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层结流体中在热外源和β效应地形效应作用下的非线性Rossby孤立波和非齐次Schrödinger方程*

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在层结流体中, 从带有地形、热外源耗散的下边界条件以及带有热外源的准地转位涡方程开始, 使用小参数展开方法和多尺度时空伸长变换推导出了具有热外源、β 效应和地形效应的强迫 Rossby 孤立波方程, 得到孤立 Rossby 振幅满足的带有地形与热外源的非齐次非线性的 Schrödinger 方程. 通过分析 Rossby 孤立波振幅的变化, 指出了热外源、β 效应和地形效应都是诱导 Rossby 孤立波产生的重要因素, 给出了切变基本流下地形、热外源和层结流体中 Rossby 的相互作用.

关键词: Rossby 波, 热外源, 地形, Schrödinger 方程

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1 引 言

天气图上的对流层中上层呈波状形式的气压场或流场中, 在北半球会有 3—5 个波, 这种波即大气长波. 由于其水平尺度与地球半径相当, 也称行星波. 1939 年, 卡尔-古斯塔夫-罗斯贝首先从理论上研究了其性质, 并做出卓越贡献, 后世为了纪念他所建立的大气长波理论, 而称大气长波为 Rossby 波. Rossby 波速与风速相当, 量级在 10 m/s, 传播具有准水平、无辐散的特点, 属于涡旋性慢波. 大尺度天气过程与大气长波关系十分密切, 所以罗斯贝波在大尺度气象研究中, 占有十分重要的地位[1,2]. Long[3] 和 Benney[4] 研究了正压流体中的 Rossby 长波, 得到了著名的振幅变化满足的 Korteweg-de Vries(KdV) 方程, 强调了非线性的重要性. Ripa[5] 讨论了两个包络孤立 Rossby 波的相互作用, 但是他没有考虑到基本流动对赤道包络孤立波的影响. Redekopp[6] 从正压流体的模式中推导了 Rossby 孤立波振幅变化满足的 KdV 方程, 极大的推广了 Long 的结果, 并且他通过对切变基本流中 Rossby 孤立波产生的研究指出了在纬向流中 Rossby 孤立波存在的必要条件[7]. Wadati[8] 从层结流体的模式中推导了 Rossby 孤立波振幅变化满足的改进的 KdV(mKdV) 方程, 也极大的推广了 Long 的结果. 谭本馗等[9] 研究了强迫耗散作用下的 Rossby 包络孤立波及其相互作用. 发现在某些参数下, 两波碰撞后性质变化不大, 而在另外一些参数下, 碰撞后两波性质显著. 宋健和杨联贵等研究了层结流体中具有 β 效应与地形效应的强迫 Rossby 孤立波, 通过分析孤立 Rossby 波振幅的变化, 发现即使基本气流没有切变, 仍可能激发出 Rossby 孤立波; 研究了强迫耗散与 β 效应地形效应作用下的非线性 Rossby 波包, 指出了 β 效应、地形效应及外源都是诱导 Rossby 孤立波产生的重要因素[10]; 研究了切变流体中 β 效应与缓变地形 Rossby 波, 得到了 Rossby 波振幅满足带有缓变地形与外源强迫的非齐次 mKdV-Burgers 方程的

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结论[11]. Maslowe等讨论了在分层流体中纬向流的切变对Rossby波的影响[12]. Ono[13,14]使用多尺度方法, 导出了层结流体中Rossby孤立波振幅变化满足的Benjamin-Davis-Ono-Burgers (BDO-Burger)方程. Boyd[15,16]采用多重尺度的方法, 导出了正压流体中Rossby孤立波振幅变化满足的KdV和mKdV方程. 赵强等在完全Coriolis力下获得线性Rossby孤立波振幅满足KdV-Burgers方程[17]. 汪萍等[18]给出了正压大气外强迫作用下非线性特征数值模拟, 解释了大气大尺度非线性运动的某些特征. 在本文中, 探讨了β推广平面近似、热外源、地形变化及其Vaisala-Brunt频率对Rossby孤立波振幅的变化.

2 方程的推导

2.1 控制方程与边界条件

Rossby波为大尺度大气运动. 这里考虑β效应下, 层结流体的无量纲形式的准地转位涡方程[19,20]

$$\left(\frac{\partial}{\partial t}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \eta}{\partial y}\frac{\partial}{\partial x}\right)\times\left[\nabla^2\eta+\beta(y)y+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial \eta}{\partial z}\right)\right]=\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}Q^*\right), \tag{1}$$

方程(1)中, η是无量纲的流函数, f是柯氏参数为常数, 推广的β平面近似取为$\beta(y)y$, $s=\dfrac{N^2}{f}$和ρ_s都是关于垂直方向z的函数, $N(z)$是Brunt-Vaisala频率, ρ_s是流体的层结密度, 其中$Q^*=\dfrac{g}{fc_{\rm P}T_0}Q$, $Q(x,y,z,t)$是外热源, g是重力加速度常数, $c_{\rm P}$是流体的定压比热容, T_0是静止流体静止时的温度, ∇^2为Laplace算子, 定义为

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \tag{2}$$

侧边界条件为刚壁条件的无量纲形式, 取

$$\frac{\partial \eta}{\partial x}=0,\quad y=0,1, \tag{3}$$

上边界条件为

$$\rho_s\eta\to 0,\quad z\to\infty, \tag{4}$$

下边界条件, 考虑地形廓线$h(x,y,t)$和湍流加热耗散的影响, 其支配方程可写为

$$\left(\frac{\partial}{\partial t}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \eta}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial \eta}{\partial z}+s\left(\frac{\partial \eta}{\partial x}\frac{\partial h}{\partial y}-\frac{\partial \eta}{\partial y}\frac{\partial h}{\partial x}\right)+\left|\frac{K}{2f}\right|^{\frac{1}{2}}s\nabla^2\eta=Q,\quad z=0, \tag{5}$$

其中K表示耗散强度.

2.2 带有强迫项的非线性Schrödinger方程

这里总的流函数为

$$\eta(x,y,z,t)=-\int^y(\overline{u}(s,z)-c_0)\mathrm{d}s+\varepsilon\psi(x,y,z,t), \tag{6}$$

其中, ε是度量非线性程度强弱的小参数, 当$\varepsilon\ll 1$时, 成为弱的非线性问题, c_0是常数, 它是切变气流中线性长波的相速度, ψ是扰动流函数.

将(6)代入方程(1)和(5)及边界条件(3)和(4), 得到扰动流函数的方程及边界调节

$$\left[\frac{\partial}{\partial t}+(\overline{u}(y,z)-c_0)\frac{\partial}{\partial x}+\varepsilon\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\right]\times\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial \psi}{\partial z}\right)\right]+\frac{\partial \psi}{\partial x}P(y,z)=\frac{1}{\varepsilon}\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}Q^*\right), \tag{7}$$

$$\left[\frac{\partial}{\partial t}+(\overline{u}(y,z)-c_0)+\varepsilon\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\right]\frac{\partial \psi}{\partial z}-\frac{\partial \psi}{\partial x}\frac{\partial \overline{u}}{\partial z}+s\left(\frac{\partial \psi}{\partial x}\frac{\partial h}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial h}{\partial x}\right)+\frac{1}{\varepsilon}s(\overline{u}(y,z)-c_0)\frac{\partial h}{\partial x}+\left|\frac{K}{2f}\right|^{\frac{1}{2}}s\nabla^2\psi=0, \tag{8}$$

$$\rho_s\psi\to 0,\quad z\to\infty, \tag{9}$$

$$\frac{\partial \psi}{\partial x}=0,\quad y=0,1, \tag{10}$$

为了使地形强迫, 外热源耗散和非线性之间达到平衡, 这里设

$$h(x,y,t)=\varepsilon^3\Omega(x,y,t), \tag{11}$$

$$Q^*(x,y,z,t)=\varepsilon^3q(x,y,z,t), \tag{12}$$

$$\left|\frac{K}{2f}\right|^{1/2}=\varepsilon^2\lambda, \tag{13}$$

$$\left|\frac{K}{2f}\right|^{1/2}s\frac{\partial \overline{u}}{\partial y}+Q^*=0, \tag{14}$$

其中λ是尺度放大后的耗散系数, 将(11), (12), (13)式代入(7)和(8)式, 得到

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}+\varepsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\right]\\&\times\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]+\frac{\partial\psi}{\partial x}P\\=&\varepsilon^2\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}q\right),\end{aligned}\tag{15}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}(y,z)-c_0)\right.\\&\left.+\varepsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\right]\frac{\partial\psi}{\partial z}\\&-\frac{\partial\psi}{\partial x}\frac{\partial\overline{u}}{\partial z}+\varepsilon^3 s\left(\frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x}\right)\\&+\varepsilon^2 s(\overline{u}-c_0)\frac{\partial\Omega}{\partial x}+\varepsilon^2\lambda s\nabla^2\psi=0,\end{aligned}\tag{16}$$

其中

$$\begin{aligned}P(y,z)=&\frac{\partial[\beta(y)y]}{\partial y}-\frac{\partial^2\overline{u}(y,z)}{\partial y^2}\\&-\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\overline{u}(y,z)}{\partial z}\right),\end{aligned}$$

在中高纬度地区, 由于外强迫的作用, 在一些大尺度波动中非线性行为变得很重要, 这时需要把波振幅看成是缓慢变化的, 所以除了快变量之外, 还需要引进如下的慢时空变量[21]

$$T_1=\varepsilon t,\quad T_2=\varepsilon^2 t,\quad X_1=\varepsilon x,\quad X_2=\varepsilon^2 x,\tag{17}$$

可作如下变换:

$$\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X_1}+\varepsilon^2\frac{\partial}{\partial X_2},\\\frac{\partial}{\partial t}&=\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2},\end{aligned}\tag{18}$$

将(18)式代入方程(15)和(16)式可得

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]\\&+\frac{\partial\psi}{\partial x}P+\varepsilon\left\{\left[\frac{\partial}{\partial T_1}+(\overline{u}-c_0)\frac{\partial}{\partial X_1}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right.\right.\\&\left.-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]\\&\left.+2\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\frac{\partial^2\psi}{\partial x\partial X_1}+\frac{\partial\psi}{\partial X_1}P\right\}\\&+\varepsilon^2\left\{\left[\frac{\partial}{\partial T_2}+(\overline{u}-c_0)\frac{\partial}{\partial X_2}+\frac{\partial\psi}{\partial X_1}\frac{\partial}{\partial y}\right.\right.\\&\left.-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial X_1}\right]\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]\\&+2\left[\frac{\partial}{\partial T_1}+(\overline{u}-c_0)\frac{\partial}{\partial X_1}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right.\\&\left.-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right]\frac{\partial^2\psi}{\partial x\partial X_1}+\frac{\partial\psi}{\partial X_2}P\\&\left.+\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi}{\partial X_1^2}+2\frac{\partial^2\psi}{\partial x\partial X_2}\right)\right\}\\&+\varepsilon^3[..]=\varepsilon^2\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}q\right),\end{aligned}\tag{19}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial X_1}\right]\frac{\partial\psi}{\partial z}-\frac{\partial\psi}{\partial x}\frac{\partial\overline{u}}{\partial z}\\&+\varepsilon\left\{\left[\frac{\partial}{\partial T_1}+(\overline{u}-c_0)\frac{\partial}{\partial X_1}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right.\right.\\&\left.\left.-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right]\frac{\partial\psi}{\partial z}-\frac{\partial\psi}{\partial X_1}\frac{\partial\overline{u}}{\partial z}\right\}\\&+\varepsilon^2\left\{\left[\frac{\partial}{\partial T_2}+(\overline{u}-c_0)\frac{\partial}{\partial X_2}+\frac{\partial\psi}{\partial X_1}\frac{\partial}{\partial y}\right.\right.\\&\left.-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial X_1}\right]\frac{\partial\psi}{\partial z}-\frac{\partial\psi}{\partial X_2}\frac{\partial\overline{u}}{\partial z}+s(\overline{u}-c_0)\frac{\partial\Omega}{\partial x}\\&\left.+\lambda s\nabla^2\psi\right\}+\varepsilon^3[..]=0,\end{aligned}\tag{20}$$

上述俩个方程中不仅有快变量x, t, 还含有了缓变量X_1, X_2, T_1, T_2.

设扰动流函数ψ具有如下的小参数展开形式[22]:

$$\psi=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots,\tag{21}$$

将(21)式代入方程(19), (20)和边界条件(9), (10)中, 得到各阶摄动问题的方程和边界条件.

对于$O(\varepsilon^0)$阶, 这里有

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_0+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi_0}{\partial z}\right)\right]\\&+\frac{\partial\psi_0}{\partial x}P=0,\end{aligned}\tag{22}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\frac{\partial\psi_0}{\partial z}\\&-\frac{\partial\psi_0}{\partial x}\frac{\partial\overline{u}}{\partial z}=0,\quad z=0,\end{aligned}\tag{23}$$

$$\rho_s\psi_0\to0,\quad z\to\infty,\tag{24}$$

$$\frac{\partial\Psi_0}{\partial x}=0,\quad y=0,1.\tag{25}$$

假设ψ_0有以下形式的解:

$$\begin{aligned}\psi_0=&A(X_1,X_2;T_1,T_2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\varPhi_0(y,z)\\&+\mathrm{c.c.},\end{aligned}\tag{26}$$

其中c.c.表示ψ_0前项的共轭, A为复振幅, k为纬向波速, ω为波频率. 同样, 引入了慢变量, 可以设

$$\begin{aligned}&\Omega(x,y,t)\\=&M(X_1,X_2;T_1,T_2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}O(y),\end{aligned}\tag{27}$$

$$\begin{aligned} & q(x,y,z,t) \\ = & V(X_1,X_2;T_1,T_2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}W(y,z). \end{aligned} \tag{28}$$

将(26)式代入(22), (23), (24), (25)式可以得到

$$\left[\frac{\partial^2}{\partial y^2}-k^2+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial}{\partial z}\right)+\frac{P}{\overline{u}-c_0-c}\right]\varPhi_0 = 0, \tag{29}$$

$$\left(\frac{\partial}{\partial z}-\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)\varPhi_0=0,\quad z=0, \tag{30}$$

$$\rho_s\varPhi_0\to 0,\quad z\to\infty, \tag{31}$$

$$\varPhi_0(y,z)=0,\quad y=0,1. \tag{32}$$

这里对上式运用分离变量法, 设$\varPhi_0(y,z) = \varPhi_{\mathrm{A}}(y)\varPhi_{\mathrm{B}}(z)$, 可得到

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-k^2+\frac{P}{\overline{u}-c_0-c}-a\right)\varPhi_{\mathrm{A}}=0, \tag{33}$$

$$\varPhi_{\mathrm{A}}(0)=0,\quad \varPhi_{\mathrm{A}}(1)=0, \tag{34}$$

$$\left[\frac{f}{\rho_s}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\rho_s}{s}\frac{\mathrm{d}}{\mathrm{d}z}\right)+a\right]\varPhi_{\mathrm{B}}=0, \tag{35}$$

$$\begin{aligned} & \rho_s\varPhi_{\mathrm{B}}\to 0,\quad z\to\infty, \\ & \left(\frac{\mathrm{d}}{\mathrm{d}z}-\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)\varPhi_{\mathrm{B}}=0,\quad z=0, \end{aligned} \tag{36}$$

其中a为设的常数, 在大气中, 大尺度Rossby波有$\overline{u}-c_0-c>0$, 故上两个方程均可有解.

这里$c=\dfrac{\omega}{k}$, 在$O(\varepsilon^0)$阶近似中, 我们确定了复振幅A的空间结构, 而对它的时间结构还没有确定, 为了得到它的时间结构, 需要更高阶的近似.

对于$O(\varepsilon^1)$阶有

$$\begin{aligned} & \left[\frac{\partial}{\partial t}-(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_1+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi_1}{\partial z}\right)\right] \\ & +\frac{\partial\psi_1}{\partial x}P=C_1, \end{aligned} \tag{37}$$

$$\begin{aligned} & \left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\frac{\partial\psi_1}{\partial z} \\ & -\frac{\partial\psi_1}{\partial x}\frac{\partial\overline{u}}{\partial z}=D_1,\quad z=0, \end{aligned} \tag{38}$$

$$\rho_s\psi_1\to 0,\quad z\to\infty,\quad \frac{\partial\psi_1}{\partial x}=0,\quad y=0,1, \tag{39}$$

这里

$$\begin{aligned} C_1 = & \frac{P\psi_0}{\overline{u}-c_0-c}\left(\frac{\partial A}{\partial T_1}+c_1\frac{\partial A}{\partial X_1}\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ & +\mathrm{i}k\left(\frac{P}{\overline{u}-c_0-c}\right)_y A^2\varPhi_0^2\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)} \\ & +\mathrm{c.c.}, \end{aligned} \tag{40}$$

$$\begin{aligned} D_1 = & -\frac{\overline{u_z}\psi_0}{\overline{u}-c_0-c}\left(\frac{\partial A}{\partial T_1}+c\frac{\partial A}{\partial X_1}\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ & -\mathrm{i}k\left(\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y A^2\varPhi_0^2\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)} \\ & +\mathrm{c.c.}, \end{aligned} \tag{41}$$

其中$c_1=c+\dfrac{2k^2(\overline{u}-c_0-c)^2}{P}$, $\left(\dfrac{P}{\overline{u}-c_0-c}\right)_y$和$\left(\dfrac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y$是对变量$y$求一阶偏导.

利用消去长期项

$$\int_0^\infty\int_0^1\frac{\varPhi_0}{\overline{u}-c_0-c}C_1\,\mathrm{d}y\mathrm{d}z=0,$$

$$\int_0^1\frac{\varPhi_0}{\overline{u}-c_0-c}D_1\,|_{z=0}\,\mathrm{d}y=0,$$

方程化解为

$$\frac{\partial A}{\partial T_1}+c_{\mathrm{g}}\frac{\partial A}{\partial X_1}=0, \tag{42}$$

其中

$$\begin{aligned} I = & \int_0^\infty\int_0^1\frac{P\varPhi_0^2}{(\overline{u}-c_0-c)^2}\mathrm{d}y\mathrm{d}z \\ & -\int_0^1\frac{\overline{u_z}\varPhi_0^2}{(\overline{u}-c_0-c)^2}\,|_{z=0}\,\mathrm{d}y, \\ I_1 = & \int_0^\infty\int_0^1 2k^2\varPhi_0^2\,\mathrm{d}y\mathrm{d}z, \\ c_{\mathrm{g}} = & c+\frac{I_1}{I}. \end{aligned}$$

在$O(\varepsilon^1)$阶问题中, 复振幅$A(X_1,X_2;T_1,T_2)$以c_{g}速度传播, 从物理意义上讲c_{g}是波的群速度.

则方程(37)和(38)可化为

$$\begin{aligned} & \left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right] \\ & \times\left[\nabla^2\psi_1+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi_1}{\partial z}\right)\right] \\ & +\frac{\partial\psi_1}{\partial x}P=\mathrm{i}kE(y,z)A^2\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)}+\mathrm{c.c.}, \end{aligned} \tag{43}$$

$$\begin{aligned} & \left[\frac{\partial}{\partial t}+(\overline{u}-c_0)\frac{\partial}{\partial x}\right]\frac{\partial\psi_1}{\partial z}-\frac{\partial\overline{u}}{\partial z}\frac{\partial\psi_1}{\partial x} \\ = & -\mathrm{i}kF(y,z)A^2\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)}+\mathrm{c.c.},\quad z=0, \end{aligned} \tag{44}$$

这里

$$E(y,z)=\varPhi_0^2\left(\frac{P}{\overline{u}-c_0-c}\right)_y,$$

$$F(y,z)=\varPhi_0^2\left(\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y,$$

其中$\overline{u}_z$表示对z求偏导, $\left(\dfrac{P}{\overline{u}-c_0-c}\right)_y$和$\left(\dfrac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y$表示对$y$求偏导. 设$\psi_1$有如下形

式的解;

$$\psi_1 = B(X_1, X_2; T_1, T_2)\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)}\Phi_1(y,z) + \text{c.c.}. \tag{45}$$

将 (45) 式代入 (43) 和 (44) 式得到

$$B\left[\frac{\partial^2}{\partial y^2} - 4k^2 + \frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial}{\partial z}\right) + \frac{P}{\overline{u}-c_0-c}\right]\Phi_1 = A^2\frac{E}{2(\overline{u}-c_0-c)}, \tag{46}$$

$$B\left(\frac{\partial}{\partial z} - \frac{\overline{u_z}}{\overline{u}-c_0-c}\right)\Phi_1 = A^2\frac{F}{2(\overline{u}-c_0-c)}, \tag{47}$$

显然, 在上述两个方程中, A 和 B 是两个相互联系的变量, 由于 A 和 B 都是慢变量 X_1, X_2, T_1, T_2 的函数, 且 B 与 A^2 成比例. 不妨设 $B = A^2$, 则

$$\psi_1 = A^2(X_1, X_2; T_1, T_2)\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)}\Phi_1 + \text{c.c.}.$$

对于 $O(\varepsilon^2)$ 阶, 有

$$\left[\frac{\partial}{\partial t} - (\overline{u}-c_0)\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_2 + \frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi_2}{\partial z}\right)\right] + \frac{\partial\psi_2}{\partial x}P = C_2, \tag{48}$$

$$\left[\frac{\partial}{\partial t} + (\overline{u}-c_0)\frac{\partial}{\partial x}\right]\frac{\partial\psi_2}{\partial z} - \frac{\partial\psi_2}{\partial x}\frac{\partial\overline{u}}{\partial z} = D_2, \quad z = 0, \tag{49}$$

$$\rho_s\psi_2 \to 0, \quad z\to\infty, \qquad \frac{\partial\psi_2}{\partial x} = 0, \quad y = 0, 1, \tag{50}$$

这里

$$\begin{aligned} C_2 =& \frac{P\psi_0}{\overline{u}-c_0-c}\left(\frac{\partial A}{\partial T_2} + c_1\frac{\partial A}{\partial X_2}\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \mathrm{i}k(2c_{\mathrm{g}} + c + 3c_0 - 3\overline{u})\Phi_0\frac{\partial^2 A}{\partial X_1^2}\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \mathrm{i}k|A|^2 A\left[\frac{1}{2}\Phi_0\left(\frac{E}{\overline{u}-c_0-c}\right)_y + \Phi_0\Phi_1\left(\frac{P}{\overline{u}-c_0-c}\right)_y + \left(\frac{E}{\overline{u}-c_0-c}\right)\frac{\partial\Phi_0}{\partial y}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ V\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}W(y,z)\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \text{c.c.} + \varDelta, \end{aligned} \tag{51}$$

$$\begin{aligned} D_2 =& -\frac{\overline{u_z}\psi_0}{\overline{u}-c_0-c}\left(\frac{\partial A}{\partial T_2} + c\frac{\partial A}{\partial X_2}\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \mathrm{i}k\left|A^2\right|A\left[\frac{1}{2}\Phi_0\left(\frac{F}{\overline{u}-c_0-c}\right)_y - \Phi_0\Phi_1\left(\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y + \left(\frac{F}{\overline{u}-c_0-c}\right)\frac{\partial\Phi_0}{\partial y}\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \lambda s\left[\frac{P}{\overline{u}-c_0-c}\Phi_0 + \frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\Phi_0}{\partial z}\right)\right]A\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &- \mathrm{i}kMs(\overline{u}-c_0)O(y)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ &+ \text{c.c.} + \varDelta, \end{aligned} \tag{52}$$

其中, $\varDelta$ 表示与 $\mathrm{e}^{-\mathrm{i}(kx-\omega t)}$, $\mathrm{e}^{\pm 2\mathrm{i}(kx-\omega t)}$ 和 $\mathrm{e}^{\pm 3\mathrm{i}(kx-\omega t)}$ 有关的其他项.

利用消去长期项

$$\int_0^\infty\int_0^1 \frac{\Phi_0}{\overline{u}-c_0-c}C_2\,\mathrm{d}y\,\mathrm{d}z = 0,$$

$$\int_0^1 \frac{\Phi_0}{\overline{u}-c_0-c}D_2\,|_{z=0}\,\mathrm{d}y = 0,$$

得到

$$\mathrm{i}\left(\frac{\partial A}{\partial T_2} + c_1\frac{\partial A}{\partial X_2}\right) + \alpha\frac{\partial^2 A}{\partial X_1^2} + \delta|A|^2 A + \mathrm{i}\gamma\lambda A = b_1 M + \mathrm{i}b_2 V, \tag{53}$$

其中,

$$\alpha = -\frac{I_2}{I}, \quad \delta = -\frac{I_3}{I}, \quad \gamma = \frac{I_4}{I},$$

$$b_1 = -\frac{I_5}{I}, \quad b_2 = -\frac{I_6}{I},$$

$$I_2 = k\int_0^\infty\int_0^1 \frac{2c_{\mathrm{g}} + c + 3c_0 - 3\overline{u}}{\overline{u}-c_0-c}\Phi_0^2\,\mathrm{d}y\,\mathrm{d}z,$$

$$\begin{aligned} I_3 =& k\int_0^\infty\int_0^1 \frac{\Phi_0}{\overline{u}-c_0-c}\left[\frac{1}{2}\Phi_0\left(\frac{E}{\overline{u}-c_0-c}\right)_y + \Phi_0\Phi_1\left(\frac{P}{\overline{u}-c_0-c}\right)_y + \frac{E}{\overline{u}-c_0-c}\frac{\partial\Phi_0}{\partial y}\right]\mathrm{d}y\,\mathrm{d}z \\ &- k\int_0^1 \frac{\Phi_0}{\overline{u}-c_0-c}\left[\frac{1}{2}\Phi_0\left(\frac{F}{\overline{u}-c_0-c}\right)_y - \Phi_0\Phi_1\left(\frac{\overline{u_z}}{\overline{u}-c_0-c}\right)_y + \frac{F}{\overline{u}-c_0-c}\frac{\partial\Phi_0}{\partial y}\right]\Bigg|_{z=0}\mathrm{d}y, \end{aligned}$$

$$I_4 = \int_0^1 \frac{s\Phi_0}{\overline{u}-c_0-c}\left[\frac{P\Phi_0}{\overline{u}-c_0-c} + \frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\Phi_0}{\partial z}\right)\right]\Bigg|_{z=0}\mathrm{d}y,$$

$$I_5 = k\int_0^1 \frac{s\Phi_0}{\overline{u}-c_0-c}(\overline{u}-c_0)O\Big|_{z=0}\mathrm{d}y,$$

$$I_6=\int_0^\infty\int_0^1\frac{\Phi_0}{\overline{u}-c_0-c}\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}W\right)\mathrm{d}y\mathrm{d}z.$$

方程(53)就是描述了层结流体强迫耗散, 热外源与β效应地形下Rossby波包振幅演变的非齐次非线性Schrödinger方程. 它反映了Rossby波的特性, 系数α,δ分别是频散系数和Landau系数, 它们与$P(y,z)$和$\overline{u}(y,z)$有关.

做坐标变换[23]

$$T=T_2,\quad X=\frac{1}{\varepsilon}(X_2-c_{\mathrm{g}}T_2)=X_1-c_{\mathrm{g}}T_1,$$

方程(53)化为标准的非线性Schrödinger方程

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta\left|A\right|^2A+\mathrm{i}\gamma\lambda A=b_1M+\mathrm{i}b_2V,$$

其中A, M, V均是缓变量X, T的函数.

若基本气流$\overline{u}=$常数时, 在正压流体中, 当β是纬度变量y的函数, $E\neq0$, 这时, $\delta\neq0$, 方程(53)还是非线性Schrödinger方程, 说明即使没有基本气流变化, 只要β效应存在, Rossby波振幅演变也满足非线性Schrödinger方程. 显然, 对层结流体而言, 即使没有基本气流的变化和β效应的存在Rossby波振幅演变仍满足非线性Schrödinger方程. 注意到孤立波的存在还必须有$I\neq0$, 即$P(y,z)\neq0$,这表示在基本气流的变化下、β效应下与层结流体下产生的Rossby波能保持恒稳定的波形.

3 结　论

大气阻塞形势的活动对天气及气候有着重要的影响, 大气阻塞的形成和维持与天气尺度波和地形强迫有关, 因此研究阻塞的形成和维持必须考虑外强迫的作用, 另一方面, 由于阻塞是一种大振幅波, 又必须考虑非线性的作用. Modons(偶极子)理论和KdV型Rossby孤立子理论解释的偶极子阻塞与实际观测到的偶极子阻塞有一定的一致性, 但是它们之间的差别还是很大的, 并且这种孤立子存在严重的缺陷, 即首先它必须要求满足长波近似, 其次它必须要求基本流有强的水平切变, 因此用KdV型Rossby孤立子理论来解释大气中的偶极子是不合适的. 在实际大气中, 由于存在非线性和外强迫, 故Rossby波的形状通常要发生缓慢变化, 这里应用多重尺度法, 从描写层结流体的Rossby波的位涡度方程出发, 推导出在β效应、地形效应和热外源耗散作用下Rossby波振幅变化满足的非齐次非线性Schrödinger方程, 它对KdV型Rossby孤立子理论所需要的条件都不要求. 这说明大气中Rossby波与切变基本流、β效应、地形效应和热外源耗散相互作用, 可以使大气中形成Rossby包络孤立子. 这里非齐次非线性的Schrödinger方程和包络Rossby孤立子能很好的解释偶极子阻塞的形成、维持和崩溃过程, 特别是能解释阻塞的多涡结构. 并进一步说明, 基本气流切变、非线性β效应、Brunt-Vaisala频率、地形效应和热外源都能够诱导包络Rossby孤立子.

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Nonlinear solitary Rossby waves with external heating source and β effect topographic effect in stratified flows described by the inhomogeneous Schrödinger equation*

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Abstract

Rossby waves are intrinsic in the large-scale systems of fluids, so they are the most important waves in the atmosphere and ocean. Theory and observation show that their basic characteristic is to satisfy the quasi-geostrophic and quasi-static equilibrium approximations. In stratified fluids, we discuss the long waves in a homogenous atmosphere and obtain the KdV equation, but the analysis is limited to the case that the velocity shear is small compared with a basic uniform zonal motion, and it gives no insight pertaining to the kinds of stream-line-flow patterns accompanying these waves. Here, the β-plane approximation $f = f_0 + \beta_0 y$ ($\beta_0$is a constant) is extended into $f = f_0 + \beta(y)y$, which includes a nonlinear function $\beta(y)$ taking the place of β in the β-plane approximation. Such an approximation can depict more precisely the motion of the atmosphere and ocean, especially in the middle and high latitude regions. It generalizes the theory developed by Helfrich and Pedlosky for time-dependent coherent structures in a marginally stable zonal flow by including forcing. Such forcing could be due to topography or external source. We take the basic flow to be a shear and the Väisälä-Brunt frequency N a function of variable z. For the stratified fluids, based on the lower boundary with external heating source and topography, as well as the quasi-geostrophic potential vorticity equation with external heating source, an inhomogeneous nonlinear Schrödinger equation (including topographic forcing and an external heating source) is derived by using the perturbation method and stretching transforms of time and space. It is found that the external heating source, β effect and topography effect are the important factors that could induce the nonlinear solitary Rossby by inspection of the evolution of the amplitude of Rossby waves. On the assumption that nonlinear topographic effects and the dissipation of external heating source are balanced, an inhomogenous equation in which the coefficients depend on $\beta(y)$, $\overline{u}(y,z)$ and $N(z)$ is derived. Results show that the topography, external heating source and Rossby waves will interact with a basic stream function that has a shear. In stratified fluids, the inhomogeneous nonlinear Schrödinger equation is obtained for describing the evolution of the amplitude of solitary Rossby envelop solitary waves as the change of Rossby parameter $\beta(y)$ with latitude y, topographic forcing and the external heating source.

Keywords: Rossby waves, external heating source, topography, Schrödinger equation

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