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# 軸對稱塑性變形問題*

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## 一. 引 言

在理想塑性體的一般理論中，雖然有數類問題（例如平面變形問題）在一般理論方面業已獲得相當的發展，並且被應用到特殊的具體情況的計算中，但一般的問題是很難化爲適當的邊值問題的. 在蘇聯力學家索珂洛甫斯基所著“塑性理論”[1] 中，所謂靜定的平面變形及平面應力問題得到極爲詳盡的敘述. 在這一類問題中，兩個平衡方程及一個屈伏條件加上適當的應力邊界條件就足以決定在平面上的應力分佈，因此，無需考慮尙在爭論中的應力應變關係.

但在空間問題中，就沒有那樣簡單，即使是在比較簡單的空間問題——軸對稱變形問題，從物理方面來看，都不能擁有這方便之處：問題不能是靜定的，因此必須考慮到應力應變關係.

有一些力學工作者，跟隨哈爾及卡門在 1909 年的建議[2]，引進一個不現實的屈伏條件，藉以形成與平面變形類似的問題. 對於理想塑性體來說，從如此求出的應力分佈不能求出與之適應的速度分佈，而且，應力分佈究竟是否接近眞實情況是無從肯定的.

可以證明，軸對稱的塑性變形問題一般不是雙曲型的邊値問題[3]. 因此，我們不必強求與平面變形問題的類似，而可以從其他方面着手.

在本文中，理想塑性體的軸對稱問題的應力分佈及速度分佈問題是同時被考慮到的. 主要的結果是將應力分量及速度分量用三個函數表示出來. 平衡方程、屈伏條件及應力應變關係（我們採用的是聖維南一米賽斯關係）被化作三個非線性的偏微分方程.

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最後,我們試用所謂湊合解法來處理一些較爲簡單的問題. 當然,這些解的適用性是有其限制的.

## 二. 基本方程

軸對稱變形的平衡方程是

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0, \tag{1}$$

$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}=0. \tag{2}$$

設整個物體處於塑性狀態,則有屈伏條件成立:

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\tau_{zr}^2=6k^2. \tag{3}$$

採用聖維南-米賽斯應力及應變率關係如下(假定無壓縮性):

$$\dot{e}_r=\frac{\partial u}{\partial r}=\frac{1}{3}\lambda(2\sigma_r-\sigma_\theta-\sigma_z), \tag{4}$$

$$\dot{e}_\theta=\frac{u}{r}=\frac{1}{3}\lambda(2\sigma_\theta-\sigma_z-\sigma_r), \tag{5}$$

$$\dot{e}_z=\frac{\partial w}{\partial z}=\frac{1}{3}\lambda(2\sigma_z-\sigma_r-\sigma_\theta), \tag{6}$$

$$\dot{g}_{zr}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=\lambda\tau_{zr}, \tag{7}$$

此處 $u$ 及 $w$ 爲沿 $r$ 及 $z$ 方向的速度分量,以上七式決定七個函數 $\sigma_r$, $\sigma_\theta$, $\sigma_z$, $\tau_{zr}$, $u$, $w$ 及 $\lambda$.

引進函數 $\chi(r, z)$ 使

$$\tau_{zr}=-\frac{1}{r}\frac{\partial \chi}{\partial z}. \tag{8}$$

則由(2)式可以取

$$\sigma_z=\frac{1}{r}\frac{\partial \chi}{\partial r}, \tag{9}$$

而該式卽得到滿足.

另一方面，由 (4) 及 (5) 有

$$\dot{e}_r = \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r\dot{e}_\theta). \tag{10}$$

今取

$$u = -\frac{r\Omega}{3}. \tag{11}$$

則由 (5) 有

$$2\sigma_\theta - \sigma_z - \sigma_r = \frac{3e_\theta}{\lambda} = \frac{3u}{\lambda r} = -\frac{\Omega}{\lambda}. \tag{12}$$

可由 (10)，即

$$\frac{1}{3}\lambda(2\sigma_r - \sigma_\theta - \sigma_z) = \frac{1}{3}\lambda(2\sigma_\theta - \sigma_z - \sigma_r) + \frac{r}{3}\frac{\partial}{\partial r}[\lambda(2\sigma_\theta - \sigma_z - \sigma_r)],$$

得出

$$\frac{\sigma_r - \sigma_\theta}{r} = -\frac{1}{3\lambda}\frac{\partial\Omega}{\partial r}.$$

由此式及 (9)，(12) 得出

$$\sigma_\theta = -\frac{r}{3\lambda}\frac{\partial\Omega}{\partial r} - \frac{\Omega}{\lambda} + \frac{1}{r}\frac{\partial\chi}{\partial r}, \tag{13}$$

$$\sigma_r = -\frac{2r}{3\lambda}\frac{\partial\Omega}{\partial r} - \frac{\Omega}{\lambda} + \frac{1}{r}\frac{\partial\chi}{\partial r}. \tag{14}$$

由 (11) 有

$$\frac{\partial u}{\partial z} = -\frac{r}{3}\frac{\partial\Omega}{\partial z}.$$

但從 (7) 有

$$\frac{\partial w}{\partial r} = -\frac{2\lambda}{r}\frac{\partial\chi}{\partial z} + \frac{r}{3}\frac{\partial\Omega}{\partial z}; \tag{15}$$

另一方面從 (6) 有

$$\frac{\partial w}{\partial z} = \frac{r}{3}\frac{\partial\Omega}{\partial r} + \frac{2}{3}\Omega. \tag{16}$$

而必有

$$\frac{\partial^2 w}{\partial z\,\partial r} = \frac{\partial^2 w}{\partial r\,\partial z}.$$

故得到

$$\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial^2 \Omega}{\partial r^2}-\frac{r}{3}\frac{\partial^2 \Omega}{\partial z^2}=-\frac{2}{r}\frac{\partial}{\partial z}\left(\lambda\frac{\partial \chi}{\partial z}\right). \tag{17}$$

從 (1) 得到方程

$$\frac{2}{\lambda}\left(\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial^2 \Omega}{\partial r^2}\right)+\frac{\partial}{\partial r}\left(\frac{1}{\lambda}\right)\left(\Omega+\frac{2r}{3}\frac{\partial \Omega}{\partial r}\right)$$

$$=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial r}\right)-\frac{1}{r}\frac{\partial^2 \chi}{\partial z^2}. \tag{18}$$

因

$$\sigma_r-\sigma_\theta=-\frac{r}{3\lambda}\frac{\partial \Omega}{\partial r}, \tag{19a}$$

$$\sigma_\theta-\sigma_z=-\frac{r}{3\lambda}\frac{\partial \Omega}{\partial r}-\frac{\Omega}{\lambda}, \tag{19b}$$

$$\sigma_z-\sigma_r=\frac{2r}{3\lambda}\frac{\partial \Omega}{\partial r}+\frac{\Omega}{\lambda}, \tag{19c}$$

屈伏條件變爲

$$\Omega^2+r\Omega\left(\frac{\partial \Omega}{\partial r}\right)+\frac{r^2}{3}\left(\frac{\partial \Omega}{\partial r}\right)^2+\frac{3\lambda^2}{r^2}\left(\frac{\partial \chi}{\partial z}\right)^2=3k^2\lambda^2. \tag{19}$$

因此，我們將軸對稱的塑形變形問題化爲牽涉到三個函數 $\Omega$, $\chi$ 及 $\lambda$ 的非線性偏微分方程 (17), (18), (19). 應力分量及速度分量按 (8), (9), (13), (14), (15), (16) 由 $\Omega$, $\chi$ 及 $\lambda$ 求出.

## 三. 若干特殊問題的解

由於所得非線性偏微分方程頗爲複雜，因此我們尚不能期望先樹立謹嚴的邊值問題然後求其準確解. 現在先考慮用湊合法求出若干解而給以適當的物理意義.

第一種情況是 $\tau_{zr}\equiv 0$, 即

$$\frac{\partial \chi}{\partial z}=0.$$

此時 (17), (18), (19) 分別簡化爲

$$\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial^2 \Omega}{\partial r^2}-\frac{r}{3}\frac{\partial^2 \Omega}{\partial z^2}=0; \tag{20}$$

$$\frac{2}{\lambda}\left(\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial \Omega}{\partial r}\right)+\left[\Omega+\frac{2r}{3}\frac{\partial \Omega}{\partial r}\right]\frac{\partial}{\partial r}\left(\frac{1}{\lambda}\right)=\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right), \tag{21}$$

$$\Omega^2+r\Omega\left(\frac{\partial \Omega}{\partial r}\right)+\frac{r^2}{3}\left(\frac{\partial \Omega}{\partial r}\right)^2=3k^2\lambda^2. \tag{22}$$

1. 最簡單的情况是

$$\Omega=\Omega(r), \qquad w=0.$$

顯然此時有最簡單平面極對稱的情况．此時不用（20）而逕由（16）得出

$$\frac{d\Omega}{dr}=-\frac{2\Omega}{r};$$

其解可以寫作

$$\Omega=-\frac{A}{2r^2}.$$

因此，由（11），

$$u=\frac{A}{6r}. \tag{23}$$

由（22）得到

$$\lambda=\epsilon\frac{A}{6kr^2},$$

此處 $\epsilon$ 是 $A$ 的符號，亦卽 $u$ 的符號，因爲 $\lambda$ 必須是正數．

從

$$\frac{\sigma_r-\sigma_\theta}{r}=-\frac{1}{3\lambda}\frac{d\Omega}{dr}=-\epsilon\frac{2k}{r},$$

代入（1），積分得到

$$\sigma_r=-2k\epsilon\ln\left(\frac{a}{r}\right), \tag{24}$$

此處 $a$ 是一常數，於是從（19 $a$）得出

$$\sigma_\theta=2k\epsilon\left[1-2\ln\left(\frac{a}{r}\right)\right]. \tag{25}$$

這樣應力分佈的解早在納達依的"塑性理論"一書[4]中提出．

2. 但如果 $w \not\equiv 0$ 而 $\frac{\partial w}{\partial z} = \dot{e}_z = \frac{B}{3}$（某一常數），則由（16）有

$$r\frac{d\Omega}{dr} + 2\Omega = B;$$

其解爲

$$\Omega = -\frac{A}{2r^2} + \frac{B}{2} = \frac{1}{2r^2}(Br^2 - A).$$

因此，由（11）得到

$$u = -\frac{1}{6r}(Br^2 - A), \tag{26}$$

而由（22）得到

$$\lambda = \epsilon\frac{A}{6kr^2}\sqrt{1 + 3\frac{B^2}{A^2}r^4}.$$

從

$$\frac{\sigma_r - \sigma_\theta}{r} = -\frac{1}{3\lambda}\frac{d\Omega}{dr} = -\frac{2\epsilon k}{r\sqrt{1+3\frac{B^2}{A^2}r^4}},$$

代入（1）式，積分得到

$$\sigma_r = C - \epsilon k\ln\left(\frac{1+\sqrt{1+\frac{3B^2}{A^2}r^4}}{r^2}\right). \quad (C\ \text{爲待定常數}) \tag{27}$$

而

$$\sigma_\theta = \sigma_r + \frac{2\epsilon k}{\sqrt{1+\frac{3B^2}{A^2}r^4}}. \tag{28}$$

從（6）得出

$$\sigma_z = \sigma_r + \frac{\epsilon k\left(1+\frac{3B}{A}r^2\right)}{\sqrt{1+3\frac{B^2}{A^2}r^4}}. \tag{29}$$

注意，此處得出的解可以用在厚圓筒在內外側面上受到均勻壓力的問題． 設邊界條件爲

$$\sigma_r = -p \qquad \text{在 } r = a,$$

$$\sigma_r = -q \qquad \text{在 } r = b.$$

於是有

$$p - q = \sigma k \ln \frac{b^2}{a^2} \frac{1 + \sqrt{1 + \frac{3B^2}{A^2} a^4}}{1 + \sqrt{1 + \frac{3B^2}{A^2} b^4}}. \tag{30}$$

由之可以確定常數 $A$ 及 $C$. 要確定 $B$ 的值可以事先假定 $\epsilon_z$ 的值.（關於這一問題可以參見文獻 [1]，第 70—73 頁.）

3. 設

$$\Omega = \Omega(z).$$

此即 $\sigma_\theta = \sigma_r$ 的情況. 按照 (20)，

$$\frac{d^2\Omega}{dz^2} = 0,$$

卽是說

$$\Omega = Az + B,$$

此處 $A$ 及 $B$ 是待定常數. 於是從 (22) 得到

$$\lambda = \kappa \frac{Az+B}{\sqrt{3} \cdot k},$$

此處 $\kappa$ 是 $\Omega$ 的符號. 而

$$\varkappa = -\frac{r(Az+B)}{3}. \tag{31}$$

(21) 給出

$$\frac{d}{dr}\left(\frac{1}{r} \frac{d\chi}{dr}\right) = 0,$$

卽

$$\sigma_z = \frac{1}{r} \frac{\partial \chi}{\partial r} = C, \text{ 某一常數}. \tag{32}$$

從 (13), (14) 得出

$$\sigma_\theta = C - \sqrt{3}\, \kappa k, \tag{33}$$

$$\sigma_r = C - \sqrt{3}\, \kappa k. \tag{34}$$

另外，從 (15) (16) 得到

$$w=\frac{2}{3}\left(\frac{Az^2}{2}+Bz\right)+\frac{Ar^2}{6}.$$

值得指出,若我們取速度邊界條件

$$w=0\ \text{在}\ z=0,\quad w=W\ \text{在}\ z=l,\ \text{則有}$$

$$A=0,\quad B=\frac{3W}{2l}.$$

因此,

$$u=-\frac{rW}{2l},\quad w=\frac{Wz}{l}.$$

這卽是迴轉體受均勻壓擠的情况(注意這解只當 $\frac{2}{3}B\leqslant\frac{\sigma_s}{E}$ 才有意義). 從這簡單例子可以看出雖然應力分量 $\sigma_z$, $\sigma_r=\sigma_\theta$ 是由平衡方程(21)及屈伏條件(22)決定的,但由於速度邊界條件之不同可以有不同的速度分佈.

4. 設

$$\Omega=R(r)Z(z).$$

由(20)得到

$$\frac{Z''}{Z}=\frac{3R'+rR''}{Rr}.$$

設

$$\frac{Z''}{Z}=\frac{3R'+rR''}{Rr}=-m^2.$$

則有

$$Z''+m^2Z=0,$$

$$R''+\frac{3}{r}R'+m^2R=0.$$

可以取其中一個特解如

$$\Omega=\frac{C}{r}J_0'(mr)\cos(mz),$$

此處 $J_0$ 是零階貝塞爾函數,而

$$J_0'(x)=-J_1(x).$$

$C$ 是一常數，可以令其爲正數． 於是有

$$u = -\frac{C}{3} J_0'(mr)\cos(mz)\,. \tag{35}$$

從（22）得出

$$\lambda = -\frac{C}{3kr} J_0'(mr)\cos(mz)\left\{1 + mr\frac{J_0(mr)}{J_0'(mr)} + m^2r^2\left[\frac{J_0(mr)}{J_0'(mr)}\right]^2\right\}^{\frac{1}{2}}.$$

因 $\lambda$ 必須是正數，故 $J_0'(mr)$ 及 $\cos(mz)$ 不能變更其符號． 若取

$$0 \leqslant r \leqslant \rho, \qquad -l \leqslant Z \leqslant l,$$

則必須限制

$$ml \leqslant \frac{\pi}{2}, \qquad m\rho \leqslant \alpha_1, \tag{36}$$

此處 $\alpha_1$ 是 $J_0'(x) = 0$ 的第二個根．

由（19a）得到

$$\frac{\sigma_r - \sigma_\theta}{r} = -\frac{k}{r}\frac{q+2}{(1+q+q^2)^{\frac{1}{2}}},$$

此處我們用符號

$$q = mr\frac{J_0(mr)}{J_0'(mr)}\,. \tag{37}$$

則從（1）直接積分得到

$$\sigma_r = -k\int_r^\rho \frac{q+2}{(1+q+q^2)^{\frac{1}{2}}}\frac{dr}{r}, \tag{38}$$

若 $\sigma_r$ 在 $r=\rho$ 處等於零． 故

$$\sigma_\theta = \sigma_r + \frac{k(q+2)}{(1+q+q^2)^{\frac{1}{2}}}\,. \tag{39}$$

而從（19c）得到

$$\sigma_z = \sigma_r + \frac{k(2q+1)}{(1+q+q^2)^{\frac{1}{2}}}\,. \tag{40}$$

上面用湊合解法求出的解相當於一個可能的塑性狀態，即，一圓柱兩端受適

當分佈的負荷所形成的塑性狀態. 任意常數 $m$ 是由總負荷及圓柱的半徑 $\rho$ 表現出來的[5].

第二種情况，假設 $\frac{\partial \chi}{\partial z} \neq 0$ 而 $\Omega =$ 常數. 譬如有一圓柱體，其半徑爲 $r = \rho$. 取

$$\frac{u}{U} = -\frac{r}{\rho}, \tag{41}$$

此處 $U$ 是在 $r = \rho$ 處徑向速度的絕對值，則

$$\Omega = \frac{3U}{\rho}.$$

從 (17) 得出

$$\frac{\partial}{\partial z}\left(\lambda \frac{\partial \chi}{\partial z}\right) = 0$$

或

$$\lambda \frac{\partial \chi}{\partial z} = r\, g(r),$$

其中 $g(r)$ 是 $r$ 的某一函數. 從 (19) 得到

$$\lambda = \frac{1}{\sqrt{3}k} \sqrt{\Omega^2 + 3g^2}.$$

於是

$$\frac{\partial \chi}{\partial z} = \frac{\sqrt{3}k\, r g}{\sqrt{\Omega^2 + 3g^2}}.$$

積分得到

$$\chi = \frac{\sqrt{3}\, k\, r z y}{\sqrt{\Omega^2 + 3y^2}} + \sqrt{3}\, k\, r\, h(r),$$

其中 $h(r)$ 是 $r$的某一函數. 代入 (18) 式得到

$$\Omega \frac{d}{dr}\left(\frac{1}{\sqrt{\Omega^2 + 3y^2}}\right) = z \frac{d^2}{dr^2}\left(\frac{g}{\sqrt{\Omega^2 + 3g^2}}\right) + \frac{z}{r}\frac{d}{dr}\left(\frac{g}{\sqrt{\Omega^2 + 3g^2}}\right)$$

$$- \frac{z}{r^2}\frac{g}{\sqrt{\Omega^2 + 3g^2}} + h'' + \frac{h'}{r} - \frac{h}{r^2}.$$

必須使

$$\frac{d^2}{dr^2}\left(\frac{g}{\sqrt{Q^2+3g^2}}\right)+\frac{1}{r}\frac{d}{dr}\left(\frac{g}{\sqrt{Q^2+3g^2}}\right)-\frac{1}{r^2}\frac{g}{\sqrt{Q^2+3g^2}}=c,$$

$$h''+\frac{h'}{r}-\frac{h}{r^2}=Q\frac{d}{dr}\left(\frac{1}{\sqrt{Q^2+3g^2}}\right).$$

可以取前一式的一個解如

$$\frac{g}{\sqrt{Q^2+3g^2}}=Ar$$

或

$$g=\frac{ArQ}{\sqrt{1-3A^2r^2}}.$$

從後一式得到

$$h=6A^2\left\{\frac{1}{r}\int^r\frac{r^3\,dr}{\sqrt{1-3A^2r^2}}-r\int^r\frac{r\,dr}{\sqrt{1-3A^2r^2}}\right\}.$$

而

$$\frac{\partial\chi}{\partial r}=\sqrt{3}\,k\left[2Azr-12A^2r\int^r\frac{r\,dr}{\sqrt{1-3A^2r^2}}\right]$$

或

$$\frac{1}{r}\frac{\partial\chi}{\partial r}=2\sqrt{3}\,k\left[Az-6A^2\int^r\frac{r\,dr}{\sqrt{1-3A^2r^2}}\right].$$

由 (8), (9), (13), (14) 我們得到

$$\tau_{zr}=-\sqrt{3}\,k\,Ar, \tag{42}$$

$$\sigma_z=2\sqrt{3}\,k\,A\left[z-6A\int^r\frac{r\,dr}{\sqrt{1-3A^2r^2}}\right], \tag{43}$$

$$\sigma_\theta=\sigma_r=-\sqrt{3}\,k\sqrt{1-3A^2r^2}+2\sqrt{3}\,k\,A\left[z-6A\int^r\frac{r\,dr}{\sqrt{1-3A^2r^2}}\right]. \tag{44}$$

可以將上面得到的湊合解看作代表下述情況中的塑性變形狀態: 一無限長圓柱體,其半徑爲 $r=\rho$, 緊套在一圓管之中,而這圓管以均勻徑向速度 $U$ 縮小,同時在圓柱體與圓管接觸面上有恆等於一常數的摩擦力作用.

我們注意到此處討論的情況均係假設整個物體達到塑性狀態,至於如何達到這極限狀態,則用我們的方法不能說明. 因此, 解的適用性是有其限制的.

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# ON THE PROBLEM OF AXIAL-SYMMETRIC PLASTIC DEFORMATION

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ABSTRACT

In the usual treatment of plane deformation problems in plasticity, the assumption of static determinacy is frequently made, with the advantage of obtaining stress distributions without reference to stress-strain relations. In the present paper, we consider the problem of axial-symmetric deformation, where this advantage is no longer existing. The material is assumed to be incompressible and entirely in a plastic state. The equations of equilibrium are

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0\,, \tag{1}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r} = 0\,, \tag{2}$$

and the yield condition is that of von Mises:

$$(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\,\tau_{zr}^2 = 6\,k^2. \tag{3}$$

The strain-rate components and the stress components are connected by the following relations of Saint-Venant and von Mises:

$$\dot{e}_r = \frac{\partial u}{\partial r} = \frac{1}{3}\lambda\,(2\,\sigma_r - \sigma_\theta - \sigma_z)\,, \tag{4}$$

$$\dot{e}_\theta = \frac{u}{r} = \frac{1}{3}\lambda\,(2\,\sigma_\theta - \sigma_z - \sigma_r)\,, \tag{5}$$

$$\dot{e}_z = \frac{\partial w}{\partial z} = \frac{1}{3}\lambda(2\sigma_z - \sigma_r - \sigma_\theta), \tag{6}$$

$$\dot{g}_{zr} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = \lambda\tau_{zr}, \tag{7}$$

where $u$ and $w$ are the velocity components along the $r$ and $z$ directions respectively.

A function $\chi(r, z)$ is introduced such that

$$\tau_{zr} = -\frac{1}{r}\frac{\partial\chi}{\partial z}, \tag{8}$$

$$\sigma_z = \frac{1}{r}\frac{\partial\chi}{\partial r}, \tag{9}$$

and a function $\Omega(r, z)$ such that

$$u = -\frac{r\Omega}{3}. \tag{11}$$

Then the eq. (2) is satisfied and the relation

$$\dot{e}_r = \frac{\partial u}{\partial r} - \frac{\partial}{\partial r}(r\dot{e}_\theta)$$

gives

$$\frac{\sigma_r - \sigma_\theta}{r} = -\frac{1}{3\lambda}\frac{\partial\Omega}{\partial r}.$$

Hence the stress components $\sigma_r, \sigma_\theta$ can be represented in the following manner:

$$\sigma_r = -\frac{2}{3}\frac{r}{\lambda}\frac{\partial\Omega}{\partial r} - \frac{\Omega}{\lambda} + \frac{1}{r}\frac{\partial\chi}{\partial r}, \tag{13}$$

$$\sigma_\theta = -\frac{r}{3\lambda}\frac{\partial\Omega}{\partial r} - \frac{\Omega}{\lambda} + \frac{1}{r}\frac{\partial\chi}{\partial r}. \tag{14}$$

The relation $\frac{\partial^2 w}{\partial z\,\partial r} = \frac{\partial^2 w}{\partial r\,\partial z}$ yields

$$\frac{\partial\Omega}{\partial r} + \frac{r}{3}\frac{\partial^2\Omega}{\partial r^2} - \frac{r}{3}\frac{\partial^2\Omega}{\partial z^2} = -\frac{2}{r}\frac{\partial}{\partial z}\left(\lambda\frac{\partial\chi}{\partial z}\right). \tag{17}$$

The eq. (1) now becomes

$$\frac{2}{\lambda}\left(\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial^2 \Omega}{\partial r^2}\right)+\frac{\partial}{\partial r}\left(\frac{1}{\lambda}\right)\left(\Omega+\frac{2r}{3}\frac{2\Omega}{\partial r}\right)=$$
$$=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial r}\right)-\frac{1}{r}\frac{\partial^2 \chi}{\partial z^2}\,; \qquad (18)$$

and the eq. (3) becomes

$$\Omega^2+r\Omega\left(\frac{\partial \Omega}{\partial r}\right)+\frac{r^2}{3}\left(\frac{\partial \Omega}{\partial r}\right)^2+\frac{3\lambda^1}{r^2}\left(\frac{\partial \chi}{\partial z}\right)^2=3k^2\lambda^2. \qquad (19)$$

The problem then reduces to the determination of three functions $\chi$, $\Omega$ and $\lambda$ from eqs. (17), (18), (19) and then the stress and velocity components can be deduced from them.

On account of the difficulties in directly formulating and solving the boundary value problems in terms of $\chi$, $\Omega$ and $\lambda$, we propose to give here a few simple solutions by the inverse method and to partake physical meanings to the solutions obtained.

The first case considered is $\tau_{zr}\equiv 0$. This means that $\chi=\chi(r)$. The eqs. (17), (18), (19) reduce to

$$\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial^2 \Omega}{\partial r^2}-\frac{r}{3}\frac{\partial^2 \Omega}{\partial z^2}=0\,, \qquad (20)$$

$$\frac{2}{\lambda}\left(\frac{\partial \Omega}{\partial r}+\frac{r}{3}\frac{\partial \Omega}{\partial r}\right)+\left(\Omega+\frac{2r}{3}\frac{\partial \Omega}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{\lambda}\right)=$$
$$=\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right), \qquad (21)$$

$$\Omega^2+r\Omega\left(\frac{\partial \Omega}{\partial r}\right)+\frac{r^2}{3}\left(\frac{\partial \Omega}{\partial r}\right)^2=3k^2\lambda^2. \qquad (22)$$

Four different situations are considered:

1. $\Omega=\Omega(r)$, $w=0$. This gives

$$u=\frac{A}{6r}\,, \qquad (23)$$

$$\sigma_r=-2k\epsilon\ln\left(\frac{a}{r}\right), \qquad (24)$$

$$\sigma_\theta=2k\epsilon\left[1-2\ln\left(\frac{a}{r}\right)\right], \qquad (25)$$

where $\epsilon$ is the sign of $u$. This is the simplest case of plane axial -symmetric deformation.

2. $\Omega=\Omega(r)$, $\frac{\partial w}{\partial z}\dot{e}_z=$ const. Then we have

$$u=-\frac{1}{6r}(Br^2-A), \tag{26}$$

$$\sigma_r=C-\epsilon k\ln\left(\frac{1+\sqrt{1+\frac{3B^2}{A^2}r^4}}{r^2}\right), \tag{27}$$

$$\sigma_\theta=\sigma_r+\frac{2\epsilon k}{\sqrt{1+\frac{3B^2}{A^2}r^4}}, \tag{28}$$

$$\sigma_z=\sigma_r+\frac{\epsilon k\left(1+\frac{3B}{A}r\right)}{\sqrt{1+\frac{3B^2}{A^2}r^4}}, \tag{29}$$

where $A$, $B$ and $C$ are arbitrary constants. This is interpreted as the case of a thick circular tube under internal and external pressures.

3. $\Omega=\Omega(z)$. This gives

$$u=-\frac{r}{3}(Az+B), \qquad w=\frac{2}{3}\left(\frac{Az^2}{2}+Bz\right)+\frac{Ar^2}{6}, \tag{31}$$

$$\sigma_z=C, \tag{32}$$

$$\sigma_r=\sigma_\theta=C-\sqrt{3}\,\kappa k, \qquad \kappa=-\operatorname{sign}u \tag{33}$$

where $A$, $B$, $C$ are arbitrary constants. This is interpreted as the case of a body of revolution undergoing uniform axial compression.

4. $\Omega=R(r)Z(z)$. A particular solution gives

$$u=-\frac{C}{3}J_0'(mr)\cos(mz), \tag{35}$$

$$\sigma_r=-k\int_r^\rho\frac{q+2}{(1+q+q^2)^{\frac{1}{2}}}\frac{dr}{r}, \tag{38}$$

$$\sigma_\theta=\sigma_r+\frac{k(q+2)}{(1+q+q^2)^{\frac{1}{2}}}, \tag{39}$$

$$\sigma_z = \sigma_r + \frac{k(2q+1)}{(1+q+q^2)^{\frac{1}{2}}}, \tag{40}$$

where $q = mr \dfrac{J_0(mr)}{J_0'(mr)}$, and $C$ and $m$ are arbitrary constants. The solution is interpreted as the case of a circular cylinder, of radius $\rho$, compressed by appropriately distributed axial loads at the ends..

The second case considered is $\dfrac{\partial\chi}{\partial z} \not\equiv 0$ (i.e. $\tau_{zr} \not\equiv 0$) but $\Omega =$ const. We take

$$\frac{u}{U} = -\frac{r}{\rho}, \tag{41}$$

where $U$ and $\rho$ are constants. We obtain

$$\tau_{zr} = -\sqrt{3}\, A k r, \tag{42}$$

$$\sigma_z = 2\sqrt{3}\, k A \left[ z - 6A \int^r \frac{r\,dr}{\sqrt{1-3A^2r^2}} \right], \tag{43}$$

$$\sigma_\theta = \sigma_r = -\sqrt{3}\, k \sqrt{1-3A^2r^2} + \\ + 2\sqrt{3}\, k A \left[ z - 6A \int^r \frac{r\,dr}{\sqrt{1-3A^2r^2}} \right]. \tag{44}$$

The solution is interpreted as the case of an infinitely long circular cylinder (with radius $\rho$) tightly collared in a tube, the friction on the contact surface being assumed constant throughout.