial\tau}\right.\\ &\left.+\frac{1}{b_1}\bar{F}_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\frac{\partial u_{21}}{\partial\tau}\right]\mathrm{d}\tau,\end{aligned}\tag{36}$$

其中 u_{12}, u_{1j} $(j=0.1)$, u_{12} 分别由 (27), (28), (31), (32), (35) 式决定.

由上所得的结果, 再由行波变换 $s=x+\bar{c}t$, 我们能分别得到 Schrödinger 扰动耦合系统 (1), (2) 的一个孤波解的一次, 二次行波近似解析式 $u_{ij\mathrm{app}}(x,t)$ $(i,j=1,2)$ 分别为

$$\begin{aligned}&u_{11\mathrm{app}}(x,t)\\ =&\ \bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\\ &-\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\\ &-\int_0^{x+\bar{c}t}\left[a_3u_{10}u_{20}-F_1\left(\frac{\partial u_{10}}{\partial\eta},\varepsilon\right)\right]\\ &\times\left[\exp\sqrt{\frac{a_2}{a_1}}\left(x+\bar{c}t-\eta\right)\right.\\ &\left.+\exp\left(-\sqrt{\frac{a_2}{a_1}}\left(x+\bar{c}t-\eta\right)\right)\right]\mathrm{d}\eta,\end{aligned}\tag{37}$$

$$\begin{aligned}&u_{21\mathrm{app}}\\ =&\ \frac{b_2}{b_1\bar{c}}\left(\bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right)\\ &-\bar{k}_3\bar{\sigma}^2\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\\ &+\int_0^{x+\bar{c}t}\left[\frac{b_2}{b_1}\frac{\partial u_{11}}{\partial\tau}-\frac{\partial u_{20}}{\partial\tau}\right.\\ &\left.+\frac{b_2}{b_1}\frac{\partial u_{10}}{\partial\tau}+\frac{1}{b_1}F_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\right]\mathrm{d}\tau,\end{aligned}\tag{38}$$

$$\begin{aligned}&u_{12\mathrm{app}}(x,t)\\ =&\ \bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\\ &-\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\\ &-\int_0^{x+\bar{c}t}\left[a_3u_{10}u_{20}-F_1\left(\frac{\partial u_{10}}{\partial\eta},\varepsilon\right)\right]\\ &\times\left[\exp\sqrt{\frac{a_2}{a_1}}\left(x+\bar{c}t-\eta\right)\right.\end{aligned}$$

$$+\exp\left(-\sqrt{\frac{a_2}{a_1}}(x+\bar{c}t-\eta)\right)\bigg]\mathrm{d}\eta$$
$$-\int_0^{x+\bar{c}t}\bigg[a_3(u_{10}u_{21}+u_{11}u_{20})$$
$$-\bar{F}_1\left(\frac{\partial u_{10}}{\partial\eta},\varepsilon\right)\frac{\partial u_{11}}{\partial\eta}\bigg]$$
$$\times\bigg[\exp\sqrt{\frac{a_2}{a_1}}(x+\bar{c}t-\eta)$$
$$+\exp\left(-\sqrt{\frac{a_2}{a_1}}(x+\bar{c}t-\eta)\right)\bigg]\mathrm{d}\eta, \qquad (39)$$

$$u_{22\mathrm{app}}(x+\bar{c}t)$$
$$=\frac{b_2}{b_1\bar{c}}\left(\bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right.$$
$$\left.-\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right)$$
$$+\int_0^{x+\bar{c}t}\bigg[\frac{b_2}{b_1}\frac{\partial u_{11}}{\partial\tau}-\frac{\partial u_{20}}{\partial\tau}$$
$$+\frac{b_2}{b_1}\frac{\partial u_{10}}{\partial\tau}+\frac{1}{b_1}F_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\bigg]\mathrm{d}\tau$$
$$+\frac{1}{b_1\bar{c}}\int_0^{x+\bar{c}t}\left[b_2\frac{\partial u_{11}}{\partial\tau}+F_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\right]\mathrm{d}\tau$$
$$+\frac{1}{\bar{c}}\int_0^{x+\bar{c}t}\bigg[\frac{b_2}{b_1}\frac{\partial u_{12}}{\partial\tau}-\frac{\partial u_{21}}{\partial\tau}$$
$$+\frac{b_2}{b_1}\frac{\partial u_{11}}{\partial\tau}+\frac{1}{b_1}\bar{F}_2\left(\frac{\partial u_{20}}{\partial\tau},\varepsilon\right)\frac{\partial u_{21}}{\partial\tau}\bigg]\mathrm{d}\tau. \qquad (40)$$

用同样的方法, 我们还能得到更高次的近似解

$$u_{ij\mathrm{app}}(x,t), \quad (i=1,2,\ j=3,4,\cdots).$$

4 举 例

为简单起见, 考虑一个特殊的Schrödinger扰动耦合系统, 其扰动项为

$$F_i\left(\frac{\partial u_i}{\partial t},\varepsilon\right)=\varepsilon\exp\left(-\left(\frac{\partial u_i}{\partial t}\right)^2\right)\quad(i=1,2),$$

其中ε为小参数. 这时系统(1), (2)为

$$a_1\frac{\partial^2u_1}{\partial x^2}-a_2u_1+a_3u_1u_2$$
$$=\varepsilon\exp\left(-\left(\frac{\partial u_1}{\partial t}\right)^2\right), \qquad (41)$$

$$b_1\frac{\partial u_2}{\partial t}-b_2\frac{\partial u_1}{\partial x}=\varepsilon\exp\left(-\left(\frac{\partial u_2}{\partial t}\right)^2\right). \qquad (42)$$

我们引入行波变换$s=x+\bar{c}t$. 这时系统(41), (42)为

$$a_1\frac{\partial^2u_1}{\partial s^2}-a_2u_1+a_3u_1u_2$$
$$=\varepsilon\exp\left(-\left(\bar{c}\frac{\partial u_1}{\partial s}\right)^2\right). \qquad (43)$$

$$b_1\bar{c}\frac{\partial u_2}{\partial s}-b_2\frac{\partial u_1}{\partial s}=\varepsilon\exp\left(-\left(\bar{c}\frac{\partial u_2}{\partial s}\right)^2\right). \qquad (44)$$

由映射(24)和系统(43), (44)式, 选取零次近似$u_{10}(s)$, $v_{10}(s)$为

$$u_{10}(s)=\bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}s\right)$$
$$-\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}s\right), \qquad (45)$$

$$u_{20}(s)=\frac{b_2}{b_1\bar{c}}u_{10}(s)$$
$$=\frac{b_2}{b_1\bar{c}}\left(\bar{k}_0-\bar{k}_1\bar{\sigma}\tanh\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right.$$
$$\left.-\bar{k}_3\bar{\sigma}^2\tanh^2\left(\bar{\sigma}\left(x+\bar{c}t\right)\right)\right). \qquad (46)$$

由(27), (28)和(31),(32)式, 有

$$u_{11}(s)=-\int_0^s\bigg[a_3u_{10}u_{20}$$
$$-\varepsilon\exp\left(-\left(\bar{c}\frac{\partial u_1}{\partial s}\right)^2\right)\bigg]$$
$$\times\bigg[\exp\sqrt{\frac{a_2}{a_1}}(s-\eta)$$
$$+\exp\left(-\sqrt{\frac{a_2}{a_1}}(s-\eta)\right)\bigg]\mathrm{d}\eta, \qquad (47)$$

$$u_{21}(s)=\frac{1}{b_1\bar{c}}\int_0^s\bigg[b_2\frac{\partial u_{11}}{\partial\tau}$$
$$+\varepsilon\exp\left(-\left(\bar{c}\frac{\partial u_2}{\partial s}\right)^2\right)\bigg]\mathrm{d}\tau, \qquad (48)$$

$$u_{12}(s)=-\int_0^s\bigg[a_3(u_{10}u_{21}+u_{11}u_{20})$$
$$-\varepsilon\exp\left(-\left(\bar{c}\frac{\partial u_1}{\partial s}\right)^2\right)\frac{\partial u_{11}}{\partial\eta}\bigg]$$
$$\times\bigg[\exp\sqrt{\frac{a_2}{a_1}}(s-\eta)$$
$$+\exp\left(-\sqrt{\frac{a_2}{a_1}}(s-\eta)\right)\bigg]\mathrm{d}\eta, \qquad (49)$$

$$u_{22}(s)=\frac{1}{\bar{c}}\int_0^s\bigg[\frac{b_2}{b_1}\frac{\partial u_{12}}{\partial\tau}-\frac{\partial u_{21}}{\partial\tau}+\frac{b_2}{b_1}\frac{\partial u_{11}}{\partial\tau}$$
$$+\frac{\varepsilon}{b_1}\exp\left(-\left(\bar{c}\frac{\partial u_2}{\partial s}\right)^2\right)\frac{\partial u_{21}}{\partial\tau}\bigg]\mathrm{d}\tau. \qquad (50)$$

由(45)—(50)式, 这时我们便得到Schrödinger扰动耦合系统(43), (44)如下的二次近似孤波解

$u_{12\mathrm{app}}(s)$, $u_{22\mathrm{app}}(s)$:

$$\begin{aligned}
& u_{12\mathrm{app}}(s) \\
= {} & \bar{k}_0 - \bar{k}_1 \bar{\sigma} \tanh(\bar{\sigma} s) - \bar{k}_3 \bar{\sigma}^2 \tanh^2(\bar{\sigma} s) \\
& - \int_0^s \left[a_3 u_{10} u_{20} - \varepsilon \exp\left(- \left(\bar{c} \frac{\partial u_1}{\partial s} \right)^2 \right) \right] \\
& \times \left[\exp \sqrt{\frac{a_2}{a_1}} (s - \eta) \right. \\
& \left. + \exp\left(- \sqrt{\frac{a_2}{a_1}} (s - \eta) \right) \right] \mathrm{d}\eta \\
& - \int_0^s \left[a_3 (u_{10} u_{21} + u_{11} u_{20}) \right. \\
& \left. - \varepsilon \exp\left(- \left(\bar{c} \frac{\partial u_1}{\partial s} \right)^2 \right) \frac{\partial u_{11}}{\partial \eta} \right] \\
& \times \left[\exp \sqrt{\frac{a_2}{a_1}} (s - \eta) \right. \\
& \left. + \exp\left(- \sqrt{\frac{a_2}{a_1}} (s - \eta) \right) \right] \mathrm{d}\eta,
\end{aligned} \tag{51}$$

$$\begin{aligned}
& u_{22\mathrm{app}}(s) \\
= {} & \frac{b_2}{b_1 \bar{c}} \left(\bar{k}_0 - \bar{k}_1 \bar{\sigma} \tanh(\bar{\sigma}(x + \bar{c} t)) \right. \\
& \left. - \bar{k}_3 \bar{\sigma}^2 \tanh^2(\bar{\sigma}(x + \bar{c} t)) \right) \\
& + \frac{1}{b_1 \bar{c}} \int_0^s \left[b_2 \frac{\partial u_{11}}{\partial \tau} + \varepsilon \exp\left(- \left(\bar{c} \frac{\partial u_2}{\partial s} \right)^2 \right) \right] \mathrm{d}\tau \\
& + \frac{1}{\bar{c}} \int_0^s \left[\frac{b_2}{b_1} \frac{\partial u_{12}}{\partial \tau} - \frac{\partial u_{21}}{\partial \tau} + \frac{b_2}{b_1} \frac{\partial u_{11}}{\partial \tau} \right. \\
& \left. + \frac{\varepsilon}{b_1} \exp\left(- \left(\bar{c} \frac{\partial u_2}{\partial s} \right)^2 \right) \frac{\partial u_{21}}{\partial \tau} \right] \mathrm{d}\tau,
\end{aligned} \tag{52}$$

其中 $\bar{k}_i$ $(i = 0, 2, 3)$, $\bar{\sigma}$, $\bar{c}$ 由 (14)—(18) 决定. 再将行波变换 $s = x + \bar{c} t$ 代入 (51), (52) 式, 我们便得到 Schrödinger 非线性扰动耦合系统 (41), (42) 如下的二次近似孤波行波解

$$u_{12\mathrm{app}}(x + \bar{c} t), \quad u_{22\mathrm{app}}(x + \bar{c} t).$$

我们还能用摄动理论以及不动点定理证明, Schrödinger 非线性扰动耦合系统 (41), (42) 的孤波解有如下的估计式[27,28]:

$$u_i(x, t) = u_{i2\mathrm{app}}(x, t) + O\left(\varepsilon^2\right), \quad 0 < \varepsilon \ll 1.$$

因此, 利用本文提出的近似方法得到的孤波近似解具有较好的精确度.

继续地, 利用同样的方法能够得到 Schrödinger 非线性扰动耦合系统 (41), (42) 的更高次近似的孤波行波解

5 结 论

孤波理论出自于一类复杂的自然现象. 因此我们需要简化它为基本模型. 利用近似方法去求解这类模型是孤波理论的重要方面. 本文就是利用投射法和同伦映射理论相结合地构造了一个简单而有效地得到了 Schrödinger 非线性耦合系统孤波的近似行波解.

由渐近方法求解模型的近似解, 不同于单纯的模拟得到的数值近似解, 由于渐近解是具有解析形式的结构, 因此它还可以进行微分, 积分等解析运算, 从而能进一步地了解相应孤波解的更深层的性态.

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The traveling wave approximation method for solving solitary wave in Schrödinger disturbed coupled system*

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Abstract

A class of the Schrödinger nonlinear disturbed coupled system is studied, using the specific technique to relate the exact and approximate solutions. Firstly, the corresponding non-disturbed coupled system is considered. The exact solitary wave solution is obtained by using the projection method. Then, the traveling wave approximation solution to the Schrödinger disturbed coupled system is found by using a functional mapping method.

Keywords: Schrödinger system, solitary wave, asymptotic solution

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