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論彈性體力學與受範性體力學中的一般變分原理*

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一. 引 言

在彈性體力學和受範性體力學中，關於能量的變分原理常常作為推導方程的手段和作為近似解法的根據. 這些原理大致可分成三類. 第一類從位移(或速度)出發，證明在所有適合位移(或速度)邊界條件的位移(或速度)中，真正的位移(或速度)使某一泛函達最小值. 這類原理在本質上和最小勢能原理相同或相似，因此可以總稱為最小勢能原理式的變分原理. 過去研究過這類原理的有: Locatelli[18] (1939), Качанов[2] (1942), Prager[22] (1942), Марков[7] (1947), Качанов[3] (1949), Greenberg[10] (1949), Phillips[20] (1949), 錢令希[1] (1950), Hill[14] (1950), Yamamota[32] (1952), 以及其他幾位學者.

第二類變分原理從應力出發，證明在所有適合平衡方程，應力邊界條件，在受範性體力學中則再加上適合屈服條件的應力中，真正的應力狀態使某一泛函達最小值. 這類原理可以說是線性彈性力學中的卡氏最小功原理的推廣，因此可以總稱為餘能原理. 過去研究過這類變分原理的學者有: Engesser[9] (1889), Haar and v. Kármán[11] (1909), Westergaad[31] (1941), Качанов[2] (1942), Prager[22] (1942), Sadowsky[28] (1943), Handelman[12] (1944), Лурье[6] (1946), Prager[23] (1946), Hill[14] (1948), Hodge and Prager[17] (1948), Philippidis[20] (1948), Reissner[25] (1948), Качанов[3] (1949), Greenberg[10] (1949), Phillips[21] (1949), 錢令希[1] (1950), Hill[14] (1950), Yamamota[32] (1952), 及其他幾位學者.

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在第三類變分原理,把位移和應力看作是彼此獨立無關的函數而讓它們同時獨立地變化. Reissner[26](1950) 曾這樣建立了一個變分原理, 證明這個變分原理相當於平衡方程、應變與應力的關係式以及邊界條件. 之後, de Veubeke[30] (1951) 補充了 Reissner 的結果.

在這三類的變分原理中, 前兩者的優點在於明確地指出某泛函是最小或最大. 對於作爲近似解法的根據來說,那末前兩者的缺點在於: (i) 事實上往往不容易適合必須適合的方程和邊界條件; (ii) 如果利用勢能原理式的變分原理以近似地求解位移或速度, 那末與這些位移或速度對應的應力便不適合平衡方程, 因此這樣求得的應力是否妥善便成了問題; 同樣,如果利用餘能原理式的變分原理以近似地求解應力,那末這樣求得的應力便不適合應變協調方程, 因而亦就沒有與它對應的連續位移或速度.

第三類變分原理的優點首先在於它在理論上是比較更一般性的,前兩類的變分原理都可包括在它的特殊形式中. 同時對於作爲近似解法的根據來說, 這類變分原理使人有更大的自由來選擇近似解答. 這類變分原理又使人能同時求得應力和位移,因而對於混合邊界值問題便特別有用.

本文的目的便在於在彈性體力學、彈性薄板的彎曲理論以及範性體力學中建立允許位移(或速度)和應力同時變化的一般變分原理. 許多前人的結果都可包括在我們的一般變分原理中.

本文第二部分建立了彈性體力學中的一般變分原理. 我們假定物體的應變是很微小的,但是它的彈性可以是任意的,即可不服從虎克定律. 事實上, 正如 Новожилов[8]所指出的, 這種彈性體還包括一系列在加荷載過程中有硬化作用的範性體. 我們首先把三個位移分量、六個應變分量和六個應力分量等十五個函數看作是彼此獨立無關的函數而建立了廣義餘能原理和廣義勢能原理. 接着我們又把三個位移分量和六個應力分量等九個函數看作是彼此獨立無關的函數而建立了廣義餘能原理和廣義勢能原理的別種表示法.

本文第四部分建立了範性體力學中的一般變分原理. 我們仍假定物體的應變很微小, 但是物體的屈服條件以及應變與應力的關係則是相當一般的. 我們將應變與應力的關係分爲兩類: 一類是變形式的關係, 另一類是流動式的關係. 在變形式的關係中,我們將位移和應力(其中應力適合屈服條件)同時變化而建立了廣義餘能原理和廣義勢能原理. 在流動式的關係中, 我們將速度和應力的變

化率(其中應力適合屈服條件)同時變化而建立了同樣的兩個變分原理.

本文第三部分建立了彈性薄板彎曲理論中的一般變分原理. 在這裏我們不但允許薄板的彈性可以是任意的，並且還允許產生大撓度. 我們將薄板內的三個內力矩,薄板平面內的三個內力以及三個位移等九個函數看作是彼此獨立無關的函數而建立了廣義勢能原理. 這一事實表明，推廣廣義勢能原理以適用於大應變問題是有可能的. 接着我們便利用這個原理以變分法導得薄板彎曲理論中的卡門方程. 在卡門方程中，一個待求函數是撓度，另一個待求函數是應力函數,兩者的性質不同. 因此無論用平常的勢能原理或餘能原理,都無法用變分法導得卡門方程. 只有我們的廣義勢能原理,允許位移與應力同時變化,才提供了這個可能性.

二. 彈性體力學中的一般變分原理

在本節中我們將建立彈性體力學的一般變分原理. 我們假定物體的變形很微小,但是物體的彈性可是任意的,卽可不服從虎克定律.

(一) 廣義餘能原理

今設有一彈性體,它的彈性可以是任意的，卽應力與應變之間的關係可不服從虎克定律. 設 U 是彈性體中單位體積的應變能. U 可以用應力分量來表示,亦可以用應變分量來表示. 今後我們規定 U 用應變分量 e_x, e_y, e_z, γ_{yz}, γ_{xz}, γ_{xy} 表示,卽 $U=U(e_x, e_y, e_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy})$.

命

$$L_U = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X\right)u + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y\right)v +$$

$$+ \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z\right)w + e_x\sigma_x + e_y\sigma_y + e_z\sigma_z +$$

$$+ \gamma_{yz}\tau_{yz} + \gamma_{xz}\tau_{xz} + \gamma_{xy}\tau_{xy} - U(e_x, \cdots \gamma_{xy}). \qquad (1)$$

我們稱 L_U 爲以應變能表示的廣義餘能,簡稱廣義餘能，因爲當應力適合平衡方程

$$\left.\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X=0,\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y=0,\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0\end{aligned}\right\}\tag{2}$$

時，L_U 的算式簡化爲

$$L_U=e_x\sigma_x+\cdots+\gamma_{xy}\tau_{xy}-U.\tag{3}$$

這便是錢令希教授在著作 [1] 中所下的餘能的定義.

設彈性體所佔的空間是 V，它的表面是 S. 設在表面的一部分 S_σ 上，指定表面力，卽已知

在 S_σ 上：

$$\left.\begin{aligned}&p_x\equiv\sigma_x\cos(nx)+\tau_{xy}\cos(ny)+\tau_{xz}\cos(nz)=\bar p_x,\\&p_y\equiv\tau_{xy}\cos(nx)+\sigma_y\cos(ny)+\tau_{yz}\cos(nz)=\bar p_y,\\&p_z\equiv\tau_{xz}\cos(nx)+\tau_{yz}\cos(ny)+\sigma_z\cos(nz)=\bar p_z;\end{aligned}\right\}\tag{4}$$

而在另一部分 S_u 上，指定位移，卽已知

$$\text{在 } S_u \text{ 上:}\qquad u=\bar u,\quad v=\bar v,\quad w=\bar w.\tag{5}$$

S_σ 和 S_u 的總和便是整個表面 S.

廣義餘能原理 把 $u, v, w, e_x, e_y, e_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 等十五個函數看作是彼此獨立無關的函數，並且讓它們的變分不受任何限制，那末下面的變分式：

$$\delta\left\{\iiint_V L_U\,dV-\iint_{S_u}(\bar u\,p_x+\bar v\,p_y+\bar w\,p_z)\,dS-\right.$$
$$\left.-\iint_{S_\sigma}\Big[u(p_x-\bar p_x)+v(p_y-\bar p_y)+w(p_z-\bar p_z)\Big]\,dS\right\}=0\tag{6}$$

相當於彈性力學中全部基本方程，卽平衡方程：

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+X=0,\\
&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+Y=0,\\
&\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+Z=0;
\end{aligned}\right\} \tag{7}$$

應力和應變的關係式:

$$\left.\begin{aligned}
&\sigma_x=\frac{\partial U}{\partial e_x}, \qquad \sigma_y=\frac{\partial U}{\partial e_y}, \qquad \sigma_z=\frac{\partial U}{\partial e_z},\\
&\tau_{yz}=\frac{\partial U}{\partial \gamma_{yz}}, \qquad \tau_{xz}=\frac{\partial U}{\partial \gamma_{xz}}, \qquad \tau_{xy}=\frac{\partial U}{\partial \gamma_{xy}};
\end{aligned}\right\} \tag{8}$$

應變與位移的關係式:

$$\left.\begin{aligned}
&e_x=\frac{\partial u}{\partial x}, \qquad e_y=\frac{\partial v}{\partial y}, \qquad e_z=\frac{\partial w}{\partial z},\\
&\gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \qquad \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, \qquad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x};
\end{aligned}\right\} \tag{9}$$

以及邊界條件:

$$\text{在 } S_u \text{ 上:} \qquad u=\bar{u}, \qquad v=\bar{v}, \qquad w=\bar{w}, \tag{10a}$$

$$\text{在 } S_\sigma \text{ 上:} \qquad p_x=\bar{p}_x, \qquad p_y=\bar{p}_y, \qquad p_z=\bar{p}_z. \tag{10b}$$

證. 將 (1) 式規定的 L_U 代入 (6) 式,然後將算子 δ 作用於積分號內的各函數;稍加整理後得

$$\iiint_V\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+X\right)\delta u\, dV+\iiint_V\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+Y\right)\delta v\, dV+$$
$$+\iiint_V\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+Z\right)\delta w\, dV+\iiint_V\left(\sigma_x-\frac{\partial U}{\partial e_x}\right)\delta e_x\, dV+$$
$$+\iiint_V\left(\sigma_y-\frac{\partial U}{\partial e_y}\right)\delta e_y\, dV+\iiint_V\left(\sigma_z-\frac{\partial U}{\partial e_z}\right)\delta e_z\, dV+$$
$$+\iiint_V\left(\tau_{yz}-\frac{\partial U}{\partial \gamma_{yz}}\right)\delta \gamma_{yz}\, dV+\iiint_V\left(\tau_{xz}-\frac{\partial U}{\partial \gamma_{xz}}\right)\delta \gamma_{xz}\, dV+$$

$$+\iiint_V\left(\tau_{xy}-\frac{\partial U}{\partial\gamma_{xy}}\right)\delta\gamma_{xy}\,dV+\iiint_V\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{xy}}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)u\,dV+$$

$$+\iiint_V\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)v\,dV+\iiint_V\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)w\,dV+$$

$$+\iiint_V(e_x\,\delta\sigma_x+e_y\,\delta\sigma_y+e_z\,\delta\sigma_z+\gamma_{yz}\,\delta\tau_{yz}+\gamma_{xz}\,\delta\tau_{xz}+\gamma_{xy}\,\delta\tau_{xy})\,dV-$$

$$-\iint_{S_u}(\bar u\,\delta p_x+\bar v\,\delta p_y+\bar w\,\delta p_z)\,dS-\iint_{S_\sigma}(u\,\delta p_x+v\,\delta p_y+w\,\delta p_z)\,dS-$$

$$-\iint_{S_\sigma}\Big[(p_x-\bar p_x)\,\delta u+(p_y-\bar p_y)\,\delta v+(p_z-\bar p_z)\,\delta w\Big]\,dS=0\,. \tag{11}$$

因爲 u, v, w 的變分不受任何限制,所以從 (11) 式的前三個積分便得平衡方程 (7), 從 (11) 式的最後一個積分得邊界條件 (10b). 又因爲 $e_x,\cdots\gamma_{xy}$ 等六個應變分量的變分不受任何限制,所以從 (11) 式第四個到第九個積分得應力與應變的關係式 (8). 這樣 (11) 式剩下的便爲關於應力分量的變分:

$$\iiint_V\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{xy}}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)u\,dV+\iiint_V\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{yz}}{\partial z}\right)v\,dV+$$

$$+\iiint_V\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)w\,dV+$$

$$+\iiint_V(e_x\,\delta\sigma_x+e_y\,\delta\sigma_y+e_z\,\delta\sigma_z+\gamma_{yz}\,\delta\tau_{yz}+\gamma_{xz}\,\delta\tau_{xz}+\gamma_{xy}\,\delta\tau_{xy})\,dV-$$

$$-\iint_{S_u}(\bar u\,\delta p_x+\bar v\,\delta p_y+\bar w\,\delta p_z)\,dS-\iint_{S_\sigma}(u\,\delta p_x+v\,\delta p_y+w\,\delta p_z)\,dS=0\,. \tag{12}$$

利用附錄公式 (104), 可得

$$\iiint_V\left\{\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{xy}}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)u+\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{yz}}{\partial z}\right)v+\right.$$
$$\left.+\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)w\right\}dV=$$
$$=-\iiint_V\left\{\frac{\partial u}{\partial x}\,\delta\sigma_x+\frac{\partial v}{\partial y}\,\delta\sigma_y+\frac{\partial w}{\partial z}\,\delta\sigma_z+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\delta\tau_{yz}+\right.$$
$$\left.+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\delta\tau_{xz}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\delta\tau_{xy}\right\}dV+$$
$$+\iint_S(u\,\delta p_x+v\,\delta p_y+w\,\delta p_z)\,dS\,. \tag{13}$$

將此代入 (12) 式,加以整理,並注意到

$$\iint_S (\quad) dS = \iint_{S_u} (\quad) dS + \iint_{S_\sigma} (\quad) dS, \tag{14}$$

我們得到

$$\begin{aligned}\iiint_V \Big\{ &\left(e_x - \frac{\partial u}{\partial x}\right)\delta\sigma_x + \left(e_y - \frac{\partial v}{\partial y}\right)\delta\sigma_y + \left(e_z - \frac{\partial w}{\partial z}\right)\delta\sigma_z + \\ &+ \left(\gamma_{yz} - \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\delta\tau_{yz} + \left(\gamma_{xz} - \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\delta\tau_{xz} + \\ &+ \left(\gamma_{xy} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\delta\tau_{xy}\Big\} dV + \\ &+ \iint_{S_u} \Big[(u - \bar u)\,\delta p_x + (v - \bar v)\,\delta p_y + (w - \bar w)\,\delta p_z \Big] dS = 0.\end{aligned} \tag{15}$$

因爲 $\sigma_x, \cdots \tau_{xy}$ 的變分不受任何限制，所以從 (15) 式便得應變與位移的關係式 (9) 和邊界條件 (10a)。

由此可知變分式 (6) 在所設條件下相當於彈性體力學中的全部基本方程。

在上述廣義餘能原理中，u, v, w, $e_x, \cdots \gamma_{xy}$, $\sigma_x, \cdots \tau_{xy}$ 等十五個函數和它們的變分不受任何限制,因此可以認爲廣義餘能原理是彈性體力學中最一般的變分原理中的一個。 如果我們給 u, v, w, $e_x, \cdots \gamma_{xy}$, $\sigma_x, \cdots \tau_{xy}$ 和它們的變分以種種限制[當然這些限制必須適合方程 (7) 至 (10)]，那末我們便可從廣義餘能原理引出一系列較爲特殊的但仍相當一般的變分原理。 今將重要的特例舉示於下：

1. 假定應力分量和它們的變分適合平衡方程和應力邊界條件 (10b)。 在這種場合，L_U 的算式簡化爲

$$L_U = e_x\,\sigma_x + e_y\,\sigma_y + \cdots + \gamma_{xy}\,\tau_{xy} - U, \tag{16}$$

而變分式 (6) 化爲

$$\delta\left\{ \iiint_V L_U\, dV - \iint_{S_u} (\bar u\, p_x + \bar v\, p_y + \bar w\, p_z)\, dS \right\} = 0. \tag{17}$$

這是餘能原理的一種表示法。 但是變分式(17)還比錢令希教授在著作 [1] 中所說明的餘能原理普遍,因爲我們未假定應力和應變適合物性方程 (8)。

2. 假定應力分量和它們的變分適合平衡方程和應力邊界條件 (10b),並且應力和應變適合關係式 (8). 在這種場合, L_U 的算式 (16), 即餘能的算式可用應力分量表示,亦可用應變分量表示:

$$L_U = L_U\,(e_x\,, \cdots \gamma_{xy}) = L_U\,(\sigma_x\,, \cdots \tau_{xy})\,, \tag{18}$$

而變分式 (6) 簡化爲

$$\delta \left\{ \iiint_V L_U\, dV - \iint_{S_u} (\bar{u}\, p_x + \bar{v}\, p_y + \bar{w}\, p_z)\, dS \right\} = 0\,. \tag{19}$$

這便是錢令希教授在著作 [1] 中所說明的餘能原理.

3. 假定應力、應變和位移適合方程 (7), (8), (9), 但不適合邊界條件(10). 在這種場合變分式 (6) 化爲

$$\iint_{S_u} \left[(u - \bar{u})\, \delta p_x + (v - \bar{v})\, \delta p_y + (w - \bar{w})\, \delta p_z \right] dS - \\ - \iint_{S_\sigma} \left[(p_x - \bar{p}_x)\, \delta u + (p_y - \bar{p}_y)\, \delta v + (p_z - \bar{p}_z)\, \delta w \right] dS = 0\,. \tag{20}$$

這個變分式特別適宜於作爲近似地求解混合邊值問題的根據,因爲在這類問題中適合方程 (7), (8), (9) 並不是最大的困難,而欲適合邊界條件 (10) 才是最大的困難.

(二)廣義勢能原理

命

$$H_U = U\,(e_x, e_y, \cdots \gamma_{xy}) - (Xu + Y\,v + Z\,w) + \sigma_x \left(\frac{\partial u}{\partial x} - e_x \right) + \\ + \sigma_y \left(\frac{\partial v}{\partial y} - e_y \right) + \sigma_z \left(\frac{\partial w}{\partial z} - e_z \right) + \tau_{yz} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \gamma_{yz} \right) + \\ + \tau_{xz} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \gamma_{xz} \right) + \tau_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \gamma_{xy} \right). \tag{21}$$

我們稱 H_U 爲以應變能表示的彈性體中單位體積的廣義勢能,簡稱廣義勢能,因爲當應變和位移適合方程 (9) 時, H_U 便是平常所說的勢能.

從 (1) 和 (21) 兩式易知

$$L_U + H_U = \frac{\partial}{\partial x}(\sigma_x u + \tau_{xy} v + \gamma_{xz} w) + \frac{\partial}{\partial y}(\tau_{xy} u + \sigma_y v + \tau_{yz} w) + + \frac{\partial}{\partial z}(\tau_{xz} u + \tau_{yz} v + \sigma_z w). \tag{22}$$

因此利用附錄公式（104）我們便得

$$\iiint_V (L_U + H_U)\, dV = \iint_S (p_x u + p_y v + p_z w)\, dS. \tag{23}$$

應用這個 H_U 和 L_U 的關係式，不難從 L_U 的變分原理（6）引出關於 H_U 的變分原理如下：

廣義勢能原理　把 $u, v, w, e_x, e_y, e_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 等十五個函數看作是彼此獨立無關的函數，並且讓它們的變分不受任何限制，那末下面的變分式：

$$\delta\left\{\iiint_V H_U\, dV - \iint_{S_\sigma} (u\,\bar{p}_x + v\,\bar{p}_y + w\,\bar{p}_z)\, dS - - \iint_{S_u}\left[p_x(u-\bar{u}) + p_y(v-\bar{v}) + p_z(w-\bar{w})\right] dS\right\} = 0 \tag{24}$$

相當於彈性力學中全部基本方程，卽平衡方程（7），應力和應變的關係式（8），應變與位移的關係式（9）以及邊界條件（10）.

廣義勢能原理當然亦可以從 H_U 的算式獲得直接的證明.

因爲在廣義勢能原理中，應力、應變、位移以及它們的變分不受任何限制，因此可以認爲廣義勢能原理亦是彈性體力學中最一般的變分原理中的一個.

如果給位移、應變和應力以種種限制［當然這些限制必須適合方程（7）至（10）］，那末我們可從廣義勢能原理引出一系列較爲特殊的但仍相當普遍的變分原理.

例如，假定應變和位移適合方程（9），並且位移又適合邊界條件（10a）；在這種場合，

$$H_U = U(e_x, e_y, \cdots \gamma_{xz}) - (X u + Y v + Z w), \tag{25}$$

而變分式（24）化爲

$$\delta\left\{\iiint_V\left[U-(Xu+Yv+Zw)\right]dV-\iint_{S_\sigma}(u\bar{p}_x+v\bar{p}_y+w\bar{p}_z)\,dS\right\}=0\,. \quad (26)$$

這便是平常所說的勢能原理.

（三） 廣義餘能原理與廣義勢能原理的別種表示法

設 V 是彈性體中單位體積的餘能. V 可以用應力分量表示,亦可以用應變分量表示. 在本文中我們規定 V 一概用應力分量表示,即 $V=V(\sigma_x, \sigma_y, \cdots\tau_{xy})$.

命

$$L_V=\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X\right)u+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+Y\right)v+$$
$$+\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z\right)w+V, \quad (27)$$

$$H_V=\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\sigma_z\frac{\partial w}{\partial z}+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+$$
$$+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-(Xu+Yv+Zw)-V\,. \quad (28)$$

如果位移、應變和應力是適合平衡方程、應力和應變的關係式以及應變與位移的關係式的一組解，那末很顯明的，$L_V=L_U$，$H_V=H_U$. 因此 L_V 和 H_V 仍是彈性體中單位體積的廣義餘能和廣義勢能，不過以與 L_U 和 H_U 不同的形式表示出來.

L_V 和 H_V 有與 L_U 和 H_U 相同的關係式,即

$$L_V+H_V=\frac{\partial}{\partial x}(\sigma_x u+\tau_{xy}v+\tau_{xz}w)+\frac{\partial}{\partial y}(\tau_{xy}u+\sigma_y v+\tau_{yz}w)+$$
$$+\frac{\partial}{\partial z}(\tau_{xz}u+\tau_{yz}v+\sigma_z w), \quad (29)$$

$$\iiint_V(L_V+H_V)\,dV=\iint_S(p_x u+p_y v+p_z w)\,dS\,. \quad (30)$$

廣義餘能原理 把 u, v, w, σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy} 等九個函數看作是彼此獨立無關的函數,並且讓它們的變分不受任何限制,那末下面的變分式:

$$\delta\left\{\iiint_V L_V\,dV - \iint_{S_u}(\bar{u}\,p_x + \bar{v}\,p_y + \bar{w}\,p_z)\,dS - \right.$$
$$\left. - \iint_{S_\sigma}\Big[u(p_x - \bar{p}_x) + v(p_y - \bar{p}_y) + w(p_z - \bar{p}_z)\Big]\,dS\right\} = 0 \quad (31)$$

相當於平衡方程 (7), 位移與和應力的關係式:

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial V}{\partial \sigma_x}, && \frac{\partial v}{\partial y} = \frac{\partial V}{\partial \sigma_y}, && \frac{\partial w}{\partial z} = \frac{\partial V}{\partial \sigma_z}, \\ &\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial V}{\partial \tau_{yz}}, && \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial V}{\partial \tau_{xz}}, && \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial V}{\partial \tau_{xy}} \end{aligned}\right\} \quad (32)$$

以及邊界條件 (10).

廣義勢能原理　把 $u, v, w, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 等九個函數看作是彼此獨立無關的函數;並讓它們的變分不受任何限制,那末下面的變分式:

$$\delta\left\{\iiint_V H_V\,dV - \iint_{S_\sigma}(u\,\bar{p}_x + v\,\bar{p}_y + w\,\bar{p}_z)\,dS - \right.$$
$$\left. - \iint_{S_u}\Big[p_x(u - \bar{u}) + p_y(v - \bar{v}) + p_z(w - \bar{w})\Big]\,dS\right\} = 0 \quad (33)$$

相當於平衡方程 (7), 位移與應力的關係式 (32) 以及邊界條件 (10).

這兩個原理的證明和第(一)小節相似,因此這裏不再寫出來.

以變分式 (33) 表示的廣義勢能原理,便是 de Veubeke[30]補充 E. Reissner[26]的結果所得的變分原理.

三. 彈性薄板彎曲理論中的一般變分原理

上述彈性力學中的一般變分原理,稍加推廣後還可應用於彈性薄板的彎曲理論. 薄板的彈性可不必服從虎克定律,同時薄板的撓度亦可不必微小,即可以產生大撓度.

(一) 一般變分原理

命 M_x, M_y, M_{xy} 爲薄板中的內力矩, N_x, N_y, N_{xy} 爲薄板平面內的內力,即薄膜應力. 命 w 是薄板的撓度, u, v 是各點在薄板平面內的位移. 命 $V_B(M_x,$

M_y, M_{xy}) 和 $V_m(N_x, N_y, N_{xy})$ 各爲薄板的單位面積的彎曲餘能和薄膜餘能.如果薄板的彈性服從虎克定律,那末

$$V_B = \frac{1}{2D(1-\mu^2)}\left[M_x^2 + M_y^2 - 2\mu M_x M_y + 2(1+\mu) M_{xy}^2\right], \tag{34}$$

$$V_m = \frac{1}{2hE}\left[N_x^2 + N_y^2 - 2\mu N_x N_y + 2(1+\mu) N_{xy}^2\right]. \tag{35}$$

設 X, Y, q 爲外力. 在薄板彎曲理論中,與 (28) 式對應的廣義勢能 H 爲

$$\begin{aligned} H = &- M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_x\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \\ &+ N_y\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + N_{xy}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - \\ &- (Xu + Yv + qw) - V_B - V_m. \end{aligned} \tag{36}$$

設薄板所佔的區域是 S, 它的邊界是 C. 爲了說明簡單起見,我們假定在邊界上 $w = 0$. 邊界上的法向彎矩 M_n 和薄膜內力 p_x, p_y 的算式各爲

$$M_n = M_x \cos^2(nx) + M_y^2 \cos^2(ny) - 2M_{xy}\cos(nx)\cos(ny), \tag{37}$$

$$\left.\begin{aligned} p_x &= N_x \cos(nx) + N_{xy}\cos(ny), \\ p_y &= N_{xy}\cos(nx) + N_y \cos(ny). \end{aligned}\right\} \tag{38}$$

設在邊界的一部分 C_M 上已知 $M_n = \overline{M}_n$, 一部分 C_w 上已知 $\frac{\partial w}{\partial n} = \alpha$, 一部分 C_N 上已知 $p_x = \bar{p}_x$ 和 $p_y = \bar{p}_y$, 一部分 C_u 上已知 $u = \bar{u}$, 和 $v = \bar{v}$. C_M 和 C_w 的總和以及 C_N 和 C_u 的總和都是整個邊界 C.

廣義勢能原理 把 M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, u, v, w 等九個函數看作是彼此獨立無關的函數,並且除使 w 及其變分 δw 在邊界 C 上等於零外,其他各函數以及它們的變分不受任何限制,那末下面的變分式:

$$\begin{aligned} \delta\Big\{ &\iint_S H\,dx\,dy + \int_{C_M} \overline{M}_n \frac{\partial w}{\partial n}\,ds + \int_{C_w} M_n\left(\frac{\partial w}{\partial n} - \alpha\right) ds - \\ &- \int_{C_N} (u\,\bar{p}_x + v\,\bar{p}_y)\,ds - \int_{C_u}\left[p_x(u-\bar{u}) + p_y(v-\bar{v})\right] ds\Big\} = 0 \end{aligned} \tag{39}$$

相當於薄板彎曲理論中的三組基本方程，即平衡方程：

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}+q+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+$$
$$+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-X\frac{\partial w}{\partial x}-Y\frac{\partial w}{\partial y}=0. \tag{40}$$

$$\left.\begin{aligned}\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+X=0,\\ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+Y=0,\end{aligned}\right\} \tag{41}$$

位移和應力的關係式：

$$-\frac{\partial^2 w}{\partial x^2}=\frac{\partial V_B}{\partial M_x},\quad -\frac{\partial^2 w}{\partial y^2}=\frac{\partial V_B}{\partial M_y},\quad 2\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial V_B}{\partial M_{xy}}, \tag{42}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2=\frac{\partial V_m}{\partial N_x},\quad \frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2=\frac{\partial V_m}{\partial N_y},\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}=\frac{\partial V_m}{\partial N_{xy}},\end{aligned}\right\} \tag{43}$$

以及邊界條件：

在 C_M 上：　$M_n=\overline{M}_n,$ (44a)

在 C_w 上：　$\frac{\partial w}{\partial n}=\alpha,$ (44b)

在 C_N 上：　$p_x=\bar{p}_x,\quad p_y=\bar{p}_y,$ (45a)

在 C_u 上：　$u=\bar{u},\quad v=\bar{v}.$ (45b)

證. 將 H 的算式 (36) 代入 (39)，然後加以整理，得

$$\iint_S\left\{-\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial V_B}{\partial M_x}\right)\delta M_x-\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial V_B}{\partial M_y}\right)\delta M_y+\right.$$
$$\left.+\left(2\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial V_B}{\partial M_{xy}}\right)\delta M_{xy}\right\}dx\,dy+$$

$$+\iint_S\left\{\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial V_m}{\partial N_x}\right]\delta N_x+\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{\partial V_m}{\partial N_y}\right]\delta N_y+\right.$$

$$\left.+\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial V_m}{\partial N_{xy}}\right]\delta N_{xy}\right\}dx\,dy+$$

$$+\iint_S\left\{N_x\frac{\partial\delta u}{\partial x}+N_y\frac{\partial\delta v}{\partial y}+N_{xy}\left(\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right)-X\,\delta u-Y\,\delta v\right\}dx\,dy+$$

$$+\iint_S\left\{-M_x\frac{\partial^2\delta w}{\partial x^2}-M_y\frac{\partial^2\delta w}{\partial y^2}+2\,M_{xy}\frac{\partial^2\delta w}{\partial x\partial y}+N_x\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}+N_y\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial y}+\right.$$

$$\left.+N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial x}\right)-q\,\delta w\right\}dx\,dy+$$

$$+\int_{C_M}\overline{M}_n\frac{\partial\delta w}{\partial n}\,ds+\int_{C_w}M_n\frac{\partial\delta w}{\partial n}\,ds+\int_{C_w}\left(\frac{\partial w}{\partial n}-\alpha\right)\delta M_n\,ds-$$

$$+\int_{C_N}(\bar{p}_x\,\delta u+\bar{p}_y\,\delta v)\,ds-\int_{C_u}(p_x\,\delta u+p_y\,\delta v)\,ds-$$

$$-\int_{C_u}\left[\delta p_x(u-\bar{u})+(v-\bar{v})\,\delta p_y\right]ds=0. \tag{46}$$

因為 M_x, M_y, M_{xy} 的變分不受任何限制，所以從 (46) 式的第一個積分可導得方程 (42)，從 (46) 式第七個積分可導得方程(44b)． 又因為 N_x, N_y, N_{xy} 的變分不受任何限制,所以從 (46) 式的第二個積分可導得方程 (43)，從 (46) 式最後一個積分可導得方程 (45b)． 最後因為 u, v, w 的變分獨立無關,因此 (46) 式剩下的積分可化為兩個式子:

$$\iint_S\left\{N_x\frac{\partial\delta u}{\partial x}+N_y\frac{\partial\delta v}{\partial y}+N_{xy}\left(\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right)-X\,\delta u-Y\,\delta v\right\}dx\,dy-$$

$$-\int_{C_N}(\bar{p}_x\,\delta u+\bar{p}_y\,\delta v)\,ds-\int_{C_u}(p_x\,\delta u+p_y\,\delta v)\,ds=0, \tag{47}$$

$$\iint_S\left\{-M_x\frac{\partial^2\delta w}{\partial x^2}-M_y\frac{\partial^2\delta w}{\partial y^2}+2\,M_{xy}\frac{\partial^2\delta w}{\partial x\partial y}+N_x\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}+N_y\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial y}+\right.$$

$$\left.+N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial x}\right)-q\,\delta w\right\}dx\,dy+$$

$$+\int_{C_M}\overline{M}_n\frac{\partial\delta w}{\partial n}\,ds+\int_{C_w}M_n\frac{\partial\delta w}{\partial n}\,ds=0. \tag{48}$$

利用附錄公式 (107) 可知

$$\iint_S \left[N_x \frac{\partial \delta u}{\partial x} + N_y \frac{\partial \delta v}{\partial y} + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right] dx\, dy = \int_C (p_x \delta u + p_y \delta v)\, ds -$$
$$-\iint_S \left\{\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\delta u + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial x}\right)\delta v\right\} dx\, dy, \tag{49}$$

所以（47）式化爲

$$\int_{C_N} \left[(p_x - \bar{p}_x)\, \delta u + (p_y - \bar{p}_y)\, \delta v\right] ds -$$
$$-\iint_S \left\{\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X\right)\delta u + \left(\frac{\partial N_{xz}}{\partial x} + \frac{\partial N_y}{\partial y} + Y\right)\delta v\right\} dx\, dy = 0. \tag{50}$$

從這個式子便可導出方程（41）和（45a）.

利用附錄公式（111）和（110）可知

$$\iint_S \left\{-M_x \frac{\partial^2 \delta w}{\partial x^2} - M_y \frac{\partial^2 \delta w}{\partial y^2} + 2\, M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y}\right\} dx\, dy =$$
$$= -\int_C M_n \frac{\partial \delta w}{\partial n}\, ds + \iint_S \left(-\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_y}{\partial y^2} + 2\, \frac{\partial^2 M_{xy}}{\partial x \partial y}\right)\delta w\, dx\, dy, \tag{51}$$

$$\iint_S \left\{N_x \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + N_y \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right)\right\} dx dy =$$
$$= -\iint_S \left\{N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2\, N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) + \right.$$
$$\left. + \frac{\partial w}{\partial y}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\right\} \delta w\, dx\, dy. \tag{52}$$

因此方程（48）化爲

$$\iint_S \delta w \left\{-\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_y}{\partial y^2} + 2\, \frac{\partial^2 M_{xy}}{\partial x \partial y} - q - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - \right.$$
$$\left. - 2\, N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) - \frac{\partial w}{\partial y}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\right\} dx\, dy +$$
$$+ \int_{C_M} (M_n - \overline{M}_n)\, \frac{\partial \delta w}{\partial n}\, ds = 0. \tag{53}$$

從這個方程便可導出方程（44a）和下面的方程：

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} +$$
$$+ \frac{\partial w}{\partial x}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right) = 0. \tag{54}$$

利用已經證明的方程（41），可知方程（54）與（40）相同.

由此可知變分式（39）在所設條件下相當於方程（40）到（45）.

在本小節所述的廣義勢能原理中，除了對 w 及其變分作了一個不重要的限制之外，其他各函數以及它們的變分不受任何限制，由此可見這個變分原理是彈性薄板彎曲理論中的一個很普遍的變分原理.

從變分式（39）還可導出一系列與它等效的變分式. 這裏我們擬指出其中之一.

命

$$F = -M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2 M_{xy} \frac{\partial^2 w}{\partial x \partial y} +$$
$$+ \frac{N_x}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{N_y}{2}\left(\frac{\partial w}{\partial y}\right)^2 + N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} -$$
$$-\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X\right)u - \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + Y\right)v - qw - V_B - V_m; \tag{55}$$

則易知

$$H - F = \frac{\partial}{\partial x}(u N_x + v N_{xy}) + \frac{\partial}{\partial y}(u N_{xy} + v N_y). \tag{56}$$

因此根據附錄公式（107）得

$$\iint_S (H - F)\, dx\, dy = \int_C (u p_x + v p_y)\, ds. \tag{57}$$

將此式代入（39），我們便得關於 F 的一個變分原理如下：

原理 把 M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, u, v, w 等九個函數看作是彼此獨立無關的函數，並且除使 w 及其變分 δw 在邊界上等於零外，其他各函數的變分不受任何限制，那末下面的變分式：

$$\delta\left\{\iint_S F\,dx\,dy+\int_{C_M}\overline{M}_n\frac{\partial w}{\partial n}\,ds+\int_{C_w}M_n\left(\frac{\partial w}{\partial n}-\alpha\right)ds+\right.$$
$$\left.+\int_{C_N}\Big[u(p_x-\bar p_x)+v(p_y-\bar p_y)\Big]\,ds+\int_{S_u}(\bar u\,p_x+\bar v\,p_y)\,ds\right\}=0 \qquad (58)$$

相當於方程（40）至（45）.

（二） 變分式（58）的特例　卡門方程的變分推導

上一小節中的變分式（39）和（58）是彈性薄板彎曲理論中很普遍的變分原理. 如果給 M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, u, v, w 以種種限制[當然這些限制必須適合方程（40）至（45）], 那末我們可導得一系列較爲特殊的但仍相當普遍的變分原理. 這裏將舉出一個實用上重要的特例.

設 M_x, M_y, M_{xy}, w 和它們的變分已經適合方程（42）. 在這個條件下，

$$U_B=-M_x\frac{\partial^2 w}{\partial x^2}-M_y\frac{\partial^2 w}{\partial y^2}+2\,M_{xy}\frac{\partial^2 w}{\partial x\partial y}-V_B \qquad (59)$$

便是薄板的單位面積的彎曲應變能，並且 U_B 可以用 w 的偏導數表示. 再設 N_x, N_y, N_{xy} 以及它們的變分已經適合方程（41）. 在這兩個條件下，F 的算式（55）簡化爲

$$F=U_B+\frac{N_x}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{N_y}{2}\left(\frac{\partial w}{\partial y}\right)^2+N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-q\,w-V_m. \qquad (60)$$

這個 F 的算式在實用上比較有效而方便，因爲欲適合方程（42）和（41）並無多大困難.

利用 F 的算式（60）和變分式（58），可以導得在彈性薄板彎曲理論中著名的卡門方程（例如見[29]）. 設外力 $X=Y=0$. 在這個條件下，爲了適合方程（41）可取

$$N_x=\frac{\partial^2\phi}{\partial y^2},\qquad N_y=\frac{\partial^2\phi}{\partial x^2},\qquad N_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}. \qquad (61)$$

設薄板的彈性服從虎克定律，那末

$$U_B=\frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2\,(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}, \qquad (62)$$

$$V_n = \frac{1}{2Eh}\left\{\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)^2 - 2(1+\mu)\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right]\right\}. \tag{63}$$

因此 F 的算式化爲

$$\begin{aligned} F = \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\} + \\ +\frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-q\,w- \\ -\frac{1}{2Eh}\left\{\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)^2 - 2(1+\mu)\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right]\right\}. \end{aligned} \tag{64}$$

將此代入變分式 (58), 然後經過一個繁長的計算後, 可得下面的方程:

$$\left.\begin{aligned} D\left\{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right\} = q + \\ +\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}, \\ \frac{1}{Eh}\left\{\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right\} = -\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2, \end{aligned}\right\} \tag{65}$$

和相關的邊界條件. 方程 (65) 便是著名的卡門方程.

在卡門方程 (65) 中, 一個未知函數是撓度 w, 另一個未知函數是應力函數 ϕ, 兩者的性質不同. 因此可以看出, 無論用平常的勢能原理或餘能原理, 都無法導得卡門方程. 只有我們的廣義勢能原理, 允許應力和位移同時變化, 才使人有可能用變分法導出卡門方程.

四. 受範性體力學中一般變分原理

在本節中, 我們將討論受範性體力學中的一般變分原理.

(一) 變形式應變與應力的關係

我們假定物體在彈性狀態中單位體積的餘能爲 $V(\sigma_x\cdots\tau_{xy})$. 因此在彈性狀態時, 應變與應力的關係爲

$$
\left.\begin{aligned}
e_x &= \frac{\partial V}{\partial \sigma_x}, & e_y &= \frac{\partial V}{\partial \sigma_y}, & e_z &= \frac{\partial V}{\partial \sigma_z}, \\
\gamma_{yz} &= \frac{\partial V}{\partial \tau_{yz}}, & \gamma_{xz} &= \frac{\partial V}{\partial \tau_{xz}}, & \gamma_{xy} &= \frac{\partial V}{\partial \tau_{xy}}.
\end{aligned}\right\} \tag{66}
$$

設物體的屈服條件為

$$
S(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}) = 常數. \tag{67}
$$

我們假定在物體進入受範性狀態後，其中的應變可分為兩部分：一部分是彈性應變，它們仍由（66）式決定，即

$$
\left.\begin{aligned}
e_x^e &= \frac{\partial V}{\partial \sigma_x}, & e_y^e &= \frac{\partial V}{\partial \sigma_y}, & e_z^e &= \frac{\partial V}{\partial \sigma_z}, \\
\gamma_{yz}^e &= \frac{\partial V}{\partial \tau_{yz}}, & \gamma_{xz}^e &= \frac{\partial V}{\partial \tau_{xz}}, & \gamma_{xy}^e &= \frac{\partial V}{\partial \tau_{xy}};
\end{aligned}\right\} \tag{68}
$$

另一部分是受範性應變，它們與應力的關係為

$$
\frac{e_x^p}{\dfrac{\partial S}{\partial \sigma_x}} = \frac{e_y^p}{\dfrac{\partial S}{\partial \sigma_y}} = \frac{e_z^p}{\dfrac{\partial S}{\partial \sigma_z}} = \frac{\gamma_{yz}^p}{\dfrac{\partial S}{\partial \tau_{yz}}} = \frac{\gamma_{xz}^p}{\dfrac{\partial S}{\partial \tau_{xz}}} = \frac{\gamma_{xy}^p}{\dfrac{\partial S}{\partial \tau_{xy}}}. \tag{69}
$$

若命上式中每一個分數都等於 λ（λ 為一未定函數，並非該物體的物性常數），則得

$$
\left.\begin{aligned}
e_x^p &= \lambda \frac{\partial S}{\partial \sigma_x}, & e_y^p &= \lambda \frac{\partial S}{\partial \sigma_y}, & e_z^p &= \lambda \frac{\partial V}{\partial \sigma_z}, \\
\gamma_{yz}^p &= \lambda \frac{\partial S}{\partial \tau_{yz}}, & \gamma_{xz}^p &= \lambda \frac{\partial V}{\partial \tau_{xz}}, & \gamma_{xy}^p &= \lambda \frac{\partial V}{\partial \tau_{xy}}.
\end{aligned}\right\} \tag{70}
$$

將（68），（70）兩式相加，最後得受範性狀態中應變與應力的關係如下：

$$
\left.\begin{aligned}
e_x &= \frac{\partial V}{\partial \sigma_x} + \lambda \frac{\partial S}{\partial \sigma_x}, & e_y &= \frac{\partial V}{\partial \sigma_y} + \lambda \frac{\partial S}{\partial \sigma_y}, & e_z &= \frac{\partial V}{\partial \sigma_z} + \lambda \frac{\partial S}{\partial \sigma_z}, \\
\gamma_{yz} &= \frac{\partial V}{\partial \tau_{yz}} + \lambda \frac{\partial S}{\partial \tau_{yz}}, & \gamma_{xz} &= \frac{\partial V}{\partial \tau_{xz}} + \lambda \frac{\partial S}{\partial \tau_{xz}}, & \gamma_{xy} &= \frac{\partial V}{\partial \tau_{xy}} + \lambda \frac{\partial S}{\partial \tau_{xy}}.
\end{aligned}\right\} \tag{71}
$$

例如,取

$$\left.\begin{aligned} V &= \frac{1}{2E}(\sigma_x+\sigma_y+\sigma_z)^2+\frac{1}{2G}(\tau_{yz}^2+\tau_{xz}^2+\tau_{xy}^2-\sigma_y\sigma_z-\sigma_z\sigma_x-\sigma_x\sigma_y),\\ S &= \frac{1}{2G}\left\{\frac{1}{6}\Big[(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+(\sigma_x-\sigma_y)^2\Big]+\tau_{yz}^2+\tau_{xz}^2+\tau_{xy}^2\right\},\end{aligned}\right\}\quad(72)$$

便可得 Huber 和 v. Mises 的屈服條件和 Hencky 的應變與應力的關係.

(二) 廣義餘能原理

今設有一受範性體,它的屈服條件是(67),單位體積的餘能是 $V(\sigma_x, \sigma_y, \cdots\tau_{xy})$. 和(27)式相同,命

$$L=\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X\right)u+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y\right)v+ \\ +\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z\right)w+V. \quad(73)$$

我們仍稱 L 爲物體中單位體積的廣義餘能. 假定外界荷載恰使物體中每一點都進入受範性狀態,並且是在加荷載過程中. 在這些假定下便有下列變分原理:

廣義餘能原理 設 u, v, w 以及它們的變分不受任何限制;設 $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 以及它們的變分除必須適合屈服條件(67)外,不再受其他限制;那末下面的變分式:

$$\delta\left\{\iiint_V L\,dV-\iint_{S_u}(\bar u p_x+\bar v p_y+\bar w p_z)\,dS- \\ -\iint_{S_\sigma}\Big[u(p_x-\bar p_x)+v(p_y-\bar p_y)+w(p_z-\bar p_z)\Big]dS\right\}=0. \quad(74)$$

相當於平衡方程:

$$\left.\begin{aligned} \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X=0,\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y=0,\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0,\end{aligned}\right\}\quad(75)$$

應變與應力的關係式:

$$
\left.\begin{aligned}
&\frac{\partial u}{\partial x}=\frac{\partial V}{\partial \sigma_x}+\lambda\frac{\partial S}{\partial \sigma_x}, && \frac{\partial v}{\partial y}=\frac{\partial V}{\partial \sigma_y}+\lambda\frac{\partial S}{\partial \sigma_y},\\
&\frac{\partial w}{\partial z}=\frac{\partial V}{\partial \sigma_z}+\lambda\frac{\partial S}{\partial \sigma_z}, && \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\frac{\partial V}{\partial \tau_{yz}}+\lambda\frac{\partial S}{\partial \tau_{yz}},\\
&\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial V}{\partial \tau_{xz}}+\lambda\frac{\partial S}{\partial \tau_{xz}}, && \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial V}{\partial \tau_{xy}}+\lambda\frac{\partial S}{\partial \tau_{xy}},
\end{aligned}\right\}\tag{76}
$$

以及邊界條件:

$$\text{在 } S_u \text{ 上：}\qquad u=\bar{u},\quad v=\bar{v},\quad w=\bar{w},\tag{77a}$$

$$\text{在 } S_\sigma \text{ 上：}\qquad p_x=\bar{p}_x,\quad p_y=\bar{p}_y,\quad p_z=\bar{p}_z.\tag{77b}$$

證. 將 L 的算式 (73) 代入 (74) 式,並將算子 δ 作用於積分號內的各函數,稍加整理後,我們得

$$
\begin{aligned}
&\iiint_V\left\{\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+X\right)\delta u+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+Y\right)\delta v+\right.\\
&\qquad\left.+\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+Z\right)\delta w\right\}dV+\\
&+\iiint_V\left\{u\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{xy}}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)+v\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{yz}}{\partial z}\right)+\right.\\
&\qquad\left.+w\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)\right\}dV+\\
&+\iiint_V\left\{\frac{\partial V}{\partial\sigma_x}\delta\sigma_x+\frac{\partial V}{\partial\sigma_y}\delta\sigma_y+\frac{\partial V}{\delta\sigma_z}\delta\sigma_z+\frac{\partial V}{\partial\tau_{yz}}\delta\tau_{yz}+\frac{\partial V}{\partial\tau_{xz}}\delta\tau_{xz}+\right.\\
&\qquad\left.+\frac{\partial V}{\partial\tau_{xy}}\delta\tau_{xy}\right\}dV-\\
&-\iint_{S_u}(\bar{u}\delta p_x+\bar{v}\delta p_y+\bar{w}\delta p_z)\,dS-\iint_{S_\sigma}(u\delta p_x+v\delta p_y+w\delta p_z)\,dS-\\
&-\iint_{S_\sigma}\Big[(p_x-\bar{p}_x)\,\delta u+(p_y-\bar{p}_y)\,\delta v+(p_z-\bar{p}_z)\,\delta w\Big]\,dS=0.
\end{aligned}\tag{78}
$$

因爲 u, v, w 的變分不受任何限制,所以從 (78) 式第一個積分便得方程 (75), 從 (78) 式最後一個積分得方程 (77b). 於是 (78) 式剩下的積分爲

$$\iiint_V \left\{ u\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{xy}}{\partial y}+\frac{\partial\delta\tau_{xz}}{\partial z}\right)+v\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{yz}}{\partial z}\right)+\right.$$

$$\left. + w\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)\right\} dV +$$

$$+\iiint_V \left\{ \frac{\partial V}{\partial\sigma_x}\delta\sigma_x+\frac{\partial V}{\partial\sigma_y}\delta\sigma_y+\frac{\partial V}{\partial\sigma_z}\delta\sigma_z+\frac{\partial V}{\partial\tau_{yz}}\delta\tau_{yz}+\frac{\partial V}{\partial\tau_{xy}}\delta\tau_{xz}+\right.$$

$$\left.+\frac{\partial V}{\partial\tau_{xy}}\delta\tau_{xy}\right\} dV - \iint_{S_u}(\bar{u}\delta p_x+\bar{v}\delta p_y+\bar{w}\delta p_z)\,dS -$$

$$-\iint_{S_u}(u\delta p_x+v\delta p_y+w\delta p_z)\,dS = 0\,. \tag{79}$$

利用公式（13），上式化爲

$$\iiint_V \left\{ \left(\frac{\partial V}{\partial\sigma_x}-\frac{\partial u}{\partial x}\right)\delta\sigma_x+\left(\frac{\partial V}{\partial\sigma_y}-\frac{\partial v}{\partial y}\right)\delta\sigma_y+\left(\frac{\partial V}{\partial\sigma_z}-\frac{\partial w}{\partial z}\right)\delta\sigma_z+\right.$$

$$+\left(\frac{\partial V}{\partial\tau_{yz}}-\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)\delta\tau_{yz}+\left(\frac{\partial V}{\partial\tau_{xz}}-\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\delta\tau_{xz}+$$

$$\left.+\left(\frac{\partial V}{\partial\tau_{xy}}-\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\delta\tau_{xy}\right\} dV +$$

$$+\iint_{S_u}\Big[(u-\bar{u})\,\delta p_x+(v-\bar{v})\,\delta p_y+(w-\bar{w})\,\delta p_z\Big]\,dS = 0\,. \tag{80}$$

如果應力分量的變分不受任何限制，那末從（80）式的第一個積分將導得（32）式。 但是在這裏應力分量和它們的變分必須適合屈服條件（67），所以

$$\frac{\partial S}{\partial\sigma_x}\delta\sigma_x+\frac{\partial S}{\partial\sigma_y}\delta\sigma_y+\frac{\partial S}{\partial\sigma_z}\delta\sigma_z+\frac{\partial S}{\partial\tau_{yz}}\delta\tau_{yz}+\frac{\partial S}{\partial\tau_{xz}}\delta\tau_{xz}+\frac{\partial S}{\partial\tau_{xy}}\delta\tau_{xy}=0\,. \tag{81}$$

從（80）式的第一個積分和（81）式，可以導得

$$\frac{\dfrac{\partial u}{\partial x}-\dfrac{\partial V}{\partial\sigma_x}}{\dfrac{\partial S}{\partial\sigma_x}}=\frac{\dfrac{\partial v}{\partial y}-\dfrac{\partial V}{\partial\sigma_y}}{\dfrac{\partial S}{\partial\sigma_y}}=\frac{\dfrac{\partial w}{\partial z}-\dfrac{\partial V}{\partial\sigma_z}}{\dfrac{\partial S}{\partial\sigma_z}}=\frac{\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}-\dfrac{\partial V}{\partial\tau_{yz}}}{\dfrac{\partial S}{\partial\tau_{yz}}}=$$

$$=\frac{\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}-\dfrac{\partial V}{\partial\tau_{xz}}}{\dfrac{\partial S}{\partial\tau_{xz}}}=\frac{\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}-\dfrac{\partial V}{\partial\tau_{xy}}}{\dfrac{\partial S}{\partial\tau_{xy}}}\,. \tag{82}$$

設上式中每一個分數都等於 λ，便得方程（76）.

應力分量的變分雖然必須適合（81）式，但是可以證明，δp_x，δp_y，δp_z 仍是綫性無關的，因此從（80）式的最後一個積分可以導得方程（77b）.

由此可知變分式（74）在所設條件下相當於方程（75），（76），（77）.

在我們的廣義餘能原理中，u，v，w，σ_x，σ_y，σ_z，τ_{yz}，τ_{xz}，τ_{xy} 等十五個函數中，除應力分量以及它們的變分必須適合屈服條件外，不再受其他限制，由此可見廣義餘能原理是受範性體力學中一個很普遍的原理．如果給 u，v，w，σ_x，σ_y，σ_z，τ_{yz}，τ_{xz}，τ_{xy} 以種種可能的限制，那末我們便可從廣義餘能原理引出一系列較為特殊的但仍相當普遍的變分原理.

例如，假定應力分量除適合屈服條件外再適合平衡方程（75）；在這個條件下，L 的的算式（73）簡化為

$$L = V\,. \tag{83}$$

這樣我們得

餘能原理　在所有適合屈服條件、平衡方程以及應力邊界條件的應力狀態中，眞正的應力狀態適合下列變分式：

$$\delta\left\{\iiint_V V\,dV - \iint_{S_u}(\bar{u}p_x + \bar{v}p_y + \bar{w}p_z)\,dS\right\} = 0\,. \tag{84}$$

對於一個沒有彈性應變的受範性體，從（68）式可知 $V = 0$．對於這種受範性體，變分式（84）化為

$$\delta\iint_{S_u}(\bar{u}p_x + \bar{v}p_y + \bar{w}p_z)\,dS = 0\,. \tag{85}$$

此式表明，Sadowsky 的最大受範性阻抗原理可推廣到一切（依照我們規定的應變與應力的關係）無彈性應變的受範性體．［再參考公式（104）］.

（三）　廣義勢能原理

設

$$H = \sigma_x\frac{\partial u}{\partial x} + \sigma_y\frac{\partial v}{\partial y} + \sigma_z\frac{\partial w}{\partial z} + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \\ + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - (Xu + Yv + Zw) - V\,. \tag{86}$$

和以前一樣，我們仍舊稱 H 爲物體中單位體積的廣義勢能.

採用與第二節第（二）小節相同的方法，我們可從上一小節的廣義餘能原理引出廣義勢能原理如下:

廣義勢能原理 設 u, v, w 以及它們的變分不受任何限制：設 $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 以及它們的變分除必須適合屈服條件（67）之外，不再受其他限制；那末下面的變分式:

$$\delta\left\{\iiint_V H\,dV-\iint_{S_\sigma}(u\bar{p}_x+v\bar{p}_y+w\bar{p}_z)dS-\right.$$
$$\left.-\iint_{S_u}\Big[p_x(u-\bar{u})+p_y(v-\bar{v})+p_z(w-\bar{w})\Big]dS\right\}=0 \tag{87}$$

相當於方程（75），（76），（77）.

廣義勢能原理亦是範性體力學中的一個很普遍的原理. 如果我們給 $u, v, w, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 以種種可能的限制，那末我們便可從廣義勢能原理引出一系列較爲特殊的但相當普遍的變分原理.

例如，假定位移和應力已經適合方程（76）：在這個假定下，

$$U=\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\sigma_z\frac{\partial w}{\partial z}+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+$$
$$+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-V \tag{88}$$

便是物體中單位體積的應變能. 這樣，廣義勢能原理亦就化爲平常所說的勢能原理.

（四） 流動式應變與應力的關係 這類受範性體的一般變分原理

在建立流動式應變與應力的關係時，我們將作如下的假定. 茲在外文字母上加一“·”以表示該函數的變化率. 設物體的屈服條件爲（67）. 在進入受範性狀態並在加荷載過程中，物體中的應變可分爲彈性應變與受範性應變兩部分. 假定彈性應變與應力的關係仍可用單位體積的餘能 $V(\sigma_x, \sigma_y, \cdots\tau_{xy})$ 表示如(68)式. 於是彈性應變的變化率爲

$$
\left.\begin{aligned}
\dot{e}_x^e &= \frac{\partial^2 V}{\partial \sigma_x^2}\dot{\sigma}_x + \frac{\partial^2 V}{\partial \sigma_x \partial \sigma_y}\dot{\sigma}_y + \frac{\partial^2 V}{\partial \sigma_x \partial \sigma_z}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{yz}}\dot{\tau}_{yz} + \\
&\quad + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{xy}}\dot{\tau}_{xy}, \\
\dot{e}_y^e &= \frac{\partial^2 V}{\partial \sigma_y \partial \sigma_x}\dot{\sigma}_x + \frac{\partial^2 V}{\partial \sigma_y^2}\dot{\sigma}_y + \frac{\partial^2 V}{\partial \sigma_y \partial \sigma_z}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{yz}}\dot{\tau}_{yz} + \\
&\quad + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{xy}}\dot{\tau}_{xy}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{\gamma}_{xy}^e &= \frac{\partial^2 V}{\partial \tau_{xy} \partial \sigma_x}\dot{\sigma}_x + \frac{\partial^2 V}{\partial \tau_{xy} \partial \sigma_y}\dot{\sigma}_y + \frac{\partial^2 V}{\partial \tau_{xy} \partial \sigma_z}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \tau_{xy} \partial \tau_{yz}}\dot{\tau}_{yz} + \\
&\quad + \frac{\partial^2 V}{\partial \tau_{xy} \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \tau_{xy}^2}\dot{\tau}_{xy}.
\end{aligned}\right\} \tag{89}
$$

爲了書寫方便起見，命

$$
\begin{aligned}
\overline{V} = \dot{\sigma}_x &\left\{\frac{1}{2}\frac{\partial^2 V}{\partial \sigma_x^2}\dot{\sigma}_x + \frac{\partial^2 V}{\partial \sigma_x \partial \sigma_y}\dot{\sigma}_y + \frac{\partial^2 V}{\partial \sigma_x \partial \sigma_z}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{yz}}\dot{\tau}_{yz} + \right. \\
&\qquad \left. + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \sigma_x \partial \tau_{xy}}\dot{\tau}_{xy}\right\} + \\
&+ \dot{\sigma}_y\left\{\frac{1}{2}\frac{\partial^2 V}{\partial \sigma_y^2}\dot{\sigma}_y + \frac{\partial^2 V}{\partial \sigma_y \partial \sigma_z}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{yz}}\dot{\tau}_{yz} + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \sigma_y \partial \tau_{xy}}\dot{\tau}_{xy}\right\} + \\
&+ \dot{\sigma}_z\left\{\frac{1}{2}\frac{\partial^2 V}{\partial \sigma_z^2}\dot{\sigma}_z + \frac{\partial^2 V}{\partial \sigma_z \partial \tau_{yz}}\dot{\tau}_{yz} + \frac{\partial^2 V}{\partial \sigma_z \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \sigma_z \partial \tau_{xy}}\dot{\tau}_{xy}\right\} + \\
&+ \dot{\tau}_{yz}\left\{\frac{1}{2}\frac{\partial^2 V}{\partial \tau_{yz}^2}\dot{\tau}_{yz} + \frac{\partial^2 V}{\partial \tau_{yz} \partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \tau_{yz} \partial \tau_{xy}}\dot{\tau}_{xy}\right\} + \\
&+ \dot{\tau}_{xz}\left\{\frac{1}{2}\frac{\partial^2 V}{\partial \tau_{xz}^2}\dot{\tau}_{xz} + \frac{\partial^2 V}{\partial \tau_{xz} \partial \tau_{xy}}\dot{\tau}_{xy}\right\} + \frac{1}{2}\frac{\partial^2 V}{\partial \tau_{xy}^2}\dot{\tau}_{xy}^2;
\end{aligned} \tag{90}
$$

則公式（89）可以縮寫爲

$$
\left.\begin{aligned}
\dot{e}_x^e &= \frac{\partial \overline{V}}{\partial \dot{\sigma}_x}, & \dot{e}_y^e &= \frac{\partial \overline{V}}{\partial \dot{\sigma}_y}, & \dot{e}_z^e &= \frac{\partial \overline{V}}{\partial \dot{\sigma}_z}, \\
\dot{\gamma}_{yz}^e &= \frac{\partial \overline{V}}{\partial \dot{\tau}_{yz}}, & \dot{\gamma}_{xz}^e &= \frac{\partial \overline{V}}{\partial \dot{\tau}_{xz}}, & \dot{\gamma}_{xy}^e &= \frac{\partial \overline{V}}{\partial \dot{\tau}_{xy}}.
\end{aligned}\right\} \tag{91}
$$

我們再假定受範性應變的變化率與應力的關係爲

$$\left.\begin{aligned}\dot{e}^p_x=\mu\frac{\partial S}{\partial\sigma_x},\qquad \dot{e}^p_y=\mu\frac{\partial S}{\partial\sigma_y},\qquad \dot{e}^p_z=\mu\frac{\partial S}{\partial\sigma_z},\\ \dot{\gamma}^p_{yz}=\mu\frac{\partial S}{\partial\tau_{yz}},\qquad \dot{\gamma}^p_{xz}=\mu\frac{\partial S}{\partial\tau_{xz}},\qquad \dot{\gamma}^p_{xy}=\mu\frac{\partial S}{\partial\tau_{xy}},\end{aligned}\right\}\tag{92}$$

其中 μ 爲一未定函數. 將 (91), (92) 兩式相加,我們最後得應變的變化率與應力及其變化率的關係如下:

$$\left.\begin{aligned}\dot{e}_x=\frac{\partial\bar{V}}{\partial\dot{\sigma}_x}+\mu\frac{\partial S}{\partial\sigma_x},\quad \dot{e}_y=\frac{\partial\bar{V}}{\partial\dot{\sigma}_y}+\mu\frac{\partial S}{\partial\sigma_y},\quad \dot{e}_z=\frac{\partial\bar{V}}{\partial\dot{\sigma}_z}+\mu\frac{\partial S}{\partial\sigma_z},\\ \dot{\gamma}_{yz}=\frac{\partial\bar{V}}{\partial\dot{\tau}_{yz}}+\mu\frac{\partial S}{\partial\tau_{yz}},\quad \dot{\gamma}_{xz}=\frac{\partial\bar{V}}{\partial\dot{\tau}_{xz}}+\mu\frac{\partial S}{\partial\tau_{xz}},\quad \dot{\gamma}_{xy}=\frac{\partial\bar{V}}{\partial\dot{\tau}_{xy}}+\mu\frac{\partial S}{\partial\tau_{xy}}.\end{aligned}\right\}\tag{93}$$

這個流動式的關係 (93) 與以前的變形式的關係 (71) 十分相似.

例如,若取

$$\left.\begin{aligned}V&=\frac{1}{2E}(\sigma_x+\sigma_y+\sigma_z)^2+\frac{1}{2G}(\tau_{yz}^2+\tau_{xz}^2+\tau_{xy}^2-\sigma_y\sigma_z-\sigma_z\sigma_x-\sigma_x\sigma_y),\\ S&=\frac{1}{2G}\left\{\frac{1}{6}\left[(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+(\sigma_x-\sigma_y)^2\right]+\tau_{yz}^2+\tau_{xz}^2+\tau_{xy}^2\right\},\end{aligned}\right\}\tag{94}$$

那末 (94) 式便化爲 Prandtl-Reuss 的關係式. v. Mises 曾取 V 等於零, S 爲應力分量的二次式來研究過晶體的受範性力學.

命

$$L'=\left(\frac{\partial\dot{\sigma}_x}{\partial x}+\frac{\partial\dot{\tau}_{xy}}{\partial y}+\frac{\partial\dot{\tau}_{xz}}{\partial z}+\dot{X}\right)\dot{u}+\left(\frac{\partial\dot{\tau}_{xy}}{\partial x}+\frac{\partial\dot{\sigma}_y}{\partial y}+\frac{\partial\dot{\tau}_{yz}}{\partial z}+\dot{Y}\right)\dot{v}+$$
$$+\left(\frac{\partial\dot{\tau}_{xz}}{\partial x}+\frac{\partial\dot{\tau}_{yz}}{\partial y}+\frac{\partial\dot{\sigma}_z}{\partial z}+\dot{Z}\right)\dot{w}+\bar{V},\tag{95}$$

$$H'=\dot{\sigma}_x\frac{\partial\dot{u}}{\partial x}+\dot{\sigma}_y\frac{\partial\dot{v}}{\partial y}+\dot{\sigma}_z\frac{\partial\dot{w}}{\partial z}+\dot{\tau}_{yz}\left(\frac{\partial\dot{v}}{\partial z}+\frac{\partial\dot{w}}{\partial y}\right)+\dot{\tau}_{xz}\left(\frac{\partial\dot{u}}{\partial z}+\frac{\partial\dot{w}}{\partial x}\right)+$$
$$+\dot{\tau}_{xy}\left(\frac{\partial\dot{u}}{\partial y}+\frac{\partial\dot{v}}{\partial x}\right)-(\dot{X}\dot{u}+\dot{Y}\dot{v}+\dot{Z}\dot{w})-\bar{V}.\tag{96}$$

方程 (93), (95), (96) 和方程 (71), (73), (86) 的結構相同. 因此採用與上兩小節相同的方法,我們不難證明下列兩個變分原理:

廣義餘能原理　設 $\dot{u}$, $\dot{v}$, $\dot{w}$ 以及它們的變分不受任何限制；設 $\dot{\sigma}_x$, $\dot{\sigma}_y$, $\dot{\sigma}_z$, $\dot{\tau}_{yz}$, $\dot{\tau}_{xz}$, $\dot{\tau}_{xy}$ 以及它們的變分，除必須適合屈服條件

$$\frac{\partial S}{\partial \sigma_x}\dot{\sigma}_x+\frac{\partial S}{\partial \sigma_y}\dot{\sigma}_y+\frac{\partial S}{\partial \sigma_z}\dot{\sigma}_z+\frac{\partial S}{\partial \tau_{yz}}\dot{\tau}_{yz}+\frac{\partial S}{\partial \tau_{xz}}\dot{\tau}_{xz}+\frac{\partial S}{\partial \tau_{xy}}\dot{\tau}_{xy}=0 \tag{97}$$

外，不再受其他限制；設 σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy} 的變分爲零；那末下面的變分式：

$$\delta\left\{\iiint_V L'\,dV-\iint_{S_u}(\bar{\dot{u}}\dot{p}_x+\bar{\dot{v}}\dot{p}_y+\bar{\dot{w}}\dot{p}_z)\,dS-\right.$$
$$\left.-\iint_{S_\sigma}\left[\dot{u}\,(\dot{p}_x-\bar{\dot{p}}_x)+\dot{v}\,(\dot{p}_y-\bar{\dot{p}}_y)+\dot{w}(\dot{p}_z-\bar{\dot{p}}_z)\right]dS\right\}=0 \tag{98}$$

相當於平衡方程：

$$\left.\begin{aligned}
&\frac{\partial \dot{\sigma}_x}{\partial x}+\frac{\partial \dot{\tau}_{xy}}{\partial y}+\frac{\partial \dot{\tau}_{xz}}{\partial z}+\dot{X}=0,\\
&\frac{\partial \dot{\tau}_{xy}}{\partial x}+\frac{\partial \dot{\sigma}_y}{\partial y}+\frac{\partial \dot{\tau}_{yz}}{\partial z}+\dot{Y}=0,\\
&\frac{\partial \dot{\tau}_{xz}}{\partial x}+\frac{\partial \dot{\tau}_{yz}}{\partial y}+\frac{\partial \dot{\sigma}_z}{\partial z}+\dot{Z}=0.
\end{aligned}\right\} \tag{99}$$

應變與應力的關係式 (93)，以及邊界條件：

$$\text{在 } S_u \text{ 上：}\quad \dot{u}=\bar{\dot{u}},\quad \dot{v}=\bar{\dot{v}},\quad \dot{w}=\bar{\dot{w}}, \tag{100a}$$

$$\text{在 } S_\sigma \text{ 上：}\quad \dot{p}_x=\bar{\dot{p}}_x,\quad \dot{p}_y=\bar{\dot{p}}_y,\quad \dot{p}_z=\bar{\dot{p}}_z. \tag{100b}$$

廣義勢能原理　設 $\dot{u}$, $\dot{v}$, $\dot{w}$ 以及它們的變分不受任何限制；設 $\dot{\sigma}_x$, $\dot{\sigma}_y$, $\dot{\sigma}_z$, $\dot{\tau}_{yz}$, $\dot{\tau}_{xz}$, $\dot{\tau}_{xy}$ 以及它們的變分，除必須適合屈服條件 (97) 外，不再受其他限制；設 σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy} 的變分爲零；那末下面的變分式：

$$\delta\left\{\iiint_V H'\,dV-\iint_{S_\sigma}(\dot{u}\bar{\dot{p}}_x+\dot{v}\bar{\dot{p}}_y+\dot{w}\bar{\dot{p}}_z)\,dS-\right.$$
$$\left.-\iint_{S_u}\left[\dot{p}_x\,(\dot{u}-\bar{\dot{u}})+\dot{p}_y\,(\dot{v}-\bar{\dot{v}})+\dot{p}_z\,(\dot{w}-\bar{\dot{w}})\right]dS\right\}=0 \tag{101}$$

相當於方程 (99), (93) 和 (100).

公式 (98) 和 (101) 是流動式受範性體力學中很普遍的變分原理. 如果我們給應變的變化率和應力的變化率以種種限制,那末我們便可從這兩個原理引出一系列較爲特殊的但仍相當普遍的變分原理.

例如,假定應力變化率已經適合平衡方程 (99) 和邊界條件 (100b), 那末 L' 的算式 (96) 簡化爲

$$L' = \overline{V}. \tag{102}$$

而廣義餘能原理 (98) 亦就簡化爲普通的餘能原理:

$$\delta\left\{\iiint_V \overline{V}\,dV - \iint_{S_u} (\dot{u}\dot{p}_x + \dot{v}\dot{p}_y + \dot{w}\dot{p}_z)\,dS\right\} = 0. \tag{103}$$

這個變分式是 Sadowsky 的最大受範性阻抗原理的一個很自然的推廣.

又例如在變分原理 (101) 中假定方程 (93) 已經成立,那末可以證明,這個變分原理便化爲平常的勢能原理.

附 錄

設 $u', v', w', \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ 爲任意九個連續函數,則

$$\begin{aligned}\iiint_V \Big\{ & u'\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\right) + v'\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}\right) + \\ & + w'\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z}\right) + \sigma_x\frac{\partial u'}{\partial x} + \sigma_y\frac{\partial v'}{\partial y} + \sigma_z\frac{\partial w'}{\partial z} + \\ & + \tau_{yz}\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right) + \tau_{xz}\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right) + \tau_{xy}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\Big\}\,dx\,dy\,dz = \\ & = \iint_S (u'p_x + v'p_y + w'p_z)\,dS,\end{aligned} \tag{104}$$

其中

$$\left.\begin{aligned} p_x &= \sigma_x\cos(nx) + \tau_{xy}\cos(ny) + \tau_{xy}\cos(nz), \\ p_y &= \tau_{xy}\cos(nx) + \sigma_y\cos(ny) + \tau_{yz}\cos(nz), \\ p_z &= \tau_{xz}\cos(nx) + \tau_{yz}\cos(ny) + \sigma_z\cos(nz). \end{aligned}\right\} \tag{105}$$

因爲（104）式的左端可以寫爲

$$\iiint_V \left\{ \frac{\partial}{\partial x}(u'\sigma_x + v'\tau_{xy} + w'\tau_{xz}) + \frac{\partial}{\partial y}(u'\tau_{xy} + v'\sigma_y + w'\tau_{yz}) + \right.$$
$$\left. + \frac{\partial}{\partial z}(u'\tau_{xz} + v'\tau_{yz} + w'\sigma_z) \right\} dx\,dy\,dz,$$

利用高斯–奥斯脱洛葛拉特斯基的積分定理，可知

$$\left.\begin{aligned}
&\iiint_V \frac{\partial}{\partial x}(u'\sigma_x + v'\tau_{xy} + w'\tau_{xz})\,dx\,dy\,dz = \\
&\qquad = \iint_S \Big[u'\sigma_x \cos(nx) + v'\tau_{xy}\cos(ny) + w'\tau_{xz}\cos(nz)\Big]\,dS\,, \\
&\iiint_V \frac{\partial}{\partial y}(u'\tau_{xy} + v'\sigma_y + w'\tau_{yz})\,dx\,dy\,dz = \\
&\qquad = \iint_S \Big[u'\tau_{yx}\cos(nx) + v'\sigma_y\cos(ny) + w'\tau_{yz}\cos(nz)\Big]\,dS\,, \\
&\iiint_V \frac{\partial}{\partial z}(u'\tau_{xz} + v'\tau_{yz} + w'\sigma_z)\,dx\,dy\,dz = \\
&\qquad = \iint_S \Big[u'\tau_{xz}\cos(nx) + v'\tau_{yz}\cos(ny) + w'\sigma_z\cos(nz)\Big]\,dS\,.
\end{aligned}\right\} \quad (106)$$

將此三式相加，並注意到定義（105），便可見公式（104）是成立的。

公式（104）在平面上的特例顯然是

$$\iint_S \left\{ u'\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}\right) + v'\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y}\right) + \sigma_x\frac{\partial u'}{\partial x} + \sigma_y\frac{\partial v'}{\partial y} + \right.$$
$$\left. + \tau_{xy}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\right\} dx\,dy = \int_C (p_x u' + v' p_y)\,ds\,, \qquad (107)$$

其中

$$p_x = \sigma_x\cos(nx) + \tau_{xy}\cos(ny)\,, \qquad p_y = \tau_{xy}\cos(nx) + \sigma_y\cos(ny)\,. \qquad (108)$$

在公式（107）中設

$$u' = w^*\frac{\partial w}{\partial x}\,, \qquad v' = w^*\frac{\partial w}{\partial y}\,, \qquad (109)$$

並假定 w^* 在邊界 C 上等於零,則得

$$\iint_S \left\{\frac{\partial w}{\partial x}\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\right)+\frac{\partial w}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}\right)+\sigma_y\frac{\partial^2 w}{\partial x^2}+\sigma_y\frac{\partial^2 w}{\partial y^2}+\right.$$

$$\left.+2\tau_{xy}\frac{\partial^2 w}{\partial x\partial y}\right\}w^*\,dx\,dy+\iint_S\left\{\sigma_x\frac{\partial w^*}{\partial x}\frac{\partial w}{\partial x}+\sigma_y\frac{\partial w^*}{\partial y}\frac{\partial w}{\partial y}+\right.$$

$$\left.+\tau_{xy}\left(\frac{\partial w^*}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial w^*}{\partial y}\frac{\partial w}{\partial x}\right)\right\}dx\,dy=0. \tag{110}$$

設 M_x, M_y, M_{xy}, w 爲 (x, y) 的四個連續函數,其中 w 在區域 S 的邊界 C 上等於零,則

$$\iint_S\left\{-M_x\frac{\partial^2 w}{\partial x^2}-M_y\frac{\partial^2 w}{\partial y^2}+2M_{xy}\frac{\partial^2 w}{\partial x\partial y}+\right.$$

$$\left.+w\left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}\right)\right\}dx\,dy=-\int_C M_n\frac{\partial w}{\partial n}\,ds, \tag{111}$$

其中

$$M_n=M_x\cos^2(nx)+M_y\cos^2(ny)-2M_{xy}\cos(nx)\cos(ny). \tag{112}$$

因爲 (111) 式的左端可寫爲

$$I=\iint_S\left\{\frac{\partial}{\partial x}\left[w\frac{\partial M_x}{\partial x}-M_x\frac{\partial w}{\partial x}+2M_{xy}\frac{\partial w}{\partial y}\right]+\right.$$

$$\left.+\frac{\partial}{\partial y}\left[w\frac{\partial M_y}{\partial y}-M_y\frac{\partial w}{\partial y}-2w\frac{\partial M_{xy}}{\partial x}\right]\right\}dx\,dy, \tag{113}$$

利用高斯–奥斯脫洛葛拉特斯基積分定理,並注意到 w 在邊界 C 上等於零, 便得。

$$I=\int_C\left\{\left[-M_x\frac{\partial w}{\partial x}+2M_{xy}\frac{\partial w}{\partial y}\right]\cos(nx)-M_y\frac{\partial w}{\partial y}\cos(ny)\right\}ds. \tag{114}$$

因爲 w 在邊界上等於零,所以

$$\left.\begin{aligned}\frac{\partial w}{\partial x}&=\frac{\partial w}{\partial n}\cos(nx)+\frac{\partial w}{\partial s}\cos(ny)=\frac{\partial w}{\partial n}\cos(nx),\\ \frac{\partial w}{\partial y}&=\frac{\partial w}{\partial n}\cos(ny)+\frac{\partial w}{\partial s}\cos(sy)=\frac{\partial w}{\partial n}\cos(ny),\end{aligned}\right\} \tag{115}$$

這樣 (114) 式便化爲 (111) 式的右端。

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ON SOME VARIATIONAL PRINCIPLES IN THE THEORY OF ELASTICITY AND THE THEORY OF PLASTICITY

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ABSTRACT

In this paper, some general variational principles in the theory of elasticity and the theory of plasticity are established. Consider an elastic body in equilibrium with small displacement. By regarding $u, v, w, e_x, e_y, e_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ as fifteen independent functions, and letting their variations be free from any restriction, we establish two variational principles, called the principle of generalized complementary energy and the principle of generalized potential energy. Each principle is equivalent to the four sets of fundamental equations of the theory of elasticity, namely, the equations of equilibrium, the stress strain relations, the strain displacement relations and the appropriate boundary conditions. Special cases of these principles are examined. These principles are next expressed in other forms, where $u, v, w, \sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ are regarded as nine independent functions with their variations free from any restrictions.

Next we consider the bending of a thin elastic plate with supported edges under large deflection. By regarding $M_x, M_y, M_{xy}, N_x, N_y, N_{xy}, u, v, w$ as nine independent functions with the restriction that w should vanish along the contour of the plate, we establish a variational principle, called the principle of generalized potential energy, which is equivalent to the three sets of fundamental equations in the theory of bending of thin plate, namely, the equations of equilibrium, the displacement stress relations (strain stress relations) and the appropriate boundary conditions. This principle is next expressed in another form which is more convenient for application. As an illustration, von Kármán's equations for the large deflection of thin plate are derived from this principle. In von Kármán's equations, one unknown is the deflection and the other unknown is the membrane stress function. Therefore it is impossible to derive von Kármán's equations either from the principle of minimum potential energy or from the principle of complementary energy.

Finally we consider the equilibrium of a plastic body with small displacement. In the case of the deformation type of stress strain relations, we establish two variational principles, each of which is equivalent to the equations of equilibrium, a certain type of stress strain relations and the appropriate boundary conditions. In these variational principles, u, v, w and their variations are free from any restriction, and $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ and their variations satisfy a certain yield condition. In the case of the flow type of stress strain relations, we get two similar variational principles, in which $\dot{u}, \dot{v}, \dot{w}$ and their variations are free from any restriction, $\dot{\sigma}_x, \dot{\sigma}_y, \dot{\sigma}_z, \dot{\tau}_{yz}, \dot{\tau}_{xz}, \dot{\tau}_{xy}$ and their variations satisfy a certain yield condition and $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}$ have no variations.