第 46 卷 第 12 期 1997 年 12 月
1000-3290/97/46(12)/2294-06

物 理 学 报
ACTA PHYSICA SINICA

Vol. 46, No. 12, December, 1997
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# 双荷静止球对称天体 Einstein-Maxwell 方程的共形平直内解

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(1997 年 4 月 10 日收到)

严格解出了一种既荷电又荷磁,且电荷密度与磁荷密度之比为常量的静止球对称天体的共形平直内解.

PACC: 0420

## 1 引 言

过去,为了寻找静止球对称天体的内解,人们作了大量工作,并且引用了共形平直的方法来消除一些奇异性.如 Banerjee 和 Santos[1], Shi-Chang 等[2]的工作,但所得内解都是不含磁荷的情况.对于场源荷电又荷磁的情况,Kasuya[3]用群论的方法,根据对称性原理,把解直接由荷电延拓到荷磁的情况.本文在假定场源既荷电又荷磁,且电荷密度与磁荷密度之比为常量的情况下,严格解出了静态球对称天体的一种共形平直内解.从而,其结果也为 Kasuya 的"复延拓法"提供了一个可靠的理论依据.

## 2 Einstein-Maxwell 方程

### 2.1 电荷密度与磁荷密度之比为常量时的 Maxwell 方程

对于场源尚含磁荷成分的 $F_{\mu\nu}$,现定义为

$$F_{\mu\nu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu + G^+_{\mu\nu}, \tag{1}$$

式中 $G^+_{\mu\nu}$为 Dirac 弦项[4]. Maxwell 方程的协变形式为

$$F^{\mu\ \nu}_{\ ;\nu} = 4\pi \rho_e u^\mu, \tag{2}$$

$$F^{+\mu\ \nu}_{\ ;\nu} = 4\pi \rho_g u^\mu, \tag{3}$$

式中 $F^{+\mu\nu}$为 $F^{\mu\nu}$的对偶张量; $\rho_e$, $\rho_g$ 分别为电荷和磁荷密度, $u^\mu$ 为四维速度.

为了便于研究,定义一个新的实反对称张量

$$\widetilde{F}^{\mu\nu} = F^{\mu\nu}\cos\alpha + F^{+\mu\nu}\sin\alpha, \tag{4}$$

式中 $\alpha$ 为常实角度.从(2)和(3)式得到

$$\widetilde{F}^{\mu\nu}{}_{;\nu} = 4\pi(\rho_e \cos\alpha + \rho_g \sin\alpha)\, u^\mu, \tag{5}$$

$$\widetilde{F}^{+\mu\nu}{}_{;\nu} = 4\pi(-\rho_e \sin\alpha + \rho_g \cos\alpha)\, u^\mu. \tag{6}$$

令

$$\rho_e \cos\alpha + \rho_g \sin\alpha = h, \tag{7}$$

$$-\rho_e \sin\alpha + \rho_g \cos\alpha = 0, \tag{8}$$

则有 $\rho_e/\rho_g = \cot\alpha$, 为一常量, 满足我们的假定. 对应的 Maxwell 方程为

$$\widetilde{F}^{\mu\nu}{}_{;\nu} = 4\pi h u^\mu, \tag{9}$$

$$\widetilde{F}^{+\mu\nu}{}_{;\nu} = 0, \tag{10}$$

即

$$(\sqrt{-g}\,\widetilde{F}^{\mu\nu})_{,\nu} = 4\pi\sqrt{-g}\, J^\mu, \tag{11}$$

式中 $J^\mu = h u^\mu$. (11)式与只带电荷的情况类似.

## 2.2 广义电场强度和广义电荷

静态球对称度规形式为

$$\mathrm{d}s^2 = \mathrm{e}^\nu \mathrm{d}t - \mathrm{e}^\lambda \mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2). \tag{12}$$

方程(9)和(10)在球对称情况下, 仅有 $\widetilde{F}^{01}$不为零, $\widetilde{F}^{01} = -\widetilde{F}^{10}$. 从(11)式解得

$$\widetilde{F}^{01} = \frac{4\pi}{r^2}\mathrm{e}^{-(\nu+\lambda)/2}\int_0^r r^2 h \mathrm{e}^{\lambda/2}\mathrm{d}r, \tag{13}$$

$$\widetilde{F}^{01} = g_{00}\, g_{11}\, \widetilde{F}^{01} = -\frac{4\pi}{r^2}\mathrm{e}^{(\nu+\lambda)/2}\int_0^r r^2 h \mathrm{e}^{\lambda/2}\mathrm{d}r. \tag{14}$$

定义广义的电场强度

$$\widetilde{E}^2 = -\widetilde{F}^{01}\widetilde{F}_{01}, \tag{15}$$

则包含在半径为 $r$ 的球内的总广义电荷为

$$\widetilde{Q}(r) = 4\pi\int_0^r h r^2 \mathrm{e}^{\lambda/2}\mathrm{d}r = r^2\widetilde{E}. \tag{16}$$

若令

$$Q(r) = 4\pi\int_0^r \rho_e r^2 \mathrm{e}^{\lambda/2}\mathrm{d}r, \tag{17}$$

$$\Phi(r) = 4\pi\int_0^r \rho_g r^2 \mathrm{e}^{\lambda/2}\mathrm{d}r, \tag{18}$$

则由(7)和(8)式可得

$$\widetilde{Q}^2 = Q^2 + \Phi^2, \tag{19}$$

$$\widetilde{E}^2 = \frac{Q^2 + \Phi^2}{r^4}. \tag{20}$$

### 2.3 场方程

类似于只带电荷的情况,可导出场方程为

$$e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}=-8\pi\rho-\frac{(Q^2+\Phi^2)}{r^4}, \tag{21a}$$

$$\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)=-8\pi P+\frac{Q^2+\Phi^2}{r^4}, \tag{21b}$$

$$e^{-\lambda}\left[\frac{1}{4}\nu'\lambda'-\frac{1}{4}\nu'^2-\frac{1}{2}\nu''-\frac{1}{2}\left(\frac{\nu'-\lambda'}{r}\right)\right]=-8\pi P-\frac{Q^2+\Phi^2}{r^4}. \tag{21c}$$

## 3 当 $\rho_e e^{\lambda/2}=K_1$, $\rho_g e^{\lambda/2}=K_2$, $K_1K_2$ 为常量时的共形平直内解

对于静态球对称情况,共形平直条件表现为如下一个方程[2]:

$$\frac{e^{\lambda}}{r^2}-\frac{1}{r^2}-\frac{\nu'^2}{4}+\frac{\nu'\lambda'}{4}-\frac{1}{2}\nu''-\frac{1}{2r}(\lambda'-\nu')=0. \tag{22}$$

由(21c)—(21b)式得

$$e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}-\frac{1}{r^2}-\frac{1}{2r}(\lambda'+\nu')+\frac{e^{\lambda}}{r^2}\right]=\frac{2(Q^2+\Phi^2)}{r^4}. \tag{23}$$

用 $e^{-\lambda}$乘(22)式,然后与(23)式相加,得

$$\frac{Q^2+\Phi^2}{r^4}=\frac{1}{r^2}-e^{-\lambda}\left(\frac{\lambda'}{2r}+\frac{1}{r^2}\right). \tag{24}$$

设场源电荷密度与磁荷密度满足如下条件:

$$\rho_e e^{\lambda/2}=K_1=\text{const},\quad \rho_g e^{\lambda/2}=K_2=\text{const}, \tag{25}$$

则

$$K_1/K_2=\cot\alpha, \tag{26}$$

$$\frac{Q^2+\Phi^2}{r^4}=\left(\frac{4\pi}{3}K_1\right)^2r^2+\left(\frac{4\pi}{3}K_2\right)^2r^2. \tag{27}$$

为了使结果简化,令

$$C_1=\left|\frac{4\pi}{3}K_1\right|,\quad C_2=\left|\frac{4\pi}{3}K_2\right|, \tag{28}$$

则(27)式可写为

$$\frac{Q^2+\Phi^2}{r^4}=(C_1^2+C_2^2)r^2. \tag{29}$$

将(29)式代入(24)式,解得

$$e^{-\lambda}=1+ar^2+(C_1^2+C_2^2)r^4, \tag{30}$$

式中 $a$ 为积分常量.将(29)和(30)式代入(21a)式得

$$8\pi\rho=-3a-6(C_1^2+C_2^2)r^2. \tag{31}$$

作变换

$$e^{-\lambda}=z,\quad e^{\nu}=y^2,\quad r^2=x, \tag{32}$$

则方程(21)变为

$$8\pi\rho + (C_1^2 + C_2^2)x = \frac{1-z}{x} - 2\frac{dz}{dx}, \tag{33a}$$

$$8\pi P - (C_1^2 + C_2^2)x = \frac{4z}{y}\frac{dy}{dx} - \frac{1-z}{x}, \tag{33b}$$

$$8\pi P + (C_1^2 + C_2^2)x = \frac{4xz}{y}\frac{d^2y}{dx^2} + \frac{4z}{y}\frac{dy}{dx} + \frac{2x}{y}\frac{dz}{dx}\frac{dy}{dx} + \frac{dz}{dx}. \tag{33c}$$

把(30)式代入(33c),(33b)式,得

$$4[1 + ax + (C_1^2 + C_2^2)x^2]\left(\frac{d^2y}{dx^2}\right) + 2[a + 2(C_1^2 + C_2^2)]\frac{dy}{dx} - (C_1^2 + C_2^2)y = 0. \tag{34}$$

引入变量 $\eta$,使它与 $x$ 的关系满足

$$\frac{d\eta}{dx} = [1 + ax + (C_1^2 + C_2^2)x^2]^{-1/2}, \tag{35}$$

则(34)式变为

$$\frac{d^2y}{d\eta^2} - \frac{1}{4}(C_1^2 + C_2^2)y = 0. \tag{36}$$

解(36)式得

$$y = D_1\left[\exp\left(\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right) + D_2\exp\left(-\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right)\right], \tag{37}$$

$$e^{\nu} = y^2 = D_1^2\left[\exp\left(\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right) + D_2\exp\left(-\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right)\right]^2, \tag{38}$$

其中 $D_1$ 和 $D_2$ 是两个积分常量.把(30)和(32)式代入(33b)式,并利用(35)式得

$$8\pi P = a + 2(C_1^2 + C_2^2)x + 2\sqrt{C_1^2 + C_2^2}[1 + ax + (C_1^2 + C_2^2)x^2]^{1/2} \times \left[\frac{\exp\left(\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right) - D_2\exp\left(-\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right)}{\exp\left(\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right) + D_2\exp\left(-\frac{1}{2}\sqrt{C_1^2 + C_2^2}\,\eta\right)}\right]. \tag{39}$$

与既荷电又荷磁的 R-N 外解[5]比较,在 $r = r_0$ 处边界条件为

$$e^{\nu(r_0)} = e^{-\lambda(r_0)} = 1 - \frac{2M}{r_0} + \frac{Q_0^2 + \Phi_0^2}{r_0^2}, \tag{40a}$$

$$P(r_0) = 0. \tag{40b}$$

由(40a)式得

$$a = -2M/r_0^3, \tag{41}$$

$$D_1^2 = \frac{1 + ar_0^2 + (C_1^2 + C_2^2)r_0^4}{\exp\left[\sqrt{C_1^2 + C_2^2}\,\eta_{r_0}\right] + 2D_2 + D_2^2}. \tag{42}$$

由(40b)式可定出

$$D_2 = \frac{\left\{2\sqrt{C_1^2 + C_2^2}[1 + ar_0^2 + (C_1^2 + C_2^2)r_0^4]^{1/2} + [a + 2(C_1^2 + C_2^2)r_0^2]\right\}\exp\left(\sqrt{C_1^2 + C_2^2}\,\eta_{r_0}\right)}{2\sqrt{C_1^2 + C_2^2}[1 + ar_0^2 + (C_1^2 + C_2^2)r_0^4]^{1/2} - [a + 2(C_1^2 + C_2^2)r_0^2]}, \tag{43}$$

其中 $\eta_{r_0}$ 为 $\eta$ 在 $r=r_0$ 处的值.下面分两种情况求出 $\eta$ 的具体形式.

1)若 $4(C_1^2+C_2^2)-a^2=0$,(35)式变为

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=\frac{1}{1-\sqrt{C_1^2+C_2^2}\,x},\tag{44}$$

积分得

$$\eta=-\frac{1}{\sqrt{C_1^2+C_2^2}}\ln\left(1-\sqrt{C_1^2+C_2^2}\,r^2\right),\tag{45}$$

积分常量变为

$$D_2=0,\tag{46}$$

$$D_1^2=\left(1-\sqrt{C_1^2+C_2^2}\,r_0^2\right)^3.\tag{47}$$

对应的内解为

$$P=0,\tag{48a}$$

$$\rho=\frac{3}{4\pi}\sqrt{C_1^2+C_2^2}\left(1-\sqrt{C_1^2+C_2^2}\,r^2\right),\tag{48b}$$

$$\mathrm{e}^{-\lambda}=\left(1-\sqrt{C_1^2+C_2^2}\,r^2\right)^2,\tag{48c}$$

$$\mathrm{e}^{\nu}=\left(1-\sqrt{C_1^2+C_2^2}\,r_0^2\right)^3\Big/\left(1-\sqrt{C_1^2+C_2^2}\,r^2\right).\tag{48d}$$

显然,在这种情况下退化为荷电又荷磁尘埃球的内解.

2)若 $4(C_1^2+C_2^2)-a^2>0$,积分求出 $\eta$ 得

$$\eta=\frac{1}{\sqrt{C_1^2+C_2^2}}\ln\left[2(C_1^2+C_2^2)r^2+a+2\sqrt{C_1^2+C_2^2}\sqrt{1+ar^2+(C_1^2+C_2^2)r^4}\right].\tag{49}$$

将 $\eta$ 代入(38)式得内解

$$8\pi P=a+2(C_1^2+C_2^2)r^2+2\sqrt{C_1^2+C_2^2}[1+ar^2+(C_1^2+C_2^2)r^4]^{1/2}\times\left[\frac{2(C_1^2+C_2^2)r^2+a+2\sqrt{C_1^2+C_2^2}\sqrt{1+ar^2+(C_1^2+C_2^2)r^4}-D_2}{2(C_1^2+C_2^2)r^2+a+2\sqrt{C_1^2+C_2^2}\sqrt{1+ar^2+(C_1^2+C_2^2)r^4}+D_2}\right],\tag{50a}$$

$$8\pi\rho=-3a-6(C_1^2+C_2^2)r^2,\tag{50b}$$

$$\mathrm{e}^{-\lambda}=1+ar^2+(C_1^2+C_2^2)r^4,\tag{50c}$$

$$\mathrm{e}^{\nu}=D_1^2\Big\{2(C_1^2+C_2^2)r^2+a+2\sqrt{C_1^2+C_2^2}[1+ar^2+(C_1^2+C_2^2)r^4]^{1/2}+2D_2+D_2^2\left[2(C_1^2+C_2^2)r^2+a+2\sqrt{C_1^2+C_2^2}(1+ar^2+C_1^2r^4+C_2^2r^4)^{1/2}\right]^{-1}\Big\},\tag{50d}$$

其中

$$D_2=\frac{\left[2\sqrt{C_1^2+C_2^2}\sqrt{1+ar_0^2+(C_1^2+C_2^2)r_0^4}+a+2(C_1^2+C_2^2)r_0^2\right]^2}{2\sqrt{C_1^2+C_2^2}\sqrt{1+ar_0^2+(C_1^2+C_2^2)r_0^4}-a-2(C_1^2+C_2^2)r_0^2},\tag{51}$$

$$D_1^2=\frac{1+ar_0^2+(C_1^2+C_2^2)r_0^4}{D_2^2+2D_2+2(C_1^2+C_2^2)r_0^2+a+2\sqrt{C_1^2+C_2^2}\sqrt{1+ar_0^2+(C_1^2+C_2^2)r_0^4}}.\tag{52}$$

## 4 结　论

对应场源荷电又荷磁，且满足条件 $\rho_e e^{\lambda/2} = K_1$，$\rho_g e^{\lambda/2} = K_2$ 的静态球对称天体的共形平直内解，可以严格解出，其结果具有物理意义.


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# A CONFORMALLY FLAT SOLUTION OF EINSTEIN-MAXWELL EQUATIONS FOR A STATIONARY SPHERICALLY SYMMETRIC DISTRIBUTION OF SPHERE POSSESSING BOTH ELECTRIC AND MAGNETIC CHARGES


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(Received 10 April 1997)


ABSTRACT

This paper gives a conformally flat interior solution of Einstein-Maxwell equations for a stationary spherically symmetric distribution of sphere possessing both electric and magnetic charges. The solution is physically reasonable.


PACC: 0420