物 理 學 報

第 10 卷 第 1 期　1954 年 3 月

片流邊界層中氣流及熱轉移*

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一. 基本方程之變換

沿一平面或略爲彎曲邊界的二維片流壓縮性邊界層的微分方程用通用的符號表示是[1]:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0, \tag{1}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\quad 0=-\frac{\partial p}{\partial y}, \tag{2}$$

$$\begin{aligned}\rho\frac{\partial (C_p T)}{\partial t}&+\rho u\frac{\partial (C_p T)}{\partial x}+\rho v\frac{\partial (C_p T)}{\partial y}\\&=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2.\end{aligned} \tag{3}$$

設物態方程是理想氣體的物態方程:

$$p=R\rho T. \tag{4}$$

考慮下面的一般邊界條件:

$$\left.\begin{aligned}&u=0,\quad v=V(x,t),\quad T=T_w(x,t) &&\text{在 } y=0;\\&u=U(x,t),\quad T=T_1(x,t) &&\text{在 } y=\infty.\end{aligned}\right\} \tag{5}$$

此處我們把吸吮的作用考慮在內. 當然速度 $V(x,t)$ 必須保持遠較主流中的速度 $U(x,t)$ 爲小,致使邊界層的基本假設依然成立.

先考慮 (1). 在極限 o 及 y 之間對 y 積分,得出

* 1953 年 12 月 20 日收到.

$$\frac{\partial}{\partial t}\int_0^y \rho\, dy + \frac{\partial}{\partial x}\int_0^y \rho u\, dy + \rho v = \rho_w V, \tag{6}$$

此處 ρ_w 代表壁上的密度. 應用符號

$$\int_0^y \rho\, dy = \rho_s Y, \tag{7}$$

此處 ρ_s 是某一標準常數密度. 可以將 (6) 式寫作

$$\rho_s \frac{\partial Y}{\partial t} + \frac{\partial}{\partial x}\int_0^y \rho u\, dy + \rho v = \rho_w V. \tag{6'}$$

但注意

$$\int_0^y \rho u\, dy = \rho_s \left[Y_u - \int_0^Y Y \frac{\partial u}{\partial Y}\, dY \right].$$

我們引用坐標變換

$$X(x, y, t) = x, \quad Y(x, y, t) = \frac{1}{\rho_s}\int_0^y \rho\, dy, \quad \Theta(x, y, t) = t. \tag{8}$$

並引用符號

$$\bar{u}(X, Y, \Theta) = u, \quad \rho_s \bar{v}(X, Y, \Theta) = \rho u + \rho_s \frac{\partial Y}{\partial t} + \rho_s u \frac{\partial Y}{\partial x} - \rho_w V. \tag{9}$$

注意

$$\bar{v}(X, o, \Theta) = \int_0^y \frac{\partial \rho}{\partial t}\, dy \bigg|_{y=0} = 0.$$

於是

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial y} = \frac{\rho}{\rho_s}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial t} = \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} + \frac{\partial}{\partial \Theta},$$

而方程 (6') 可以寫作

$$\bar{v} + Y\frac{\partial \bar{u}}{\partial X} - \frac{\partial}{\partial X}\int_0^Y Y \frac{\partial \bar{u}}{\partial Y}\, dY = 0.$$

對 Y 求偏微分,得出

$$\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} = 0. \tag{10}$$

轉向 (2) 式,在簡單的運算後,得出新坐標中的方程

$$\frac{\partial \bar{u}}{\partial \Theta}+\bar{u}\frac{\partial \bar{u}}{\partial X}+\left(\bar{v}+\frac{\rho_w}{\rho_s}V\right)\frac{\partial \bar{u}}{\partial Y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial Y}\right), \tag{11}$$

此處寫 $p(x, y, t)=P(X, Y, \Theta)$, 並且照顧到

$$\frac{\partial P}{\partial Y}=0,$$

而且我們用符號

$$\bar{\mu}=\frac{\mu\rho}{\rho_s}. \tag{12}$$

同樣地, (4) 可以化為

$$\frac{\partial (C_p T)}{\partial \Theta}+\bar{u}\frac{\partial (C_p T)}{\partial X}+\left(\bar{v}+\frac{\rho_w}{\rho_s}V\right)\frac{\partial (C_p T)}{\partial Y}=\frac{1}{\rho}\frac{\partial P}{\partial \Theta}+\frac{\bar{u}}{\rho}\frac{\partial P}{\partial X}$$
$$+\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\bar{k}\frac{\partial T}{\partial Y}\right)+\frac{\bar{\mu}}{\rho_s}\left(\frac{\partial \bar{u}}{\partial Y}\right)^2, \tag{13}$$

此處

$$\bar{k}=\frac{k\rho}{\rho_s}. \tag{14}$$

將 (11) 乘以 $\bar{u}$ 然後加之於 (13), 得出

$$\frac{\partial\left(C_p T+\frac{\bar{u}^2}{2}\right)}{\partial \Theta}+\bar{u}\frac{\partial\left(C_p T+\frac{\bar{u}^2}{2}\right)}{\partial X}+\left(\bar{v}+\frac{\rho_w}{\sigma_s}V\right)\frac{\partial\left(C_p T+\frac{\bar{u}^2}{2}\right)}{\partial Y}$$
$$=\frac{1}{\rho}\frac{\partial P}{\partial \Theta}+\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\frac{\bar{\mu}}{\sigma}\frac{\partial\left(C_p T+\frac{\bar{u}^2}{2}\right)}{\partial Y}\right)$$
$$+\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\frac{1-\sigma}{\sigma}\bar{\mu}\bar{u}\frac{\partial \bar{u}}{\partial Y}\right)-\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\frac{\bar{\mu}}{\sigma}T\frac{\partial C_p}{\partial Y}\right).$$

假設 C_p 是差不多不規溫度 T 而變,則得到

$$\frac{\partial h}{\partial \Theta}+\bar{u}\frac{\partial h}{\partial X}+\left(\bar{v}+\frac{\rho_w}{\rho}V\right)\frac{\partial h}{\partial Y}$$
$$-\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\frac{\bar{\mu}}{\sigma}\frac{\partial h}{\partial Y}\right)=\frac{1}{\rho}\frac{\partial P}{\partial \Theta}-\frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\frac{1-\sigma}{\sigma}\bar{\mu}\bar{u}\frac{\partial \bar{u}}{\partial Y}\right), \tag{15}$$

此處

$$h = C_p T + \frac{\bar{u}^2}{2}$$

是靜止全能，而 $\sigma = \frac{C_p \mu}{k}$ 是普朗托爾（Prandtl）數.

爲簡便起見，我們假設

$$\frac{\mu}{\mu_s} = \frac{\rho_s}{\rho}, \tag{16}$$

於是 $\bar{\mu} = \frac{\mu\rho}{\rho_s} = \mu_s$, 某一常數；而（11）取狀如下：

$$\frac{\partial \bar{u}}{\partial \Theta} + \bar{u}\frac{\partial \bar{u}}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s} V\right)\frac{\partial \bar{u}}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu_s \frac{\partial^2 \bar{u}}{\partial Y^2}, \tag{17}$$

此處

$$\nu_s = \frac{\mu_s}{\rho_s}. \tag{18}$$

先將在邊界層內任一點的密度 ρ 與相應的（具有同樣 X 的）在邊界層外緣的一點的密度 ρ_1 聯系起來，有

$$\frac{\rho_1}{\rho} = \frac{T}{T_1}. \tag{19}$$

因爲 P 是不視 Y 而變的. 由於在主流中有

$$-\frac{1}{\rho_1}\frac{\partial P}{\partial X} = \frac{\partial U}{\partial \Theta} + U\frac{\partial U}{\partial X}, \tag{20}$$

方程（17）變爲

$$\frac{\partial \bar{u}}{\partial \Theta} + \bar{u}\frac{\partial \bar{u}}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s} V\right)\frac{\partial \bar{u}}{\partial Y} = \left(\frac{T}{T_1}\right)\left(\frac{\partial U}{\partial \Theta} + U\frac{\partial U}{\partial X}\right) + \nu_s \frac{\partial^2 \bar{u}}{\partial Y^2}. \tag{21}$$

一般說來，不考慮能量方程（15）是不能單獨積分運動方程（17）的.

二. 無吸吮作用的平板熱轉移問題

作爲一例，我們考流沿一平板的定常片流壓縮邊界層的熱轉移問題. 只需

假設普朗托爾數是一常數. 由於此時可以假定在整個邊界層內壓力爲一常數, (16) 這一假定相當於假定黏性係數與絕對溫度成正比.

設均勻的主流速度爲 $U^{(0)}$, 板上溫度爲 $T^{(w)}$ 而主流中溫度爲 $T^{(0)}$. 這些都是常數. 並取 $V=0$.

應用主流的 $\mu^{(0)}$ 及 $\rho^{(0)}$ (現在都是常數) 作爲標準量,則有下列微分方程:

$$\frac{\partial \bar{u}}{\partial X}+\frac{\partial \bar{v}}{\partial Y}=0, \tag{22}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial X}+\bar{v}\frac{\partial \bar{u}}{\partial Y}=\frac{\mu^{(0)}}{\rho^{(0)}}\frac{\partial^2 \bar{u}}{\partial Y^2}, \tag{23}$$

$$\bar{u}\frac{\partial h}{\partial X}+\bar{v}\frac{\partial h}{\partial Y}=\frac{\mu^{(0)}}{\rho^{(0)}}\frac{\partial}{\partial Y}\left(\frac{1}{\sigma}\frac{\partial h}{\partial Y}\right)+\frac{\mu^{(0)}}{\rho^{(0)}}\frac{\partial}{\partial Y}\left(\frac{1-\sigma}{\sigma}\bar{u}\frac{\partial h}{\partial Y}\right); \tag{24}$$

及邊界條件

$$\begin{aligned}&\bar{u}=\bar{v}=0 \quad 在\ Y=0, \quad \bar{u}=U^{(0)} \quad 在\ Y=\infty;\\ &h=C_p^{(w)}T^{(w)}=h^{(w)} \ 在\ Y=0, \ h=C_p^{(0)}T^{(0)}+\frac{U^{(0)2}}{2}=h^{(0)} \ 在\ Y=\infty.\end{aligned} \tag{25}$$

引進變數

$$\eta=\frac{1}{2}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}}}\frac{Y}{\sqrt{X}}, \tag{26}$$

並取流函數如

$$\Psi=\sqrt{\frac{U^{(0)}\mu^{(0)}X}{\rho^{(0)}}}f(\eta), \tag{27}$$

則 $\quad \bar{u}=\dfrac{\partial \Psi}{\partial Y}=\dfrac{U^{(0)}}{2}f', \quad v=-\dfrac{\partial \Psi}{\partial X}=\dfrac{1}{2}\sqrt{\dfrac{U^{(0)}\mu^{(0)}}{\rho^{(0)}X}}(\eta f'-f);$

而 (23) 化爲勃拉休斯 (Blasius) 方程[2]

$$f'''+ff''=0, \tag{28}$$

其邊界條件是

$$f=f'=0 \quad 在\ \eta=0, \quad f'=2 \quad 在\ \eta=\infty. \tag{29}$$

再取

$$h = h^{(w)} - (h^{(w)} - h^{(0)})\, g(\eta). \tag{30}$$

則 (24) 化爲

$$\left(\frac{1}{\sigma} g'\right)' + f g' = \frac{U^{(0)2}}{2H}\left(\frac{\sigma-1}{\sigma} f' f''\right), \tag{31}$$

此處

$$H = h^{(0)} - h^{(w)} = \frac{U^{(0)2}}{2} + (C_p^{(0)} T^{(0)} - C_p^{(w)} T^{(w)}). \tag{32}$$

微分方程可以寫爲

$$g'' + \sigma f g' = \frac{U^{(0)2}}{2H}(\sigma - 1) f' f'',$$

假使普朗托爾數 σ 是一常數. f 是由勃拉休斯解給出. 剩下來需要解的邊值問題是關於 g 的:

$$g'' + \sigma f g' = \frac{U^{(0)2}}{2H}(\sigma - 1) f' f'', \tag{33}$$

$$g = 0 \quad 在\ \eta = 0, \quad g = 1 \quad 在\ \eta = \infty. \tag{34}$$

微分方程

$$g'' + \sigma f g' = 0$$

的一解是如

$$g_1 = \int^{\eta} e^{-\sigma \int^{\xi} f d\xi}\, d\xi, \tag{35}$$

而 (33) 的一般解是

$$g = \frac{U^{(0)2}(\sigma-1)}{2H}\int_0^{\eta} \frac{\left[\int_{\eta}^{\xi} e^{-\sigma\int_0^{\xi} f\, ds}\, ds\right]}{e^{-\sigma\int_0^{\xi} f\, ds}} f'(\xi) f''(\xi)\, d\xi$$

$$+ A\int_0^{\eta} e^{-\sigma\int_0^{\xi} f\, ds}\, d\xi + B,$$

其中常數 A 及 B 由邊界條件 (34) 決定:

$$B = 0,$$

$$A=\frac{1-\dfrac{U^{(0)^2}(\sigma-1)}{2H}\displaystyle\int_0^{\infty}\frac{\left[\int_{\infty}^{\xi}e^{-\sigma\int_0^s f\,dt}\,ds\right]}{e^{-\sigma\int_0^{\xi}f\,ds}}\,d\xi}{\displaystyle\int_0^{\infty}e^{-\sigma\int_0^{\xi}f\,ds}\,d\xi}.$$

於是得出

$$g=\frac{U^{(0)^2}(\sigma-1)}{2H}P_\sigma(\eta)+\left(1-\frac{U^{(0)^2}(\sigma-1)}{2H}P_\sigma(\infty)\right)\frac{\alpha_\sigma(\infty)}{\alpha_\sigma(\eta)}, \tag{36}$$

此處

$$\alpha_\sigma(\eta)=\frac{1}{\displaystyle\int_0^{\eta}e^{-\sigma\int_0^{\xi}f\,d\xi}\,d\xi}, \tag{37}$$

$$P_\sigma(\eta)=\int_0^{\eta}\frac{\left[\int_{\eta}^{\xi}e^{-\sigma\int_0^s f\,dt}\,ds\right]}{e^{-\sigma\int_0^{\xi}f\,ds}}f'(\xi)\,f''(\xi)\,d\xi. \tag{38}$$

於是在板面上的溫度導數是

$$\left(\frac{\partial T}{\partial y}\right)_{y=0}=\left(\frac{\rho}{\rho^{(0)}}\frac{\partial T}{\partial Y}\right)_{Y=0}=\frac{\rho^{(w)}}{\rho^{(0)}C_p^{(w)}}\left(\frac{\partial h}{\partial Y}\right)_{Y=0}=\frac{\rho^{(w)}}{\rho^{(0)}C_p^{(w)}}\left(\frac{1}{2}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}X}}\right)Hg'$$

$$=\frac{\rho^{(w)}H}{\rho^{(0)}C_p^{(w)}}\left(\frac{1}{2}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}X}}\right)\left(1-\frac{U^{(0)^2}(\sigma-1)P_\sigma(\infty)}{2H}\right)\alpha_\sigma(\infty); \tag{39}$$

而僅計算一面的具寬度 1 和 長度 l 的面積上的平均傳熱量是

$$Q_{av}=-\frac{1}{l}\int_0^l k^{(w)}\left(\frac{\partial T}{\partial y}\right)_{y=0}dx$$

$$=\frac{k^{(w)}T^{(0)}H}{lT^{(w)}C_p^{(w)}}\left(\sqrt{\frac{U^{(0)}\rho^{(0)}l}{\mu^{(0)}}}\right)\left(P_\sigma(\infty)-\frac{2H}{U^{(0)^2}(\sigma-1)}\right)\frac{U^{(0)^2}(\sigma-1)}{2H}\alpha_\sigma(\infty).$$

於是

$$\frac{lQ_{av}}{k^{(w)}T^{(0)}}=\sqrt{R^{(0)}}\,(\sigma-1)\,\alpha_\sigma(\infty)\left[P_\sigma(\infty)-\frac{2H}{U^{(0)^2}(\sigma-1)}\right]\frac{U^{(0)^2}}{2C_p^{(w)}T^{(w)}},$$

$$=\sqrt{R^{(0)}}\left[\frac{U^{(0)^2}P_\sigma(\infty)(\sigma-1)}{2C_p^{(w)}T^{(w)}}-\frac{H}{C_p^{(w)}T^{(w)}}\right]\alpha_\sigma(\infty),$$

此處

$$R^{(0)} = \frac{U^{(0)}\rho^{(0)} l}{\mu^{(0)}}. \tag{40}$$

但

$$\frac{H}{T^{(w)} C_p^{(w)}} = \frac{U^{(0)2}}{2\, C_p^{(w)} T^{(w)}} + \left(\frac{C_p^{(0)} T^{(0)}}{C_p^{(w)}} - 1\right),$$

因此

$$\frac{l\, Q_{av}}{k^{(w)} T^{(0)}} = \sqrt{R^{(0)}}\, \alpha_\sigma(\infty) \left[\left(1 - \frac{C_p^{(0)} T^{(0)}}{C_p^{(w)}}\right) - \frac{U^{(0)2}}{2\, C_p^{(w)} T^{(w)}} + \frac{U^{(0)2} P_\sigma(\infty)(\sigma - 1)}{2\, C_p^{(w)} T^{(w)}}\right]$$

$$= \sqrt{R^{(0)}}\, \alpha_\sigma(\infty) \left[\left(1 - \frac{C_p^{(0)} T^{(0)}}{C_p^{(w)}}\right) - \frac{U^{(0)2}}{2\, C_p^{(w)} T^{(w)}} (1 - P_\sigma(\infty)(\sigma - 1))\right];$$

並且

$$\frac{U^{(0)2}}{2\, C_p^{(w)} T^{(w)}} = \frac{U^{(0)2}}{2\, C_p^{(0)} T^{(0)}} \left(\frac{C_p^{(0)} T^{(0)}}{C_p^{(w)} T^{(w)}}\right) = \frac{(\gamma - 1)\, U^{(0)2}}{2\, Q^{(0)2}} \left(\frac{C_p^{(0)} T^{(0)}}{C_p^{(w)} T^{(w)}}\right)$$

$$= (\gamma - 1)\, M^{(0)2} \left(\frac{C_p^{(0)} T^{(0)}}{C_p^{(w)} T^{(w)}}\right),$$

因此有

$$\frac{l\, Q_{av}}{k^{(w)} T^{(0)}} = \sqrt{R^{(0)}}\, \alpha_\sigma(\infty) \left\{1 - \left(\frac{C_p^{(0)} T^{(0)}}{C_p^{(w)} T^{(w)}}\right)\left[1 - \frac{\gamma - 1}{2} M^{(0)2} (1 - P_\sigma(\infty)(\sigma - 1))\right]\right\}.$$

假使我們引用一努賽特 (Nusselt) 數

$$N = \frac{l\, Q_{av}}{k^{(w)} T^{(0)}}, \tag{41}$$

得出最後的結果

$$N = \sqrt{R^{(0)}}\, \alpha_\sigma(\infty) \left\{ 1 - \frac{C_p^{(0)} T^{(0)}}{C_p^{(w)} T^{(w)}} \left[1 - \frac{\gamma - 1}{2} M^{(0)2} (1 - P_\sigma(\infty)(1 - \sigma))\right]\right\}. \tag{42}$$

現在來看 $P_\sigma(\infty)$ 如何求出. 首先,由 (28)

$$\int_0^s f\, dt = -\int_0^s \frac{f'''}{f''}\, dt = -[\log f''(s) - \log f''(0)] = -\log \frac{f''(s)}{f''(0)},$$

因此

$$e^{-\sigma \int_0^s f\, dt} = \left(\frac{f''(s)}{f''(0)}\right)^\sigma,$$

而

$$P_\sigma(\infty)=\int_0^\infty\left[\int_\infty^\xi\left(\frac{f''(s)}{f''(0)}\right)^\sigma ds\right]\left(\frac{f''(\xi)}{f''(0)}\right)^{-\sigma}f'(\xi)\,f''(\xi)\,d\xi$$

$$=-\int_0^\infty\left[\int_\xi^\infty(f''(s))^\sigma ds\right](f''(\xi))^{(1-\sigma)}f'(\xi)\,d\xi. \qquad (43)$$

從 f 已知的值．可以容易地求出 (43) 式中積分的近似值，相當於不同的 σ.

在 $\sigma=1$ 時有

$$N=\sqrt{R^{(0)}}\,\alpha_1(\infty)\left\{1-\frac{C_p^{(0)}T^{(0)}}{C_p^{(w)}T^{(w)}}\left(1-\frac{\gamma-1}{2}M^{(0)2}\right)\right\},$$

此處 $\alpha_1(\infty)=0.664$（參見 (37) 式）．故有

$$N=C_D\left\{1-\frac{C_p^{(0)}T^{(0)}}{C_p^{(w)}T^{(w)}}\left(1-\frac{\gamma-1}{2}M^{(0)2}\right)\right\}, \qquad (44)$$

此處 C_p 為摩擦阻力係數．(44) 相當於一推廣的雷諾耳 (Reynolds) 類比．

三．具吸吮作用的平板問題

其次我們考慮沿平板的定常運動，將吸吮作用考慮在內．設板面上的溫度是 $T_w(x)$．設 $U^{(0)}$, $T^{(0)}$ 是主流中的速度及溫度．即以主流中的密度為標準密度．有下列方程：

$$\frac{\partial\bar u}{\partial X}+\frac{\partial\bar v}{\partial Y}=0, \qquad (45)$$

$$\bar u\frac{\partial\bar u}{\partial X}+\left(\bar v+\frac{\rho_w}{\rho^{(0)}}V\right)\frac{\partial\bar u}{\partial Y}=\nu^{(0)}\frac{\partial^2\bar u}{\partial Y^2}, \qquad (46)$$

$$\bar u\frac{\partial h}{\partial X}+\left(\bar v+\frac{\rho_w}{\rho^{(0)}}V\right)\frac{\partial h}{\partial Y}$$

$$=\nu^{(0)}\frac{\partial}{\partial Y}\left(\frac{1}{\sigma}\frac{\partial l}{\partial Y}\right)+\nu^{(0)}\frac{\partial}{\partial Y}\left(\frac{1-\sigma}{\sigma}\bar u\frac{\partial\bar u}{\partial Y}\right), \qquad (47)$$

此處

$$\frac{\rho_w}{\rho^{(0)}}=\frac{T^{(0)}}{T_w}; \qquad (48)$$

有下列邊界條件:

$$\bar{u}=0,\quad \bar{v}=0,\quad h=C_p^{(w)}T_w \quad 在\ Y=0;$$

$$\bar{u}=U^{(0)},\quad h=C_p^{(0)}T^{(0)}+\frac{U^{(0)2}}{2}=h^{(0)} \quad 在\ Y=\infty. \tag{49}$$

要求勃拉休斯型解,必須取

$$T^{(0)}V=-\frac{T_w U^{(0)}}{2}\left(\frac{c^{(0)}}{X}\right)^{1/2}, \tag{50}$$

此處 $c^{(0)}$ 是一個長度. 因此我們是考慮一特殊情況,板面上的溫度和吸吮速度分佈之比是與由板頭計算的距離的平方根成反比; 當然, 解在板頭附近是沒有意義的.

仍取

$$\Psi=\sqrt{\frac{U^{(0)}\mu^{(0)}X}{\rho^{(0)}}}\,f(\eta),$$

此處

$$\eta=\frac{1}{2}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}}}\frac{Y}{\sqrt{X}};$$

引進一個"雷諾耳數"

$$\beta=\sqrt{\frac{U^{(0)}\rho^{(0)}c^{(0)}}{\mu^{(0)}}}, \tag{51}$$

得出方程

$$f'''+(f-\beta)f''=0, \tag{52}$$

及邊界條件

$$f(0)=f'(0)=0,\quad f'(\infty)=2. \tag{53}$$

取

$$\phi=f-\beta, \tag{54}$$

得出勃拉休斯方程

$$\phi'''+\phi\phi''=0, \tag{55}$$

及邊界條件

$$\phi(0)=-\beta,\quad \phi'(0)=0,\quad \phi'(\infty)=2. \tag{56}$$

我們注意, 假使邊界層的基本假設能沿平板的一定長度上成立, 尤其當平板的

溫度比主流溫度要高得多，則常數 $c^{(0)}$ 必須適當地小，卽是說，β 必須是一小數.

下面我們求 $\phi''(0)$ 的近似值，這值與 $\left.\frac{\partial \bar{u}}{\partial Y}\right|_{Y=0}$ 的關係如下：

$$\left.\frac{\partial \bar{u}}{\partial Y}\right|_{Y=0} = \frac{U^0}{\eta}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}X}}\,\phi''(0).$$

設

$$\phi''(0) = q, \tag{57}$$

則有近似式

$$\phi = -\beta + \frac{q\eta^2}{2}.$$

將此放入方程 (55) 內，得出

$$\frac{\phi'''}{\phi''} = \beta\eta - \frac{q\eta^3}{6}.$$

積分，得出

$$\phi'' = q\,e^{\beta\eta - \frac{q\eta^3}{6}}.$$

因此

$$\phi'(\infty) = q\int_0^\infty e^{\beta\eta - \frac{q\eta^3}{6}}\,d\eta = 2. \tag{58}$$

應用在 $\eta = \infty$ 的邊界條件（這是懷耳 (Weyl)[3] 的方法）. 因此我們需要解方程

$$\int_0^\infty e^{\beta\eta - q\frac{\eta^3}{6}}\,d\eta = \frac{2}{q} \tag{59}$$

來求出 g. 展開指數函數 $e^{\beta\eta}$ 並依項積分，得出

$$\frac{\Gamma\left(\frac{1}{3}\right)}{\left(\frac{9}{2}q\right)^{\frac{1}{3}}} + \beta\frac{\Gamma\left(\frac{2}{3}\right)}{\left(\frac{3}{4}\right)^{\frac{1}{3}}q^{\frac{2}{3}}} + \frac{\beta^2}{2}\frac{\Gamma(1)}{\left(\frac{q}{2}\right)} + \cdots = \frac{2}{q}.$$

由於 β 很小，可在方程左方取 $q = 1.328$（相當於 $\beta = 0$），得出

$$q = \frac{1.34}{\left(1 + 1.65\dfrac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\beta + \cdots\right)}.$$

四. 在迴轉體上的邊界層

下面我們指出壓縮性及非壓縮性氣體在迴轉體上的邊界層之間的關係. 由於非壓縮性的情况已有前人作過,這聯系可能是有用的.

以 x 代表沿迴轉體的子午線上的距離, y 代表沿法方向從迴轉體面的距離,其相應的速度分量為 u, v, 假使子午線的曲徑較邊界層的厚度要大的多,運動方程是

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{60}$$

$$0 = -\frac{\partial p}{\partial y}; \tag{61}$$

能量方程是

$$\rho u \frac{\partial (C_p T)}{\partial x} + \rho v \frac{\partial (C_p T)}{\partial y} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

這和平面情况完全一樣(只考慮定常運動).

連續方程是

$$\frac{\partial}{\partial x}(\rho r u) + \frac{\partial}{\partial y}(\rho r v) = 0. \tag{62}$$

對一鈍頭的迴轉體來說 $\frac{u}{r}$ 在前靜止點是有限的,因此可由 $\frac{u}{r_0}$ 代表, r_0 代表由迴轉體面上一點到對稱軸的距離. 於是連續方程變為

$$\frac{\partial}{\partial x}(\rho r_0 u) + \frac{\partial}{\partial y}(\rho r_0 v) = 0. \tag{63}$$

對 r 積分,得出

$$\frac{\partial}{\partial x}\left(r_0 \int_0^y \rho u \, dy\right) + \rho r_0 v = 0.$$

積分常數使 v 在 $y = 0$ 處等於零. 取

$$\rho_s Y = \int_0^y \rho \, dy, \tag{64}$$

則

$$\int_0^y \rho\, u\, dy = \rho_s \left[Y u - \int_0^Y Y \frac{\partial u}{\partial Y}\, d Y \right].$$

取變數變換

$$X(x, y) = x, \quad Y(x, y) = \frac{1}{\rho_s} \int_0^y \rho\, dy, \tag{65}$$

得出

$$\rho\, r_0\, v + \rho_s r_0\, Y \frac{\partial u}{\partial x} - \rho_s\, r_0 \frac{\partial}{\partial x} \int_0^Y Y \frac{\partial u}{\partial Y}\, d Y + \rho_s r_0\, u \frac{\partial Y}{\partial x}$$

$$+ \rho_s \frac{d r_0}{d x} \left(Y u - \int_0^Y Y \frac{\partial u}{\partial Y}\, d Y \right) = 0\,.$$

取

$$\bar{u}(X, Y) = u(x, y), \quad \rho_s\, \bar{v}(X, Y) = \rho\, v + \rho_s u \frac{\partial Y}{\partial X},$$

則有

$$Y \frac{\partial \bar{u}}{\partial x} + \bar{v} - \frac{\partial}{\partial x} \int_0^Y Y \frac{\partial \bar{u}}{\partial Y}\, d Y + \frac{d \ln r_0}{d x} \left(Y u - \int_0^Y Y \frac{\partial \bar{u}}{\partial Y}\, dY \right) = 0.$$

對 Y 取偏微分，得出

$$\frac{\partial (r_0\, \bar{u})}{\partial X} + \frac{\partial (r_0\, \bar{v})}{\partial Y} = 0. \tag{66}$$

這是在一迴轉體面上的非壓縮性邊界層的連續方程[4]．

運動及能量方程可以像二維情況一樣變換．

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ON THE GAS FLOW AND HEAT TRANSFER IN LAMINAR BOUNDARY LAYER FLOW

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ABSTRACT

The equations of the plane, laminar compressible boundary layer flow past a plane or slightly curved wall

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0,$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),$$

$$0=-\frac{\partial p}{\partial y},$$

$$\rho\frac{\partial (C_p T)}{\partial t}+\rho u\frac{\partial}{\partial x}(C_p T)+\rho v\frac{\partial}{\partial y}(C_p T)=\frac{\partial p}{\partial t}+$$

$$+u\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

are considered. The compressible fluid is assumed to be an ideal gas. At the wall, the effect of a small suction is allowed for. From the equation of continuity we find a natural transformation of coordinates

$$X(x,y,t)=x,\quad Y(x,y,t)=\frac{1}{\rho_s}\int_0^y \rho\, dy,\ \Theta(x,y,t)=t,$$

and a change of dependent variables

$$\bar{u}(X,Y,\Theta)=u,\quad \rho_s\bar{v}(X,Y,\Theta)=\rho v+\rho_s\frac{\partial Y}{\partial t}+\rho_s u\frac{\partial Y}{\partial x}-\rho_w V,$$

where ρ_s is some standard density and V is the velocity of suction at the wall. The equations of continuity, motion and energy are transformed respectively into

$$\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} = 0,$$

$$\frac{\partial \bar{u}}{\partial \Theta} + \bar{u}\frac{\partial \bar{u}}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s}V\right)\frac{\partial \bar{u}}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial Y}\right),$$

$$\frac{\partial(C_p T)}{\partial \Theta} + \bar{u}\frac{\partial(C_p T)}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s}V\right)\frac{\partial(C_p T)}{\partial Y} = \frac{1}{\rho}\frac{\partial P}{\partial \Theta} + \frac{\bar{u}}{\rho}\frac{\partial P}{\partial X} +$$

$$+ \frac{1}{\rho_s}\frac{\partial}{\partial Y}\left(\bar{k}\frac{\partial T}{\partial Y}\right) + \frac{\bar{\mu}}{\rho_s}\left(\frac{\partial \bar{u}}{\partial Y}\right)^2,$$

where

$$\bar{\mu} = \frac{\mu\rho}{\rho_s}, \quad \bar{k} = \frac{k\rho}{\rho_s}.$$

If we assume the viscosity of the gas to be directly proportional to the absolute temperature, the equation of motion simplifies to

$$\frac{\partial \bar{u}}{\partial \Theta} + \bar{u}\frac{\partial \bar{u}}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s}V\right)\frac{\partial \bar{u}}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} - \nu_s\frac{\partial^2 \bar{u}}{\partial Y^2},$$

where $\nu_s = \frac{\mu_s}{\rho_s}$. The equation of energy simplifies to

$$\frac{\partial h}{\partial \Theta} + \bar{u}\frac{\partial h}{\partial X} + \left(\bar{v} + \frac{\rho_w}{\rho_s}V\right)\frac{\partial h}{\partial Y} - \nu_s\frac{\partial}{\partial Y}\left(\frac{1}{\sigma}\frac{\partial h}{\partial Y}\right) =$$

$$= \frac{1}{\rho}\frac{\partial P}{\partial \Theta} - \nu_s\frac{\partial}{\partial Y}\left(\frac{1-\sigma}{\sigma}\bar{u}\frac{\partial \bar{u}}{\partial Y}\right),$$

where h denotes the stagnation enthaepy

$$h = C_p T + \frac{\bar{u}^2}{2}$$

and σ is the Prandtl number: $\sigma = \frac{C_p\mu}{k}$.

As an example, the problem of heat transfer in the steady, plane, laminar, compressible boundary layer flow past a semi-infinite plate is considered. The velocity and temperature in the mainstream are given by $U^{(0)}$ and $T^{(0)}$ respectively. The temperature at the wall is given by $T^{(w)}$. The effect of suction is not considered. The viscosity and density in the mainstream are taken as the standard quantities. The equations become

$$\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} = 0,$$

$$\bar{u}\frac{\partial \bar{u}}{\partial X} + \bar{v}\frac{\partial \bar{u}}{\partial Y} = \frac{\mu^{(0)}}{\rho^{(0)}}\frac{\partial^2 \bar{u}}{\partial Y^2},$$

$$\bar{u}\frac{\partial h}{\partial X} + \bar{v}\frac{\partial h}{\partial Y} = \frac{\mu^{(0)}}{\rho^{(0)}}\frac{\partial}{\partial Y}\left(\frac{1}{\sigma}\frac{\partial h}{\partial Y}\right) + \frac{\mu^0}{\rho_0}\frac{\partial}{\partial y}\left(\frac{1-\sigma}{\sigma}\bar{u}\frac{\partial \bar{u}}{\partial Y}\right),$$

and the boundary conditions are

$$\bar{u} = \bar{v} = 0 \quad \text{at } Y = 0, \qquad u = U^{(0)} \quad \text{at } Y = \infty;$$

$$h = C_p^{(w)} T^{(w)} = h^{(w)} \quad \text{at } Y = 0, \quad h = C_p^{(0)} T^{(0)} + \frac{U^{(0)2}}{2} \quad \text{at } Y = \infty.$$

A new variable

$$\eta = \frac{1}{2}\sqrt{\frac{U^{(0)}\rho^{(0)}}{\mu^{(0)}}}\frac{Y}{\sqrt{X}}$$

and a stream function $\Psi = \sqrt{\frac{U^{(0)}\mu^{(0)}X}{\rho^{(0)}}}\, f(\eta)$ are introduced. The equation of motion reduces to the well-known Blasius equation

$$f''' + f f'' = 0$$

with the boundary conditions

$$f = f' = 0 \quad \text{at } \eta = 0, \quad f' = 2 \quad \text{at } \eta = \infty.$$

Further, a function g is introduced such that

$$h = h^{(w)} - (h^{(w)} - h^{(0)})\, g(\eta).$$

The equation of energy becomes

$$\left(\frac{1}{\sigma}g'\right)' + f g' = \frac{U^{(0)2}}{2H}\frac{(\sigma-1)}{\sigma} f' f'',$$

where

$$H = h^{(0)} - h^{(w)}.$$

We consider the case where the Prandtl number is a constant. We find that the average amount of heat ransferred across a plate surface of breadth unity and length l is given by

$$Q_{av}=\frac{k^{(w)}T^{(0)}H}{e\,T^{(w)}}\,C_p^{(w)}\left(\sqrt{\frac{U^{(0)}\,\rho^{(0)}\,l}{\mu^{(0)}}}\right)\left(P_\sigma(\infty)-\frac{2H}{U^{(0)2}(\sigma-1)}\right)\frac{U^{(0)2}(\sigma-1)}{2H}\,\alpha_\sigma(\infty),$$

$$\text{where}\quad \alpha_\sigma(\eta)=\frac{1}{\int_0^\eta e^{-\sigma\int_0^\xi f d\xi}\,d\xi},\ P_\sigma(\eta)=\int_0^\eta\frac{\left[\int_\eta^\xi e^{-\sigma\int_0^\eta f\,dt}\,ds\right]}{e^{-\sigma\int_0^\xi f\,ds}}\,f'(\xi)\,f''(\xi)\,d\xi.$$

Introducing an appropriate Reynolds number

$$R^{(0)}=\frac{U^{(0)}\,\rho^{(0)}\,l}{\mu^{(0)}}$$

and a Nusselt number $N=\dfrac{l\,Q_{av}}{k^{(w)}T^{(0)}}$, we can put the result in the following form:

$$N=\sqrt{R^0}\,\alpha_\sigma(\infty)\left\{1-\frac{C_p^{(0)}\,T^{(0)}}{C_p^{(w)}\,T^{(w)}}\left[1-\frac{\gamma-1}{2}\,M^{(0)2}\,(1-P_\sigma(\infty)\,(1-\sigma))\right]\right\},$$

where $M^{(0)}=\dfrac{U^{(0)}}{a^{(0)}}$ and a_0 is the velocity of sound in the main stream. When $\sigma=1$, we have

$$N=C_D\left\{1-\frac{C_p^{(0)}\,T^{(0)}}{C_p^{(w)}\,T^{(w)}}\left(1-\frac{\gamma-1}{2}\,M^{(0)2}\right)\right\},$$

where C_D is the drag coefficient.

Next, we consider the steady flow past a semi-infinite plate, considering the effect of suction. The following boundary value problem has to be solved:

$$\frac{\partial\bar u}{\partial X}+\frac{\partial\bar v}{\partial Y}=0,$$

$$\bar u\,\frac{\partial\bar u}{\partial X}+\left(\bar v+\frac{\rho_w}{\rho^{(0)}}\,V\right)\frac{\partial\bar u}{\partial Y}=\nu^{(0)}\,\frac{\partial^2\bar u}{\partial Y^2},$$

$$\bar u\,\frac{\partial h}{\partial X}+\left(\bar v+\frac{\rho_w}{\rho^{(0)}}\,V\right)\frac{\partial h}{\partial Y}=\nu^{(0)}\,\frac{\partial}{\partial Y}\left(\frac{1}{\sigma}\,\frac{\partial h}{\partial Y}\right)+\nu^{(0)}\,\frac{\partial}{\partial y}\left(\frac{1-\sigma}{\sigma}\,\bar u\,\frac{\partial\bar u}{\partial Y}\right),$$

$$\bar{u}=\bar{v}=0,\ h=C_p^{(w)}\,T^{(w)}\ \text{ at } Y=0;\ \bar{u}=U^{(0)},\ h=C_p^{(0)}\,T^{(0)}\,\frac{U^{(0)2}}{2}=h^{(0)}\ \text{ at } Y=\infty.$$

In order to find a solution of the Blasius type we assume

$$T^{(0)}\,V=-\frac{T_w U^{(0)}}{2}\left(\frac{c^{(0)}}{X}\right)^{\frac{1}{2}},$$

where $c^{(0)}$ is a constant corresponding to a length. Introducing the variable η and the stream function Ψ as in the previous case, we arrive at the following boundary value problem:

$$f''' + (f-\beta)\,f'' = 0,$$

$$f(0)=f'(0)=0, \qquad f'(\infty)=2.$$

A rough estimate of the velocity gradient at the wall is made by a method proposed by H. Weyl for the original Blasius case.

Finally, a correlation between the compressible, laminar steady boundary-layer flow and the corresponding incompressible flow over a body of revolution is indicated. The procedure is similar to that in the plane case.