第 51 卷 第 6 期 2002 年 6 月
1000-3290/2002/51(06)/1156-03
物 理 学 报
ACTA PHYSICA SINICA
Vol.51,No.6,June,2002
©2002 Chin. Phys. Soc.

Poincaré-Chetaev 方程的 Noether 对称性*

葛伟宽
(湖州师范学院物理系,湖州 313000)

(2001 年 12 月 17 日收到;2002 年 1 月 10 日收到修改稿)

建立 d'Alembert-Lagrange 原理的 Poincaré-Chetaev 形式,给出原理在无限小变换下的变形形式,由此得到广义 Noether 等式以及守恒量的形式.举例说明结果的应用.

关键词:Poincaré-Chetaev 方程,无限小变换,广义 Noether 等式,守恒量
PACC:0320

1. 引 言

1901 年 Poincaré 利用无限小变换的 Lie 可迁群建立了完整力学系统一类新型运动微分方程[1].上世纪 20 年代 Chetaev 将此思想发展到变换群为非可迁的、约束是非定常的、变量是不独立的情形[2].他们建立的方程称为 Poincaré-Chetaev 方程.1994 年 Rumyatsev 指出,Poincaré-Chetaev 方程对 Hamilton 系统的近代理论有重要意义[3].文献[4]研究了 Poincaré-Chetaev 方程的代数结构和 Poisson 理论,文献[5]研究了这类方程的 Lie 对称性.

自 Noether 1918 年发现对称性与守恒量之间的潜在关系以来,Noether 对称性研究已取得重要进展[6—12].本文研究 Poincaré-Chetaev 方程的 Noether 对称性.首先,建立 Poincaré-Chetaev 形式的 d'Alembert-Lagrange 原理;其次,研究这个原理在群的无限小变换下的变形形式,利用它导出广义 Noether 等式并给出守恒量的形式;最后,举例说明结果的应用.

2. d'Alembert-Lagrange 原理的 Poincaré-Chetaev 形式

假设完整力学系统有 n 个自由度,其在空间位置由 m 个变量 x_i $(i=1,\cdots,m\geqslant n)$来确定.如果 $m=n$,那么,x_i 是独立的广义坐标;如果 $m>n$,那么,x_i 是系统不独立坐标或多余坐标.

假设用某种方法引入加在微分不可积分约束组上的参数化,广义速度有形式

$$\dot{x}_i=\zeta_i^s(t,x)\eta_s+\zeta_i(t,x),\quad \operatorname{rank}(\zeta_i^s)=n \tag{1}$$

存在无限小线性算子的封闭组

$$X_0=\frac{\partial}{\partial t}+\zeta_i\frac{\partial}{\partial x_i},$$
$$X_s=\zeta_i^s\frac{\partial}{\partial x_i}\qquad (s=1,\cdots,n), \tag{2}$$

其对易子为

$$[X_s,X_k]f=X_sX_kf-X_kX_sf$$
$$=C_{sk}^rX_rf\qquad (s,k,r=0,1,\cdots,n). \tag{3}$$

d'Alembert-Lagrange 原理可表为

$$\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}_i}+\frac{\partial L}{\partial x_i}+Q_i\right)\delta x_i=0$$
$$(i=1,\cdots,m). \tag{4}$$

由(1)式得虚位移方程

$$\delta x_i=\zeta_i^s(t,x)\delta\pi_s. \tag{5}$$

将(5)式代入(4)式,得

$$\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}_i}\zeta_i^s+\frac{\partial L}{\partial x_i}\zeta_i^s+Q_i\zeta_i^s\right)\delta\pi_s=0, \tag{6}$$

其中 $\delta\pi_s$ 为准坐标的变分.

令

$$\tilde{L}(t,x_i,\eta_s)=L(t,x_i,\zeta_i^s\eta_s+\zeta_i), \tag{7}$$

则原理(4)可表为

$$\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\eta_s}+C_{rs}^k\eta_r\frac{\partial\tilde{L}}{\partial\eta_k}\right.$$
$$\left.+C_{0s}^k\frac{\partial\tilde{L}}{\partial\eta_k}+X_s\tilde{L}+\tilde{Q}_s\right)\delta\pi_s=0, \tag{8}$$

* 国家自然科学基金(批准号:19972010)资助的课题.

其中

$$\tilde{Q}_s = Q_i \zeta_i^s. \tag{9}$$

由原理(8)中 $\delta\pi_s$ 的独立性,得到 Poincaré-Chetaev 方程[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s} = C_{rs}^k \eta_r \frac{\partial \tilde{L}}{\partial \eta_k} + C_{0s}^k \frac{\partial \tilde{L}}{\partial \eta_s} + X_s \tilde{L} + \tilde{Q}_s \quad (s = 1,\cdots,n). \tag{10}$$

称原理(8)为 d'Alembert-Lagrange 原理的 Poincaré-Chetaev 形式.

3. 原理在无限小变换下的变形形式

引进时间和准坐标下的无限小变换

$$t^* = t + \Delta t, \pi_s^*(t^*) = \pi_s(t) + \Delta\pi_s \quad (s = 1,\cdots,n), \tag{11}$$

或其展开式

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{x}, \eta),$$

$$\pi_s^*(t^*) = \pi_s(t)_s + \varepsilon\xi_s(t, \boldsymbol{x}, \eta), \tag{12}$$

其中 ε 为无限小参数,ξ_0, ξ_s 为无限小变换生成函数.注意,这里 π_s, π_s^* 只是一种记号,而 $\Delta\pi_s$ 有意义.将

$$\delta\pi_s = \Delta\pi_s - \eta_s \Delta t = \varepsilon(\xi_s - \eta_s \xi_0) \tag{13}$$

代入原理(8),得

$$\varepsilon\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s} + C_{rs}^k \eta_r \frac{\partial \tilde{L}}{\partial \eta_k} + C_{0s}^k \frac{\partial \tilde{L}}{\partial \eta_k} + X_s \tilde{L} + \tilde{Q}_s\Big)(\xi_s - \eta_s \xi_0) = 0.$$

上式可改写为

$$\varepsilon\Big\{\Big(\frac{\partial \tilde{L}}{\partial t} + \frac{\partial \tilde{L}}{\partial x_i}\dot{x}_i\Big)\xi_0 + \Big(\tilde{L} - \frac{\partial \tilde{L}}{\partial \eta_s}\eta_s\Big)\dot{\xi}_0 + \frac{\partial \tilde{L}}{\partial \eta_s}\dot{\xi}_s + \Big(C_{rs}^k \eta_r \frac{\partial \tilde{L}}{\partial \eta_k} + C_{0s}^k \frac{\partial \tilde{L}}{\partial \eta_k} + X_s \tilde{L} + \tilde{Q}_s\Big)(\xi_s - \eta_s \xi_0) + \dot{G} - \frac{\mathrm{d}}{\mathrm{d}t}\Big[\tilde{L}\xi_0 + \frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s - \eta_s\xi_0) + G\Big]\Big\} = 0, \tag{14}$$

(14)式称为 d'Alembert-Lagrange 原理的 Poincaré-Chetaev 形式在无限小变换下的变形形式.

4. 广义 Noether 定理

由变形形式(14)式容易得到下述广义 Noether 定理.

定理　对于无限小变换(12),如果存在规范函数 $G = G(t, \boldsymbol{x}, \eta)$ 满足下述广义 Noether 等式

$$\Big(\frac{\partial \tilde{L}}{\partial t} + \frac{\partial \tilde{L}}{\partial x_i}\dot{x}_i\Big)\xi_0 + \Big(\tilde{L} - \frac{\partial \tilde{L}}{\partial \eta_s}\eta_s\Big)\dot{\xi}_0 + \frac{\partial \tilde{L}}{\partial \eta_s}\dot{\xi}_s + \Big(C_{rs}^k \eta_r \frac{\partial \tilde{L}}{\partial \eta_k} + C_{0s}^k \frac{\partial \tilde{L}}{\partial \eta_k} + X_s \tilde{L} + \tilde{Q}_s\Big)(\xi_s - \eta_s \xi_0) + \dot{G} = 0, \tag{15}$$

则 Poincaré-Chetaev 方程存在如下形式守恒量

$$I = \tilde{L}\xi_0 + \frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s - \eta_s \xi_0) + G = \mathrm{const}. \tag{16}$$

5. 算　例

研究重刚体绕固定点转动问题的 Noether 对称性.

刚体的动能为

$$T = \frac{1}{2}(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2),$$

势能为

$$V = Mg(\overset{1}{\chi_0}\gamma_1 + \overset{2}{\chi_0}\gamma_2 + \overset{3}{\chi_0}\gamma_3),$$

其中 $\omega_1, \omega_2, \omega_3$ 为刚体角速度在与刚体相固联的惯性主轴上的投影;A_1, A_2, A_3 为主惯性矩;$\overset{1}{\chi_0}, \overset{2}{\chi_0}, \overset{3}{\chi_0}$ 为质心在固联轴中的坐标;$\gamma_1, \gamma_2, \gamma_3$ 为铅垂线与惯性主轴夹角的余弦.令

$$\chi_i = \gamma_i, \eta_s = \omega_s \quad (i, s = 1,2,3),$$

而 r_i 满足 Poisson 方程

$$\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1,$$

$$\dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2,$$

即

$$\dot{\chi}_1 = \eta_3\chi_2 - \eta_2\chi_3, \dot{\chi}_2 = \eta_1\chi_3 - \eta_3\chi_1,$$

$$\dot{\chi}_3 = \eta_2\chi_1 - \eta_1\chi_2,$$

可以计算得

$$\zeta_1^2 = -\chi_3, \zeta_1^3 = \chi_2, \zeta_2^1 = \chi_3,$$

$$\zeta_2^3 = -\chi_1, \zeta_3^1 = -\chi_2, \zeta_3^2 = \chi_1,$$

$$\zeta_i^i = \zeta_i = 0 \quad (i = 1,2,3),$$

以及

$$C_{12}^3 = C_{23}^1 = C_{31}^2 = 1, C_{21}^3 = C_{32}^1 = C_{13}^2 = -1,$$

$$C_{\alpha s}^k = 0, \tilde{Q}_s = 0.$$

对此问题,广义 Noether 等式(15)有如下解,

$$\xi_0 = 1, \xi_s = 0 (s = 1,2,3), G = 0.$$

此时守恒量(16)式给出

$$I = \tilde{L} - \frac{\partial \tilde{L}}{\partial \eta_s}\eta_s = \mathrm{const}.$$

它代表能量积分.

下面假设

$$A_1 = A_2, \chi_0^1 = \chi_0^2 = 0,$$

此时等式(15)有解

$$\xi_3 = 1, \xi_0 = \xi_1 = \xi_2 = 0, G = 0.$$

而守恒量(16)式给出

$$I = \frac{\partial \tilde{L}}{\partial \eta_3} = A_3 \eta_3 = \text{const}.$$

这是经典 Lagrange 情形的积分,代表对对称主轴的动量矩守恒.

[1] Poincaré H 1901 *CR Acad Paris* **132** 369

[2] Chetaev N 1927 *CR Acad Paris* **185** 1577

[3] Rumyatsev V V 1994 *P M M* **58** 3

[4] Mei F X 1999 *Application of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) p20 (in Chinese) [梅凤翔 1999 李群和李代数对约束力学系统的应用(北京:科学出版社)第 20 页]

[5] Zhao S X and Mei F X 2000 *J Beijing Institute of Technology* **20** 265 (in Chinese) [赵树信、梅凤翔 2000 北京理工大学学报 **20** 265]

[6] Djukić Dj S and Vujanović B, 1975 *Acta Mechanica* **23** 17

[7] Li Z P 1981 *Acta Phys. Sin.* **30** 1659 (in Chinese) [李子平 1981 物理学报 **30** 1659]

[8] Li Z P 1984 *Acta Phys. Sin.* **33** 814 (in Chinese) [李子平 1984 物理学报 **33** 814]

[9] Bahar L Y and Kwatny H G 1987 *Int J. Non-Linear Mech.* **22** 125

[10] Liu D 1991 *Science in China* (Ser.A) **34** 419

[11] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) p1 (in Chinese) [赵跃宇、梅凤翔 1999 力学系统的对称性与不变量(北京:科学出版社)第 1 页]

[12] Mei F X 2000 *ASME Appl. Mech. Rev.* **53** 283

Noether symmetry of Poincaré-Chetaev equations*

Ge Wei-Kuan

(*Department of Physics*, *Huzhou Teachers College*, *Huzhou* 313000, *China*)

(Received 17 December 2001; revised manuscript received 10 January 2002)

Abstract

The Poincaré-Chetaev form of the d'Alembert-Lagrange principle is established. The varied form under the infinitesimal transformations is given and the generalized Noether identity and the form of conserved quantity are obtained. An example is given to illustrate the application of the result.

Keywords: Poincaré-Chetaev equations, infinitesimal transformation, generalized Noether identity, conserved quantity

PACC: 0320

* Project supported by the National Natural Science Foundation of China (Grant No.19972010).