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经典单色短峰水波理论的三阶完备解*

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针对经典的有限水深三阶单色短峰波,考虑环境均匀流效应,赋予一新的解析解,从而与原经典解析解构成完备的解析解.这就在第二阶解、第三阶解结构上显示出多种相互作用机制.

关键词: 单色短峰波, 完备解, 均匀流, 有限水深

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1. 引 言

自从 Hsu 等[1]提出经典的有限水深三阶单色短峰波理论以来,随之便伴随着一批从不同侧面开展的短峰波工作[2—10].这些工作或者着眼于高阶深水[3—5]、有限水深[6]短峰波的数值解;或者考虑表面张力波效应[7],并且纳入环境均匀流的一维[8]、二维[9]效应;或者由短峰波的单色扩展到双色[10];或者直接上溯到该经典短峰波理论[1]的基础"源头",发现了其解与其必须满足的振幅、相位方程不相匹配的症结所在,并加以克服[11].进一步回看该经典短峰波的基础理论,可发现其最基本的第一阶解并不唯一或不完备,尚存在与它处于同一地位的另一基本解.它们可并为一处,共同构成该经典短峰波系统的一般解[12]或完备解.本文以此为动念,又将环境均匀流效应包含其中,从而得到一个更为实际短峰波系统的完备解析解.

2. 短峰波系统的构造

在 Hsu 等[1]建立的单色短峰波运动框架体系下,引入沿着 x 轴的一维均匀流 U,可得一总速度势

$$\Phi(x,y,z,t) = Ux + \phi(x,y,z,t).$$

首先将 U 无量纲化,

$$\tilde{U} = \frac{U}{\tilde{\omega}}\sqrt{\frac{k}{g}},$$

其中 k,g 和 $\tilde{\omega}$ 分别为特征波数、重力加速度和无量纲角频率.然后再连同已有的若干无量纲量[1],可得到下列新型无量纲化控制方程组(省略无量纲符号):

$$\nabla^2\phi = 0 \qquad (-h \leqslant z \leqslant \varepsilon\zeta), \tag{1}$$

$$\zeta + \omega\phi_t + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_y^2 + \phi_z^2) + \omega U\phi_x = 0 \qquad (z = \varepsilon\zeta), \tag{2}$$

$$\phi_z - \omega\zeta_t - \varepsilon(\phi_x\zeta_x + \phi_y\zeta_y) - \omega U\zeta_x = 0 \qquad (z = \varepsilon\zeta), \tag{3}$$

$$\phi_z = 0 \qquad (z = -h), \tag{4}$$

$$\phi_y = 0 \qquad (y = 0). \tag{5}$$

这里 ζ 和 h 分别表示自由表面位移和常水深;小参数 ε 代表波陡,$\varepsilon = ka$,其中 a 为短峰波的第一阶振幅.

为确定一般解的某些系数,需对原有的相位方程、振幅方程和唯一性条件[1]依次作出如下扩展:

$$\int_0^{\pi}\int_0^{2\pi}\zeta(y,t)(a\sin t + b\cos t)\cos y\,\mathrm{d}t\,\mathrm{d}y = 0, \tag{6}$$

$$\int_{-h}^{0}\int_0^{\pi}\int_0^{2\pi}\phi(y,z,t)(a\sin t + b\cos t)\cos y\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z = -\frac{1}{2}\pi^2(\tanh h)^{1/2}(a^2 + b^2), \tag{7}$$

$$\frac{[jmU(\tanh h)^{1/2} \pm (1 - mU)(j\tanh jh)^{1/2}]^2}{\tanh h} \neq l^2 \qquad (j = 2,3,\cdots; l = 1,2,\cdots), \tag{8}$$

其中 a 和 b 为第一阶无量纲波幅.

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同样将该短峰波系统的未知量 ϕ,ζ,ω 作摄动展开,并将自由表面上的运动学、动力学边界条件在平均水位 $z=0$ 处作 Taylor 级数展开.据此,可得到前三阶控制方程组($i=1,2,3$),

$$\nabla^2\phi_i=0 \qquad (-h\leqslant z\leqslant 0), \tag{9}$$

$$\zeta_i+\omega_1(\phi_{it}+U\phi_{ix})=E_i \qquad (z=0), \tag{10}$$

$$\phi_{iz}-\omega_1(\zeta_{it}+U\zeta_{ix})=F_i \qquad (z=0), \tag{11}$$

$$\phi_{iz}=0 \qquad (z=-h), \tag{12}$$

$$\phi_{iy}=0 \qquad (y=0). \tag{13}$$

这里

$$E_1=F_1=0, \tag{14}$$

$$\begin{aligned}E_2={}&-\omega_2\phi_{1t}-\omega_1\phi_{1tz}\zeta_1\\&-\frac{1}{2}[(\phi_{1x})^2+(\phi_{1y})^2+(\phi_{1z})^2]\\&-U(\omega_1\zeta_1\phi_{1xz}+\omega_2\phi_{1x}),\end{aligned} \tag{15a}$$

$$\begin{aligned}F_2={}&\omega_2\zeta_{1t}+\zeta_{1x}\phi_{1x}+\zeta_{1y}\phi_{1y}-\zeta_1\phi_{1zz}\\&+U\omega_2\zeta_{1x},\end{aligned} \tag{15b}$$

$$\begin{aligned}E_3={}&-\omega_3\phi_{1t}-2\zeta_1(\phi_{1x}\phi_{1xz}+\phi_{1y}\phi_{1yz}+\phi_{1z}\phi_{1zz})\\&-2(\phi_{1x}\phi_{2x}+\phi_{1y}\phi_{2y}+\phi_{1z}\phi_{2z})\\&-U\omega_3\phi_{1x}-\omega_1(2\zeta_2\phi_{1zt}+2\zeta_1\phi_{2zt}+\zeta_1^2\phi_{1zzt})\\&-2\omega_2(\phi_{1zt}\zeta_1+\phi_{2t})\\&-U\omega_1(2\zeta_2\phi_{1xz}+\zeta_1^2\phi_{1xzz}+2\zeta_1\phi_{2xz})\\&-2U\omega_2(\zeta_1\phi_{1xz}+\phi_{2x}),\end{aligned} \tag{16a}$$

$$\begin{aligned}F_3={}&\omega_3\zeta_{1t}-2\left(\phi_{1zz}\zeta_2+\phi_{2zz}\zeta_1+\frac{1}{2}\phi_{1zzz}\zeta_1^2\right)\\&+2\zeta_1(\phi_{1xz}\zeta_{1x}+\phi_{1yz}\zeta_{1y})\\&+2(\phi_{2x}\zeta_{1x}+\phi_{2y}\zeta_{1y}+\phi_{1x}\zeta_{2x}+\phi_{1y}\zeta_{2y})\\&+2\omega_2U\zeta_{2x}+U\omega_3\zeta_{1x}.\end{aligned} \tag{16b}$$

3. 完 备 解

求解方程组(9)—(13),可依次得到短峰波系统的前三阶完备解.

3.1. 第一阶解

短峰波系统的第一阶解为

$$\zeta_1=\cos Y(a\cos X+b\sin X), \tag{17}$$

$$\phi_1=\omega_1(1-mU)\frac{\cosh Z}{\sinh h}\cos Y(a\sin X-b\cos X), \tag{18}$$

$$\tanh h=\omega_1^2(1-mU)^2\equiv\tau^2. \tag{19}$$

这里

$$\begin{aligned}X&=mx-t,\\Y&=ny,\\Z&=z+h,\\m&=\sin\theta,\\n&=\cos\theta,\end{aligned}$$

其中 θ 为入射波或反射波与 y 轴之间的夹角. 显见,以 a 为标志的原经典解($a=1$)[1]和以 b 为标志的新解共同以线性叠加形式构成了完备第一阶解,它们自身或相互间并没有发生相互作用.但是在后续的高阶解中,由于非线性作用将显示出多种相互作用.

3.2. 第二阶解

短峰波系统的第二阶解为

$$\begin{aligned}\zeta_2={}&(\alpha_{22}^2\cos 2X+\nu_{22}^2\sin 2X)\cos 2Y\\&+\alpha_{20}^2\cos 2X+\nu_{20}^2\sin 2X+\alpha_{02}^2\cos 2Y,\end{aligned} \tag{20}$$

$$\begin{aligned}\phi_2={}&\mu_0^2t+\beta_0^2+(\beta_{22}^2\sin 2X\\&+\mu_{22}^2\cos 2X)\cos 2Y\cosh 2Z\\&+(\beta_{20}^2\sin 2X+\mu_{20}^2\cos 2X)\cosh 2mZ,\end{aligned} \tag{21}$$

$$\omega_2=0. \tag{22}$$

这里,β_0^2 为一常数,

$$\mu_0^2=\frac{1}{8}(a^2+b^2)(1-mU)(\tau-\tau^{-3}), \tag{23}$$

$$\beta_{20}^2=\frac{(a^2-b^2)K_2}{16\cosh 2mh}=\frac{(a^2-b^2)(1+\omega_m^4)K_1}{16\cosh 2mh}, \tag{24a}$$

$$\mu_{20}^2=\frac{-abK_2}{8\cosh 2mh}, \tag{24b}$$

$$\beta_{22}^2=\frac{3(a^2-b^2)(1-\tau^8)}{16\tau^7\cosh 2h}, \tag{25a}$$

$$\mu_{22}^2=\frac{-3ab(1-\tau^8)}{8\tau^7\cosh 2h}, \tag{25b}$$

$$\alpha_{22}^2=\frac{(a^2-b^2)(3-\tau^4)}{8\tau^6}, \tag{26a}$$

$$\nu_{22}^2=\frac{ab(3-\tau^4)}{4\tau^6}, \tag{26b}$$

$$\alpha_{20}^2=\frac{(a^2-b^2)[3\tau^2-(m^2-n^2)\tau^{-2}+\tau K_2]}{8}, \tag{27a}$$

$$\nu_{20}^2=\frac{ab[3\tau^2-(m^2-n^2)\tau^{-2}+\tau K_2]}{4}, \tag{27b}$$

$$\alpha_{02}^2=\frac{(a^2+b^2)(n^2-m^2+\tau^4)}{8\tau^2}, \tag{28}$$

其中

$$K_1 = \frac{K_2}{(1+\omega_m^4)},$$

$$K_2 = \frac{(1+2m^2-2n^2-3\tau^4)(1+\omega_m^4)}{\tau^3(1-m\omega_m^2\tau^{-2}+\omega_m^4)}, \quad (29)$$

$$\omega_m^2 = \tanh mh.$$

当(20)—(22)式中的系数 μ_{ij}^2, ν_{ij}^2 和 U 均为零时，此第二阶完备解即化为经典的第二阶特解.

若从量纲角度看，(23)—(28)式表征无量纲长度 a 或 b 的平方. 若从非线性相互作用角度看，便可发现以下两点：(1)涉及 μ,β 和 α 的系数，含有作用因子 $a^2 \pm b^2$. 这便首先表现出 a 或 b 的自身平方相互作用，然后它们发生彼此和差相互作用，有些类似于超谐波、亚谐波相互作用，或双色波的和频、差频相互作用. (2)涉及 μ 和 ν 的系数，含有作用因子 ab. 这便表现出 a 和 b 不可分割的彼此相互作用. 此第二阶解之完备性，当由这些本质性的作用因子来刻画.

3.3. 第三阶解

短峰波系统的第三阶解为

$$\begin{aligned}\zeta_3 =& (\alpha_{11}^3\cos X + \nu_{11}^3\sin X)\cos Y \\ &+ (\alpha_{13}^3\cos X + \nu_{13}^3\sin X)\cos 3Y \\ &+ (\alpha_{31}^3\cos 3X + \nu_{31}^3\sin 3X)\cos Y \\ &+ (\alpha_{33}^3\cos 3X + \nu_{33}^3\sin 3X)\cos 3Y,\end{aligned} \quad (30)$$

$$\begin{aligned}\phi_3 =& \mu_0^3 t + \beta_0^3 + (\beta_{13}^3\sin X + \mu_{13}^3\cos X) \\ &\times \cos 3Y\cosh\gamma_{13}Z + (\beta_{31}^3\sin 3X \\ &+ \mu_{31}^3\cos 3X)\cos Y\cosh\gamma_{31}Z \\ &+ (\beta_{33}^3\sin 3X + \mu_{33}^3\cos 3X)\cos 3Y\cosh 3Z,\end{aligned} \quad (31)$$

$$\begin{aligned}\omega_3 =& (a^2+b^2)(1-mU)^{-1}\Big\{\frac{9}{32}\tau^{-7} \\ &- \frac{1}{32}[10+4(m^2-n^2)^2]\tau^{-3} \\ &+ \frac{1}{32}(38m^2-10n^2-1)\tau - \frac{1}{8}m\tau^2\omega_m^2K_1 \\ &- \frac{1}{16}(\tau^4-4m^2+1)K_2 - \frac{1}{4}\tau^5\Big\}.\end{aligned} \quad (32)$$

这里，β_0^3 为一常数，$\gamma_{13} = \sqrt{m^2+9n^2}, \gamma_{31} = \sqrt{9m^2+n^2}$,

$$\begin{aligned}\beta_{13}^3 =& a(a^2+b^2)\{16\cosh\gamma_{13}h[\gamma_{13}\tanh\gamma_{13}h - \tau^2]\}^{-1} \\ &\times[2\tau^5 - (5+8m^2)\tau + 4(3-n^2-2n^4)\tau^{-3} \\ &- (9m^2-3n^2)\tau^{-7}],\end{aligned} \quad (33a)$$

$$\begin{aligned}\mu_{13}^3 =& -b(a^2+b^2)\{16\cosh\gamma_{13}h[\gamma_{13}\tanh\gamma_{13}h \\ &- \tau^2]\}^{-1}[2\tau^5 - (5+8m^2)\tau + 4(3-n^2 \\ &- 2n^4)\tau^{-3} - (9m^2-3n^2)\tau^{-7}],\end{aligned} \quad (33b)$$

$$\begin{aligned}\beta_{31}^3 =& a(a^2-3b^2)\{16\cosh\gamma_{31}h[\gamma_{31}\tanh\gamma_{31}h \\ &- 9\tau^2]\}^{-1}[(9-36m^2)\tau^{-7} + 36m\tau^2\omega_m^2K_1 \\ &+ (60+4m^2+8m^4)\tau^{-3} + 2(3\tau^4-8m^2 \\ &- 1)K_2 - (32m^2+31)\tau + 18\tau^5],\end{aligned} \quad (34a)$$

$$\begin{aligned}\mu_{31}^3 =& b(b^2-3a^2)\{16\cosh\gamma_{31}h[\gamma_{31}\tanh\gamma_{31}h \\ &- 9\tau^2]\}^{-1}[(9-36m^2)\tau^{-7} + 36m\tau^2\omega_m^2K_1 \\ &+ (60+4m^2+8m^4)\tau^{-3} + 2(3\tau^4-8m^2 \\ &- 1)K_2 - (32m^2+31)\tau + 18\tau^5],\end{aligned} \quad (34b)$$

$$\begin{aligned}\beta_{33}^3 =& a(a^2-3b^2)(128\cosh 3h)^{-1}(1+3\tau^4) \\ &\times[9\tau^{-13} - 22\tau^{-9} + 13\tau^{-5}],\end{aligned} \quad (35a)$$

$$\begin{aligned}\mu_{33}^3 =& b(b^2-3a^2)(128\cosh 3h)^{-1}(1+3\tau^4) \\ &\times[9\tau^{-13} - 22\tau^{-9} + 13\tau^{-5}],\end{aligned} \quad (35b)$$

$$\begin{aligned}\alpha_{11}^3 =& \frac{1}{32}a(a^2+b^2)\{3\tau^{-8} + [8-4(m^2 \\ &- n^2)^2]\tau^{-4} + 8\tau^4 - 9 + 4m\tau\omega_m^2K_2 \\ &+ 2[\tau^3 + (2m^2-1)\tau^{-1}]\},\end{aligned} \quad (36a)$$

$$\begin{aligned}\nu_{11}^3 =& \frac{1}{32}b(a^2+b^2)\{3\tau^{-8} + [8-4(m^2 \\ &- n^2)^2]\tau^{-4} + 8\tau^4 - 9 + 4m\tau\omega_m^2K_2 \\ &+ 2[\tau^3 + (2m^2-1)\tau^{-1}]\},\end{aligned} \quad (36b)$$

$$\begin{aligned}\alpha_{13}^3 =& a(a^2+b^2)\{16[\gamma_{13}\tanh\gamma_{13}h - \tau^2]\}^{-1} \\ &\times[2\tau^6 - (5+8m^2)\tau^2 + 4(3-n^2-2n^4)\tau^{-2} \\ &- (9m^2-3n^2)\tau^{-6}] + \frac{a(a^2+b^2)}{16} \\ &\times[3(n^2-m^2)\tau^{-8} + 9\tau^{-4} \\ &+ (n^2-5m^2-6) + 2\tau^4],\end{aligned} \quad (37a)$$

$$\begin{aligned}\nu_{13}^3 =& \frac{b(a^2+b^2)}{16}\{16[\gamma_{13}\tanh\gamma_{13}h - \tau^2]\}^{-1} \\ &\times[2\tau^6 - (5+8m^2)\tau^2 + 4(3-n^2-2n^4)\tau^{-2} \\ &- (9m^2-3n^2)\tau^{-6}] + \frac{a(a^2+b^2)}{16} \\ &\times[3(n^2-m^2)\tau^{-8} + 9\tau^{-4} \\ &+ (n^2-5m^2-6) + 2\tau^4],\end{aligned} \quad (37b)$$

$$\begin{aligned}\alpha_{31}^3 =& \frac{a(a^2-3b^2)}{16}[3n^2\tau^{-8} + 21\tau^{-4} - 10 + n^2 \\ &+ 6\tau^4 + 12m\tau\omega_m^2K_1 + 2(\tau^3 - m^2\tau^{-1})K_2] \\ &+ \{16[\gamma_{31}\tanh\gamma_{31}h - 9\tau^2]\}^{-1}3a(a^2-3b^2) \\ &\times\{(9-36m^2)\tau^{-6} + (60+4m^2+8m^4)\tau^{-2}\end{aligned}$$

$$+36m\tau^3\omega_m^2K_1+2[3\tau^5-(8m^2+1)\tau]K_2$$

$$-(32m^2+31)\tau^2+18\tau^6\},\tag{38a}$$

$$\nu_{31}^3=\frac{b(3a^2-b^2)}{16}[3n^2\tau^{-8}+21\tau^{-4}-10+n^2$$

$$+6\tau^4+12m\tau\omega_m^2K_1+2(\tau^3-m^2\tau^{-1})K_2]$$

$$+\{16[\gamma_{31}\tanh\gamma_{31}h-9\tau^2]\}^{-1}3a(a^2-3b^2)$$

$$\times\{(9-36m^2)\tau^{-6}+(60+4m^2+8m^4)\tau^{-2}$$

$$+36m\tau^3\omega_m^2K_1+2[3\tau^5-(8m^2+1)\tau]K_2$$

$$-(32m^2+31)\tau^2+18\tau^6\},\tag{38b}$$

$$\alpha_{33}^3=a(a^2-3b^2)[16(\tanh3h-3\tau^2)]^{-1}$$

$$\times(-27\tau^{-6}+66\tau^{-2}-39\tau^2)$$

$$+\frac{1}{16}a(a^2-3b^2)(-3\tau^{-8}+21\tau^{-4}-15),\tag{39a}$$

$$\nu_{33}^3=b(3a^2-b^2)[16(\tanh3h-3\tau^2)]^{-1}$$

$$\times(-27\tau^{-6}+66\tau^{-2}-39\tau^2)$$

$$+\frac{1}{16}a(a^2-3b^2)(-3\tau^{-8}+21\tau^{-4}-15).\tag{39b}$$

同样,当(30)—(32)式中的系数 μ_{ij}^3,ν_{ij}^3 和 U 均为零时,此第三阶完备解即化为相应的经典之特解.

从量纲角度看,(33)—(39)式表征无量纲长度 a 或 b 的三次方. 若再从非线性相互作用角度看,则该第三阶完备解要比上述的第二阶完备解表现得更为丰富多样,这自然是由多种作用因子来体现. 显然,这些丰富多样的作用因子,也就预示着谐波相互作用和多色波频率相互作用的丰富多样. 若在第一阶解中不纳入新解 b 的作用,则这些丰富多样性便无从谈起.

至此,已给出该扩展短峰波系统的前三阶完备解析解,大大扩充了经典短峰波系统之解的内涵和形式. 依据该完备解析解,就可得出一系列运动学、动力学的物理变量(波高、波陡、相速度、波压力等)的表达式,进而可望应用于近海、海洋工程实践.

4. 结 论

对于一个比较真实的物理模型系统,其解析解集合应该是完备的,可含一个或数个解,且各具要义. 通过考察和分析,本文给出经典短峰波系统的完备解,并且添加了普遍的环境流效应. 这将触动、影响到自该经典短峰波系统建立以来而随之产生的与其密切关联的众多短峰波的工作,进而须对它们进行修正和改造,使之更加精确和完善.

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A complete third-order solution for classical monochromatic short-crested water waves*

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Abstract

A new solution for classical monochromatic short-crested waves in finite water depth is presented by including the ambient uniform currents. With the aid of the original classical solution, we obtain a complete solution and show a variety of interaction mechanisms for the second-and third-order solution structures.

Keywords: monochromatic short-crested waves, complete solution, uniform current, finite water depth

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