

邊緣簡支 $30^{\circ}-60^{\circ}-90^{\circ}$ 三角板的 平衡、穩定與振動問題*

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一. 引言

三角形薄板問題已經解決的不多。在邊緣簡支的情形，截至目前為止，只解決了等腰直角三角板和等邊三角板的問題。至於 $30^{\circ}-60^{\circ}-90^{\circ}$ 三角板的問題，據作者所知，還沒有人討論過。過去研究過簡支等腰直角三角板彎曲問題和穩定問題的作者有：伽遼金^[1-4]，納達埃^[6] 和賽斯^[7]。過去研究過等邊三角板彎曲問題和穩定問題的作者有：馬科斯^[11]，富諾斯基-克里格^[12]，孫^[13]，賽斯^[7,8] 和斯蒂芬生^[14]。斯蒂芬生的結果是包含在他用複變函數研究厚板平衡問題的解答中，最近在尚尼津的專著^[15]中，也載有簡支等邊三角板穩定問題的解答。孫^[13]，雪弗爾和哈佛斯^[16]等作者都研究過等邊三角板的振動問題。賽斯^[9]在 1947 年也研究過直邊平板的振動問題，但是作者沒有看到這篇文章。

馬科斯^[1]首先提出把薄板彎曲問題轉化為兩個薄膜問題來處理的方法。均佈載荷下的薄膜問題可比擬為柱體的扭轉問題。因此三角形柱體的扭轉問題的解決可以幫助三角形板彎曲問題的解決。等邊三角形柱體的扭轉問題是聖維南^[17]在 1855 年解決的。等腰直角三角形柱體的扭轉問題是伽遼金^[5]在 1918 年解決的。後來高盧索夫^[18]也曾討論過這個問題。賽斯^[10]在 1934 年解決的流體動力學問題，事實上也同時找到了 $30^{\circ}-60^{\circ}-90^{\circ}$ 三角形柱體的扭轉問題中的扭曲函數。到 1939 年，海^[19]採用反映法與橢圓函數處理過這種柱體的扭轉問題，並算出了截面的抗扭剛度。但是海和賽斯的結果都很繁雜，利用他們的結果去求解均佈載荷下 $30^{\circ}-60^{\circ}-90^{\circ}$ 三角板的彎曲問題，困難很大。

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1) 見書 [11] 第 9 頁至第 13 頁。

本文採取胡海昌同志^[20]建議的反映法並利用三角級數首先解決了 $30^\circ - 60^\circ - 90^\circ$ 三角形柱體的扭轉問題，得到的結果比海和賽斯的簡單很多，因此本文繼續用它來解決了均佈載荷下 $30^\circ - 60^\circ - 90^\circ$ 三角板的彎曲問題，計算出板在斜邊中線上的撓度數字結果。本文還解決了這種三角板在集中載荷下的彎曲問題，在拉壓與彎曲聯合下的平衡問題，以及這種板的穩定問題與振動問題；求得板在屈曲時的最小臨界壓力及板在自然振動時的基本固有頻率。文中採用的符號除某些必要的變更外與鐵木辛哥著書 [21], [22] 中採用的相同。

二. 三角形柱體的扭轉問題

根據扭轉理論，本節問題須求出應力函數 ϕ ，使滿足方程

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \quad (1)$$

及邊界條件

$$\text{在邊界上} \quad \phi = 0. \quad (2)$$

於是，我們便得到剪應力

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad (3)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x}, \quad (4)$$

及扭矩

$$M_t = 2 \iint \phi \, dx \, dy. \quad (5)$$

又根據薄膜比擬理論，上述問題可以當作薄膜平衡問題來處理，即求出薄膜的撓度 z ，使滿足方程

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = p(x, y), \quad (6)$$

其中

$$p(x, y) = -\frac{q}{s} \quad (7)$$

及邊界條件

$$\text{在邊界上} \quad z = 0; \quad (8)$$

這裏的 q 是薄膜承擔的橫向均佈載荷， s 是薄膜內的均勻張力。

薄膜垂下的體積爲

$$v = \iint z \, dx \, dy. \quad (9)$$

兩個問題的物理量間具有下面的對應關係：

$$z \sim \phi,$$

$$\frac{q}{s} \sim 2G\theta, \quad (10)$$

$$v \sim \frac{1}{2} M_t.$$

現在我們來討論三角形薄膜的平衡問題。若從方程(6), (7) 及邊界條件(8)考慮直接求解不容易，因此，採用胡海昌同志^[20]建議的反映法。我們將三角形薄膜反映成一張矩形薄膜，承擔如圖1所示的均勻張力與橫向載荷。可以證明，這張矩形薄膜的撓度函數在其定義域 ΔOAB 內就給出我們所需要的解答。根據線性理論的特點，這張矩形薄膜的撓度 z 可以從圖 2_1 和圖 2_{11} 兩張矩形薄膜的撓度疊加得到。

我們在字母的右下角附註‘I’或‘II’，區別屬於圖 2_1 板或圖 2_{11} 板的物理量，則

$$z = z_1 + z_{11}; \quad (11)$$

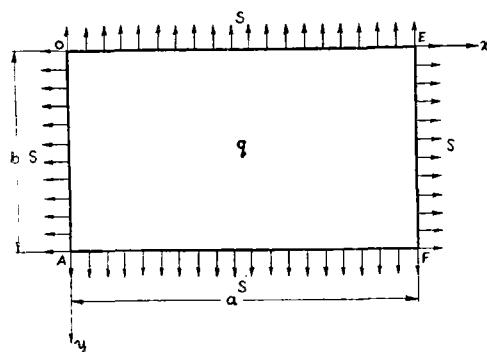


圖 2_1

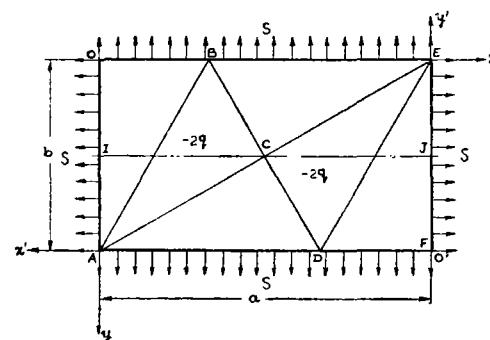


圖 2_{11}

其中 z_1 和 z_{11} 必須滿足下面的方程及邊界條件：

$$\left. \begin{aligned} \frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} &= -\frac{q}{s}, \\ \text{在邊界上} \quad z_1 &= 0; \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} \frac{\partial^2 z_{II}}{\partial x^2} + \frac{\partial^2 z_{II}}{\partial y^2} &= p_{II}(x, y), \\ \text{在邊界上 } z_{II} &= 0. \end{aligned} \right\} \quad (13)$$

於是我們的問題就轉變為找尋問題 (12) 及 (13) 的解答。

問題 (12) 已有現成的答案¹⁾, 它是

$$z_1 = \frac{4qb^2}{\pi^3 s} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^3} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \sin \frac{m\pi y}{b}. \quad (14)$$

本文將一貫採用符號 α_m , β_m 來代表:

$$\alpha_m = \frac{m\pi b}{2a}, \quad (15)$$

$$\beta_m = \frac{m\pi a}{2b}. \quad (16)$$

為使問題 (13) 的求解手續大為簡化, 我們增加一個 $o'x'y'$ 坐標系 (圖 2_{II}), oxy 與 $o'x'y'$ 兩坐標系的換算關係是

$$\left. \begin{aligned} x' &= a - x, \\ y' &= b - y. \end{aligned} \right\} \quad (17)$$

我們在字母的右上角附註 ‘◦’ 或 ‘◦’ 區別屬於圖 2_{II} 薄膜 $\square OIJE$ 部分或 $\square IAFJ$ 部分的物理量。因為這張薄膜對於 IJ 線是斜對稱的, 所以我們如果已知道它在 $\square OIJE$ 部分的撓度式子

$$z_{II}^{\circ} = f(x, y), \quad (18a)$$

那末它在 $\square IAFJ$ 部分的撓度式子就能立即寫出來, 即

$$z'_{II} = f(x', y'). \quad (18b)$$

根據 (13), z_{II}° 和 z'_{II} 應滿足方程

$$\frac{\partial^2 z_{II}^{\circ}}{\partial x^2} + \frac{\partial^2 z_{II}^{\circ}}{\partial y^2} = p_{II}^{\circ}(x, y), \quad (19a)$$

$$\frac{\partial^2 z'_{II}}{\partial x'^2} + \frac{\partial^2 z'_{II}}{\partial y'^2} = p'_{II}(x', y'), \quad (19b)$$

1) 見書 [21] 第 276 頁。

及邊界條件

$$\text{在邊界 } JE, EO, OI \text{ 上} \quad z_{II}^o(x, y) = 0, \quad (20a)$$

$$\text{在邊界 } IA, AF, FJ \text{ 上} \quad z'_{II}(x', y') = 0; \quad (20b)$$

此外尚須滿足連續條件

$$z_{II}^o\left(x, \frac{b}{2}\right) = z'_{II}\left(x', \frac{b}{2}\right), \quad (21)$$

$$\frac{\partial}{\partial y} z_{II}^o\left(x, \frac{b}{2}\right) = -\frac{\partial}{\partial y'} z'_{II}\left(x', \frac{b}{2}\right). \quad (22)$$

但根據斜對稱關係 (18) 及坐標換算關係 (17)，連續條件 (21)、(22) 可改寫為

$$z_{II}^o\left(x, \frac{b}{2}\right) = z_{II}^o\left(a - x, \frac{b}{2}\right), \quad (23)$$

$$\frac{\partial}{\partial y} z_{II}^o\left(x, \frac{b}{2}\right) = -\frac{\partial}{\partial y} z_{II}^o\left(a - x, \frac{b}{2}\right). \quad (24)$$

於是，我們已經把邊值問題 (13) 歸結到從 (19a)，(20a)，(23) 及 (24) 求出 z_{II}^o 的問題。現在我們來解決這個問題，為此，把 $p_{II}^o(x, y)$ 及 $z_{II}^o(x, y)$ 展成下面的三角級數：

$$p_{II}^o = \sum_{m=1}^{\infty} p_{IIm}^o(y) \sin \frac{m\pi x}{a}, \quad (25)$$

$$\text{其中 } p_{IIm}^o(y) = \frac{4q}{\pi sm} \left[(-1)^{m+1} \cos \frac{m\pi}{3b} y + (-1)^m \cos \frac{m\pi}{b} y + 2 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right]; \quad (26)$$

以及

$$z_{II}^o = \sum_{m=1}^{\infty} z_{IIm}^o(y) \sin \frac{m\pi x}{a}, \quad (27)$$

其中 $z_{IIm}^o(y)$ 為待定函數。(27) 式在邊界 OI, JE 上已經滿足了邊界條件 (20a)

今將 (25) 和 (27) 式代入方程 (19a)，邊界條件 (20a) 及連續條件 (23)，(24) 後，我們得到 z_{IIm}^o 必須滿足的方程，邊界條件及連續條件如下：

$$\left. \begin{array}{l} \frac{d^2 z_{IIm}^o}{dy^2} - \frac{m^2 \pi^2}{a^2} z_{IIm}^o = p_{IIm}^o, \\ \text{邊界條件} \quad z_{IIm}^o(0) = 0, \\ \text{連續條件} \quad z_{IIm}^o\left(\frac{b}{2}\right) = 0, \quad m \text{ 為偶數}, \\ \quad \quad \quad \frac{\partial}{\partial y} z_{IIm}^o\left(\frac{b}{2}\right) = 0, \quad m \text{ 為奇數}. \end{array} \right\} \quad (28)$$

將 (26) 式代入並經過運算後，我們得到 (28) 的解為

$$z_{11m}^{\circ} = \frac{6q b^2}{\pi^3 s m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \frac{3}{2} \cos \frac{m\pi y}{3b} + \right. \\ \left. + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right], \quad m \text{ 為奇數}, \quad (29a)$$

$$z_{11m}^{\circ} = \frac{6q b^2}{\pi^3 s m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} + \frac{3}{2} \cos \frac{m\pi y}{3b} - \right. \\ \left. - \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right], \quad m \text{ 為偶數}. \quad (29b)$$

再將 (29) 式代入 (27) 式後，我們就求得了 z_{11}° ，即

$$z_{11}^{\circ} = \frac{6q b^2}{\pi^3 s} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \frac{3}{2} \cos \frac{m\pi y}{3b} + \right. \right. \\ \left. + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} + \\ \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} + \frac{3}{2} \cos \frac{m\pi y}{3b} - \right. \right. \\ \left. \left. - \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} \right\}. \quad (30a)$$

又根據斜對稱關係 (18) 及坐標換算關係 (17)，我們從 (30a) 式很容易寫出

$$z'_{11} = \frac{6q b^2}{\pi^3 s} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi}{a} (y-b)}{\cosh \alpha_m} - \right. \right. \\ \left. - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) - \frac{1}{2} \cos \frac{m\pi}{b} y + 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} + \\ \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi}{a} (y-b)}{\sinh \alpha_m} - \right. \right. \\ \left. \left. - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} \right\}. \quad (30b)$$

將 (30) 式和 (14) 式代入 (11) 式，我們就找到了如圖 1 所示矩形薄膜的撓度，

即

$$\begin{aligned}
 z = & \frac{4qb^2}{\pi^3 s} \left\{ \sum_{m=1,3,5\dots}^{\infty} \frac{1}{m^3} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \sin \frac{m\pi y}{b} + \right. \\
 & + \frac{3}{2} \sum_{m=1,3,5\dots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \right. \\
 & - \frac{3}{2} \cos \frac{m\pi y}{3b} + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \left. \right] \sin \frac{m\pi x}{a} + \\
 & + \frac{3}{2} \sum_{m=2,4,6\dots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} + \right. \\
 & \left. \left. + \frac{3}{2} \cos \frac{m\pi y}{3b} - \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} \right\}, \\
 & \quad \left(0 \leqslant y \leqslant \frac{b}{2} \right), \tag{31a}
 \end{aligned}$$

$$\begin{aligned}
 z = & \frac{4qb^2}{\pi^3 s} \left\{ \sum_{m=1,3,5\dots}^{\infty} \frac{1}{m^3} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \sin \frac{m\pi y}{b} + \right. \\
 & + \frac{3}{2} \sum_{m=1,3,5\dots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi}{a} (y-b)}{\cosh \alpha_m} - \right. \\
 & - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) - \frac{1}{2} \cos \frac{m\pi y}{b} + 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \left. \right] \sin \frac{m\pi x}{a} + \\
 & + \frac{3}{2} \sum_{m=2,4,6\dots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi}{a} (y-b)}{\sinh \alpha_m} - \right. \\
 & \left. - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} \right\}, \\
 & \quad \left(\frac{b}{2} \leqslant y \leqslant b \right). \tag{31b}
 \end{aligned}$$

以上兩個式子也給出了三角形薄膜的撓度。三角形薄膜垂下的體積可採用第一個式子，通過積分

$$\int_0^a dx \int_0^{b/2} z dy$$

的運算並應用下列兩級數之和^[23]：

$$\sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{96},$$

$$\sum_{m=1,3,5\cdots}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^5} = -\frac{5}{1536} \pi^5,$$

得到

$$\begin{aligned} v = & \frac{4qb^4}{s} \left\{ \frac{17\sqrt{3}}{192} - \frac{1}{\pi^5} \sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^5} \left[(2 \tanh \beta_m - 9 \tanh \alpha_m) + \right. \right. \\ & \left. \left. + \frac{(-1)^{\frac{m-1}{2}} 9\sqrt{3}}{\cosh \alpha_m} + 27\sqrt{3} \sin \frac{m\pi}{3} \right] \right\}. \end{aligned} \quad (32)$$

從(31)式並根據對應關係(10)，我們也可以立即寫出本節扭轉問題中的應力函數 ϕ ，即

$$\begin{aligned} \phi = & \frac{8G\theta b^2}{\pi^3} \left\{ \sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^3} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \sin \frac{m\pi y}{b} + \right. \\ & + \frac{3}{2} \sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \right. \\ & - \frac{3}{2} \cos \frac{m\pi y}{3b} + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \left. \right] \sin \frac{m\pi x}{a} + \\ & + \frac{3}{2} \sum_{m=2,4,6\cdots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} + \right. \\ & \left. + \frac{3}{2} \cos \frac{m\pi y}{3b} - \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} \left. \right\}, \\ & \left(0 \leqslant y \leqslant \frac{b}{2} \right), \end{aligned} \quad (33a)$$

$$\begin{aligned} \phi = & \frac{8G\theta b^2}{\pi^3} \left\{ \sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^3} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \sin \frac{m\pi y}{b} + \right. \\ & + \frac{3}{2} \sum_{m=1,3,5\cdots}^{\infty} \frac{1}{m^3} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi}{a} (y-b)}{\cosh \alpha_m} - \right. \\ & - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) - \frac{1}{2} \cos \frac{m\pi y}{b} + 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \left. \right] \sin \frac{m\pi x}{a} + \end{aligned}$$

$$\begin{aligned}
& + \frac{3}{2} \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^3} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi}{a} (y-b)}{\sinh \alpha_m} - \right. \\
& \left. - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} \Big\}, \\
& \quad \left(\frac{b}{2} \leqslant y \leqslant b \right). \quad (33b)
\end{aligned}$$

於是，從 (3) 及 (4) 兩式，我們得到剪應力的式子如下：

$$\begin{aligned}
\tau_{xx} = & \frac{8bG\theta}{\pi^2} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \cos \frac{m\pi y}{b} + \right. \\
& + \frac{\sqrt{3}}{2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \cosh \frac{m\pi y}{a}}{\cosh \alpha_m} + \right. \\
& \left. + \frac{\sqrt{3}}{2} \sin \frac{m\pi y}{3b} - \frac{\sqrt{3}}{2} \sin \frac{m\pi y}{b} - \sqrt{3} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} + \\
& + \frac{\sqrt{3}}{2} \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi y}{a}}{\sinh \alpha_m} - \right. \\
& \left. - \frac{\sqrt{3}}{2} \sin \frac{m\pi y}{3b} + \frac{\sqrt{3}}{2} \sin \frac{m\pi y}{b} - \sqrt{3} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} \Big\}, \\
& \quad \left(0 \leqslant y \leqslant \frac{b}{2} \right), \quad (34a)
\end{aligned}$$

$$\begin{aligned}
\tau_{xx} = & \frac{8bG\theta}{\pi^2} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \right] \cos \frac{m\pi y}{b} + \right. \\
& + \frac{\sqrt{3}}{2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - \sqrt{3} (-1)^{\frac{m-1}{2}} \cosh \frac{m\pi}{a} (y-b)}{\cosh \alpha_m} + \right. \\
& \left. + \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3b} (y-b) + \frac{\sqrt{3}}{2} \sin \frac{m\pi y}{b} + \sqrt{3} \sin \frac{m\pi}{3} \cos \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} + \\
& + \frac{\sqrt{3}}{2} \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi}{a} (y-b)}{\sinh \alpha_m} + \right. \\
& \left. + \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3b} (y-b) - \frac{\sqrt{3}}{2} \sin \frac{m\pi}{b} y - \sqrt{3} \sin \frac{m\pi}{3} \cos \frac{m\pi}{3b} (y-b) \right] \sin \frac{m\pi x}{a} \Big\}, \\
& \quad \left(\frac{b}{2} \leqslant y \leqslant b \right), \quad (34b)
\end{aligned}$$

$$\begin{aligned} \tau_{yz} = & \frac{8bG\theta}{\pi^2} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \sin \frac{m\pi y}{b} - \right. \\ & - \frac{\sqrt{3}}{2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \right. \\ & - \frac{3}{2} \cos \frac{m\pi y}{3b} + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \left. \right] \cos \frac{m\pi x}{a} - \\ & - \frac{\sqrt{3}}{2} \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} + \right. \\ & + \frac{3}{2} \cos \frac{m\pi y}{3b} - \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \left. \right] \cos \frac{m\pi x}{a} \left. \right\}, \\ & \quad \left(0 \leqslant y \leqslant \frac{b}{2} \right), \quad (35a) \end{aligned}$$

$$\begin{aligned} \tau_{yz} = & \frac{8bG\theta}{\pi^2} \left\{ \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} \sin \frac{m\pi y}{b} - \right. \\ & - \frac{\sqrt{3}}{2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\cosh \frac{m\pi}{2a} (2y-b) - \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi}{a} (y-b)}{\cosh \alpha_m} - \right. \\ & - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) - \frac{1}{2} \cos \frac{m\pi y}{b} + 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \left. \right] \cos \frac{m\pi x}{a} - \\ & - \frac{\sqrt{3}}{2} \sum_{m=2,4,6,\dots}^{\infty} \frac{1}{m^2} \left[\frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi}{a} (y-b)}{\sinh \alpha_m} - \right. \\ & - \frac{3}{2} \cos \frac{m\pi}{3b} (y-b) + \frac{1}{2} \cos \frac{m\pi y}{b} - 3 \sin \frac{m\pi}{3} \sin \frac{m\pi}{3b} (y-b) \left. \right] \cos \frac{m\pi x}{a} \left. \right\}, \\ & \quad \left(\frac{b}{2} \leqslant y \leqslant b \right). \quad (35b) \end{aligned}$$

又從 (32) 式並考慮對應關係 (10), 則扭矩為

$$\begin{aligned} M_t = & 16G\theta b^4 \left\{ \frac{17\sqrt{3}}{192} - \frac{1}{\pi^5} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^5} \left[(2 \tanh \beta_m - 9 \tanh \alpha_m) + \right. \right. \\ & + \frac{(-1)^{\frac{m-1}{2}} 9\sqrt{3}}{\cosh \alpha_m} + 27\sqrt{3} \sin \frac{m\pi}{3} \left. \right] \left. \right\}, \quad (36) \end{aligned}$$

抗扭剛度爲

$$C = \frac{M_t}{\theta} = 16Gb^4 \left\{ \frac{17\sqrt{3}}{192} - \frac{1}{\pi^5} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^5} \left[(2 \tanh \beta_m - 9 \tanh \alpha_m) + \right. \right. \\ \left. \left. + \frac{(-1)^{\frac{m-1}{2}} 9\sqrt{3}}{\cosh \alpha_m} + 27\sqrt{3} \sin \frac{m\pi}{3} \right] \right\}. \quad (37)$$

我們知道在 (33)–(37) 諸式中的 a 與 b 有一定的比值，即

$$a = \sqrt{3} b, \quad (38)$$

其中 b 是三角形 60° 角對邊的長度。將這比值代入 (37) 式進行數字計算，我們有

$$C = 0.007914 Gb^4. \quad (39)$$

這個結果與海^[19] 的結果相符合。

三. 均佈載荷下三角板的彎曲問題

根據薄板小撓度彎曲理論，板的撓度 w 應適合方程

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x, y)}{D}. \quad (40)$$

在本節問題中板上的載荷是均佈的，即

$$p(x, y) = q,$$

且板在邊緣是簡支的，即 w 應滿足邊界條件：

$$\left. \begin{array}{l} \text{在邊界上} \quad w = 0, \\ \frac{\partial^2 w}{\partial n^2} = 0; \end{array} \right\} \quad (41)$$

其中 n 是指邊界的法線方向。 (40) 式中的 D 是抗彎剛度，即

$$D = \frac{Eh^3}{12(1-\nu^2)}; \quad (42)$$

這裏的 E 是板料的楊氏彈性係數， ν 是板料的泊松係數， h 是板的厚度。

和上節所說的理由一樣，採用反映法，我們把三角板反映成一塊邊緣簡支的矩形板，承擔如圖 3 所示的分佈載荷。這塊矩形板平衡時的撓度也給出了上述三角板的

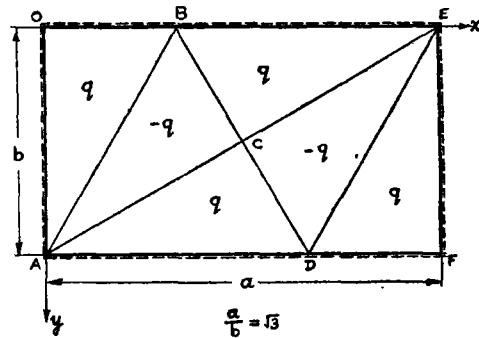


圖 3

撓度。而它又可以從圖 4_I 及圖 4_{II} 兩塊矩形板的撓度疊加起來得到，即

$$w = w_I + w_{II}. \quad (43)$$

這裏附註在字母右下角的記號 'I' 或 'II' 的意義與上一節的類似。於是，我們必須解

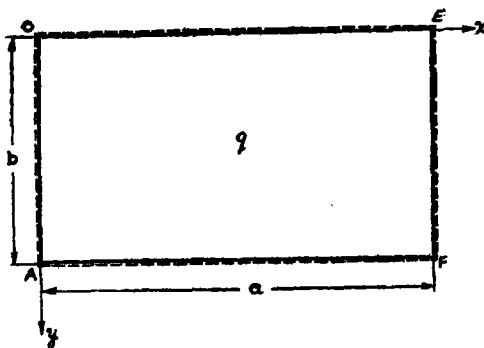


圖 4I

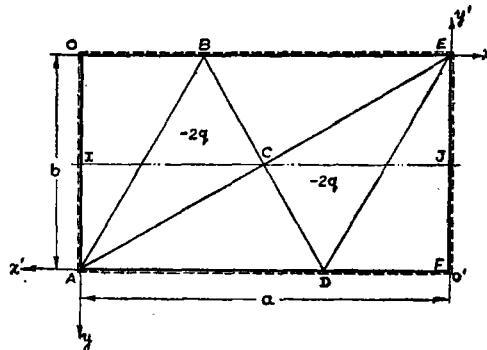


圖 4II

決兩塊矩形板的彎曲問題，即

$$\left. \begin{aligned} & \frac{\partial^4 w_I}{\partial x^4} + 2 \frac{\partial^4 w_I}{\partial x^2 \partial y^2} + \frac{\partial^4 w_I}{\partial y^4} = \frac{q}{D}, \\ & \text{在邊界上 } \quad w_I = 0, \\ & \frac{\partial^2 w_I}{\partial n^2} = 0; \end{aligned} \right\} \quad (44)$$

$$\left. \begin{aligned} & \frac{\partial^4 w_{II}}{\partial x^4} + 2 \frac{\partial^4 w_{II}}{\partial x^2 \partial y^2} + \frac{\partial^4 w_{II}}{\partial y^4} = \frac{p_{II}(x, y)}{D}, \\ & \text{在邊界上 } \quad w_{II} = 0, \\ & \frac{\partial^2 w_{II}}{\partial n^2} = 0. \end{aligned} \right\} \quad (45)$$

問題 (44) 已有現成的答案¹⁾，它是

$$w_I = \frac{4qb^4}{D\pi^5} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^5} \left[1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} + \frac{m\pi x}{2b} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} - \frac{\beta_m \sinh \frac{m\pi x}{b}}{2 \cosh^2 \beta_m} \right] \sin \frac{m\pi y}{b}. \quad (46)$$

現在我們來討論問題 (45) 的解。根據馬科斯²⁾提出的方法，薄板彎曲問題可以

1) 見書 [22]，第 128 頁。

2) 與第 215 頁註 1) 相同。

化為兩個連貫的薄膜問題來處理。於是我們把問題 (45) 轉化為兩個連貫的薄膜問題如下：

$$\left. \begin{aligned} \frac{\partial^2 M_{II}}{\partial x^2} + \frac{\partial^2 M_{II}}{\partial y^2} &= -p_{II}(x, y), \\ \text{在邊界上 } M_{II} &= 0 \end{aligned} \right\} \quad (47a)$$

及

$$\left. \begin{aligned} \frac{\partial^2 w_{II}}{\partial x^2} + \frac{\partial^2 w_{II}}{\partial y^2} &= -\frac{M_{II}}{D}, \\ \text{在邊界上 } w_{II} &= 0. \end{aligned} \right\} \quad (47b)$$

和上節的方法一樣，我們增加一個 $o'x'y'$ 坐標系（圖 4_{II}），它與 oxy 坐標系的換算關係還是 (17) 式。圖 4_{II} 這塊板的情況對於 IJ 線是斜對稱的，因此 (47a)，(47b) 中的 M_{II} 和 w_{II} 也是斜對稱的。我們仍舊在字母的右上角附註記號 ‘◦’ 或 ‘◦’，區別板 $\square OIJE$ 部分或 $\square IAFJ$ 部分的物理量。於是把上節 (30a) 式中的字母 $z_{II}^◦$ 及 $\frac{q}{s}$ 分別用字母 $M_{II}^◦$ 及 $-q$ 替換後，我們得到用無窮三角級數表達的 $M_{II}^◦(x, y)$ 式子，而式子 $M'_{II}(x', y')$ 可寫出為

$$M'_{II}(x', y') \equiv M_{II}^◦(x, y).$$

這兩個 $M_{II}^◦(x, y)$ 和 $M'_{II}(x', y')$ 式子就是問題 (47a) 的解，將它們代入 (47b) 後，我們得到 $w_{II}^◦$ 和 w'_{II} 應滿足的方程

$$\frac{\partial^2 w_{II}^◦}{\partial x^2} + \frac{\partial^2 w_{II}^◦}{\partial y^2} = -\frac{M_{II}^◦}{D}, \quad (48a)$$

$$\frac{\partial^2 w'_{II}}{\partial x^2} + \frac{\partial^2 w'_{II}}{\partial y^2} = -\frac{M'_{II}}{D}, \quad (48b)$$

及邊界條件

$$\text{在邊界 } JE, EO, OI \text{ 上 } w_{II}^◦(x, y) = 0, \quad (49a)$$

$$\text{在邊界 } IA, AF, FJ \text{ 上 } w'_{II}(x', y') = 0; \quad (49b)$$

此外它們還應該滿足連續條件

$$w_{II}^◦\left(x, \frac{b}{2}\right) = w'_{II}\left(x', \frac{b}{2}\right), \quad (50)$$

$$\frac{\partial}{\partial y} w_{II}^◦\left(x, \frac{b}{2}\right) = -\frac{\partial}{\partial y'} w'_{II}\left(x', \frac{b}{2}\right). \quad (51)$$

因為 $w_{II}^◦$ 和 w'_{II} 具有斜對稱關係，即若

$$w_{II}^◦ = g(x, y), \quad (52a)$$

則

$$w_{II}' = g(x', y'). \quad (52b)$$

所以連續條件 (50), (51) 可以改寫為

$$w_{II}^{\circ}\left(x, \frac{b}{2}\right) = w_{II}^{\circ}\left(a - x, \frac{b}{2}\right), \quad (53)$$

$$\frac{\partial}{\partial y} w_{II}^{\circ}\left(x, \frac{b}{2}\right) = - \frac{\partial}{\partial y} w_{II}^{\circ}\left(a - x, \frac{b}{2}\right). \quad (54)$$

於是問題 (45) 已經歸結為從 (48a), (49a), (53) 及 (54) 求 w_{II}° 的問題。現在我們來討論 w_{II}° 的求法，為此將 $w_{II}^{\circ}(x, y)$ 展成三角級數

$$w_{II}^{\circ} = \sum_{m=1}^{\infty} w_{IIm}^{\circ}(y) \sin \frac{m\pi x}{a}, \quad (55)$$

其中 $w_{IIm}^{\circ}(y)$ 是待定函數。顯然，這樣的級數展開式在邊界 OI, JE 上已滿足邊界條件 (49a)。將它代入方程 (48a)，邊界條件 (49a) 及連續條件 (53) 和 (54) 後，我們得到了 w_{IIm}° 必須滿足的方程，邊界條件及連續條件如下：

$$\left. \begin{array}{l} \frac{\partial^2 w_{IIm}^{\circ}}{\partial y^2} - \frac{m^2 \pi^2}{a^2} w_{IIm}^{\circ} = - \frac{M_{IIm}^{\circ}}{D}, \\ \text{邊界條件} \quad w_{IIm}^{\circ}(0) = 0, \\ \text{連續條件:} \quad w_{IIm}^{\circ}\left(\frac{b}{2}\right) = 0, \quad m \text{ 為偶數,} \\ \frac{\partial}{\partial y} w_{IIm}^{\circ}\left(\frac{b}{2}\right) = 0, \quad m \text{ 為奇數; } \end{array} \right\} \quad (56)$$

其中 M_{IIm}° 是把上節 (29) 式中的字母 z_{IIm}° 及 $\frac{q}{s}$ 用字母 M_{IIm}° 及 $-q$ 分別替換後得到的式子。

問題 (56) 的解答是不難找出的，將此解答代入 (55) 式，我們有 w_{II}° 的式子如下：

$$\begin{aligned} w_{II}^{\circ} = & \frac{9qb^4}{\pi^5 D} \left\{ \sum_{m=1, 3, 5, \dots}^{\infty} \frac{1}{m^5} \left[\frac{(-1)^{\frac{m-1}{2}} \sqrt{3} (2 + \alpha_m \tanh \alpha_m) \sinh \frac{m\pi y}{a} + 2 \cosh \frac{m\pi}{2a} (2y - b)}{\cosh \alpha_m} + \right. \right. \\ & + \alpha_m \frac{\sinh \frac{m\pi y}{a}}{\cosh^2 \alpha_m} - \frac{m\pi y}{a} \frac{\sinh \frac{m\pi}{2a} (2y - b) + (-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi y}{a}}{\cosh \alpha_m} - \\ & \left. \left. - \frac{9}{4} \cos \frac{m\pi y}{3b} + \frac{1}{4} \cos \frac{m\pi y}{b} - \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} - \right\} \end{aligned}$$

$$\begin{aligned}
 & - \sum_{m=2,4,6}^{\infty} \frac{1}{m^5} \left[\frac{(-1)^{\frac{m}{2}} (2 + \alpha_m \coth \alpha_m) \sinh \frac{m\pi y}{a} - 2 \sinh \frac{m\pi}{2a} (2y - b)}{\sinh \alpha_m} \right. \\
 & - \alpha_m \frac{\sinh \frac{m\pi y}{a}}{\sinh^2 \alpha_m} + \frac{m\pi y}{a} \frac{\cosh \frac{m\pi}{2a} (2y - b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi y}{a}}{\sinh \alpha_m} - \\
 & \left. - \frac{9}{4} \cos \frac{m\pi y}{3b} + \frac{1}{4} \cos \frac{m\pi y}{b} + \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b} \right] \sin \frac{m\pi x}{a} \Big\}. \quad (57a)
 \end{aligned}$$

根據斜對稱關係 (52) 及坐標換算關係 (17)，我們又可寫出 w'_{11} ，即

$$\begin{aligned}
 w'_{11} = & \frac{9qb^4}{\pi^5 D} \left\{ \sum_{m=1,3,5}^{\infty} \frac{1}{m^5} \left[\frac{2 \cosh \frac{m\pi}{2a} (2y - b) - (-1)^{\frac{m-1}{2}} \sqrt{3} (2 + \alpha_m \tanh \alpha_m) \sinh \frac{m\pi}{a} (y - b)}{\cosh \alpha_m} \right. \right. \\
 & - \alpha_m \frac{\sinh \frac{m\pi}{a} (y - b)}{\cosh^2 \alpha_m} + \frac{m\pi (y - b)}{a} \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi (y - b)}{a} - \sinh \frac{m\pi}{2a} (2y - b)}{\cosh \alpha_m} - \\
 & - \frac{9}{4} \cos \frac{m\pi (y - b)}{3b} - \frac{1}{4} \cos \frac{m\pi y}{b} + \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi (y - b)}{3b} \Big] \sin \frac{m\pi x}{a} + \\
 & + \sum_{m=2,4,6}^{\infty} \frac{1}{m^5} \left[\frac{2 \sinh \frac{m\pi}{2a} (2y - b) - (-1)^{\frac{m}{2}} (2 + \alpha_m \coth \alpha_m) \sinh \frac{m\pi (y - b)}{a}}{\sinh \alpha_m} + \right. \\
 & + \alpha_m \frac{\sinh \frac{m\pi (y - b)}{a}}{\sinh^2 \alpha_m} - \frac{m\pi (y - b)}{a} \frac{\cosh \frac{m\pi}{2a} (2y - b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi (y - b)}{a}}{\sinh \alpha_m} - \\
 & \left. \left. - \frac{9}{4} \cos \frac{m\pi (y - b)}{3b} + \frac{1}{4} \cos \frac{m\pi y}{b} - \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi (y - b)}{3b} \right] \sin \frac{m\pi x}{a} \right\}. \quad (57b)
 \end{aligned}$$

將 (57) 式和 (46) 式代入 (43) 式後的結果就給出了如圖 3 所示矩形板的撓度式子，並且這式子也表達了三角板的撓度。今將三角板的撓度 w 及其三階以下的各個偏導數列成一表：

(I) $0 \leq y \leq \frac{b}{2}$

$$\begin{aligned}
 w = & \frac{9qb^4}{D\pi^5} \left[\sum_{m=1,3,5}^{\infty} \frac{K_m(y)}{m^5} \sin \frac{m\pi x}{a} + \right. \\
 & + \sum_{m=2,4,6}^{\infty} \frac{S_m(y)}{m^5} \sin \frac{m\pi x}{a} + \frac{4}{9} \sum_{m=1,3,5}^{\infty} \frac{E_m(x)}{m^5} \sin \frac{m\pi y}{b} \Big], \quad (58a)
 \end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{3\sqrt{3}qb^3}{D\pi^4} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m(y)}{m^4} \cos \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m(y)}{m^4} \cos \frac{m\pi x}{a} + \frac{4\sqrt{3}}{9} \sum_{m=1,3,5,\dots}^{\infty} \frac{F_m(x)}{m^4} \sin \frac{m\pi y}{b} \right], \\ \frac{\partial w}{\partial y} &= \frac{3\sqrt{3}qb^3}{D\pi^4} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{L_m(y)}{m^4} \sin \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{T_m(y)}{m^4} \sin \frac{m\pi x}{a} + \frac{4\sqrt{3}}{9} \sum_{m=1,3,5,\dots}^{\infty} \frac{E_m(x)}{m^4} \cos \frac{m\pi y}{b} \right], \\ \frac{\partial^2 w}{\partial x^2} &= -\frac{3qb^2}{D\pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m(y)}{m^3} \sin \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m(y)}{m^3} \sin \frac{m\pi x}{a} - \frac{4}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{G_m(x)}{m^3} \sin \frac{m\pi y}{b} \right], \\ \frac{\partial^2 w}{\partial x \partial y} &= \frac{3qb^2}{D\pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{L_m(y)}{m^3} \cos \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{T_m(y)}{m^3} \cos \frac{m\pi x}{a} + \frac{4}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{F_m(x)}{m^3} \cos \frac{m\pi y}{b} \right], \\ \frac{\partial^2 w}{\partial y^2} &= \frac{3qb^2}{D\pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{M_m(y)}{m^3} \sin \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{U_m(y)}{m^3} \sin \frac{m\pi x}{a} - \frac{4}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{E_m(x)}{m^3} \sin \frac{m\pi y}{b} \right], \\ \frac{\partial^3 w}{\partial x^3} &= -\frac{\sqrt{3}qb}{D\pi^2} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m(y)}{m^2} \cos \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m(y)}{m^2} \cos \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{H_m(x)}{m^2} \sin \frac{m\pi y}{b} \right], \\ \frac{\partial^3 w}{\partial x^2 \partial y} &= -\frac{\sqrt{3}qb}{D\pi^2} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{L_m(y)}{m^2} \sin \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{T_m(y)}{m^2} \sin \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{G_m(x)}{m^2} \cos \frac{m\pi y}{b} \right], \\ \frac{\partial^3 w}{\partial x \partial y^2} &= \frac{\sqrt{3}qb}{D\pi^2} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{M_m(y)}{m^2} \cos \frac{m\pi x}{a} + \right. \\ &\quad \left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{U_m(y)}{m^2} \cos \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{F_m(x)}{m^2} \sin \frac{m\pi y}{b} \right],\end{aligned}$$

$$\frac{\partial^3 w}{\partial y^3} = \frac{\sqrt{3} q b}{D \pi^2} \left[\sum_{m=1,3,5 \dots}^{\infty} \frac{N_m(y)}{m^2} \sin \frac{m\pi x}{a} + \right. \\ \left. + \sum_{m=2,4,6 \dots}^{\infty} \frac{V_m(y)}{m^2} \sin \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1,3,5 \dots}^{\infty} \frac{E_m(x)}{m^2} \cos \frac{m\pi y}{b} \right];$$

其中

$$E_m(x) = 1 - \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} + \frac{m\pi x}{2b} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} - \beta_m \frac{\sinh \frac{m\pi x}{b}}{2 \cosh^2 \beta_m},$$

$$F_m(x) = - \frac{\sinh \frac{m\pi}{2b} (2x-a)}{2 \cosh \beta_m} + \frac{m\pi x}{2b} \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} - \beta_m \frac{\cosh \frac{m\pi x}{b}}{2 \cosh^2 \beta_m},$$

$$G_m(x) = \frac{m\pi x}{2b} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} - \beta_m \frac{\sinh \frac{m\pi x}{b}}{2 \cosh^2 \beta_m},$$

$$H_m(x) = \frac{1}{2} \frac{\sinh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} + \frac{m\pi x}{2b} \frac{\cosh \frac{m\pi}{2b} (2x-a)}{\cosh \beta_m} - \beta_m \frac{\cosh \frac{m\pi x}{b}}{2 \cosh^2 \beta_m},$$

$$K_m(y) = \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} (2 + \alpha_m \tanh \alpha_m) \sinh \frac{m\pi y}{a} + 2 \cosh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} + \\ + \alpha_m \frac{\sinh \frac{m\pi y}{a}}{\cosh^2 \alpha_m} - \frac{m\pi y}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) + (-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi y}{a}}{\cosh \alpha_m} - \\ - \frac{9}{4} \cos \frac{m\pi y}{3b} + \frac{1}{4} \cos \frac{m\pi y}{b} - \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b},$$

$$L_m(y) = \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} (1 + \alpha_m \tanh \alpha_m) \cosh \frac{m\pi y}{a} + \sinh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} + \\ + \alpha_m \frac{\cosh \frac{m\pi y}{a}}{\cosh^2 \alpha_m} - \frac{m\pi y}{a} \frac{\cosh \frac{m\pi}{2a} (2y-b) + (-1)^{\frac{m-1}{2}} \sqrt{3} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} + \\ + \frac{3}{4} \sqrt{3} \sin \frac{m\pi y}{3b} - \frac{\sqrt{3}}{4} \sin \frac{m\pi y}{b} - \frac{3\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b},$$

$$M_m(y) = \left[(-1)^{\frac{m-1}{2}} \sqrt{3} \tanh \alpha_m + \frac{1}{\cosh \alpha_m} \right] \frac{\alpha_m \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} -$$

$$\begin{aligned}
& - \frac{m\pi y}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) + (-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi y}{a}}{\cosh \alpha_m} + \\
& + \frac{3}{4} \cos \frac{m\pi y}{3b} - \frac{3}{4} \cos \frac{m\pi y}{b} + \frac{3}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b}, \\
N_m(y) = & \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} (\alpha_m \tanh \alpha_m - 1) \cosh \frac{m\pi y}{a} - \sinh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} + \\
& + \alpha_m \frac{\cosh \frac{m\pi y}{a}}{\cosh^2 \alpha_m} - \frac{m\pi y}{a} \frac{\cosh \frac{m\pi y}{2a} (2y-b) + \sqrt{3} (-1)^{\frac{m-1}{2}} \sinh \frac{m\pi y}{a}}{\cosh \alpha_m} - \\
& - \frac{\sqrt{3}}{4} \sin \frac{m\pi y}{3b} + \frac{3}{4} \sqrt{3} \sin \frac{m\pi y}{b} + \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b}, \\
S_m(y) = & - \frac{(-1)^{\frac{m}{2}} (2 + \alpha_m \coth \alpha_m) \sinh \frac{m\pi y}{a} - 2 \sinh \frac{m\pi}{2a} (2y-b)}{\sinh \alpha_m} + \alpha_m \frac{\sinh \frac{m\pi y}{a}}{\sinh^2 \alpha_m} - \\
& - \frac{m\pi y}{a} \frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi y}{a}}{\sinh \alpha_m} + \\
& + \frac{9}{4} \cos \frac{m\pi y}{3b} - \frac{1}{4} \cos \frac{m\pi y}{b} - \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b}, \\
T_m(y) = & - \frac{(-1)^{\frac{m}{2}} (1 + \alpha_m \coth \alpha_m) \cosh \frac{m\pi y}{a} - \cosh \frac{m\pi}{2a} (2y-b)}{\sinh \alpha_m} + \alpha_m \frac{\cosh \frac{m\pi y}{a}}{\sinh^2 \alpha_m} - \\
& - \frac{m\pi y}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} - \\
& - \frac{3\sqrt{3}}{4} \sin \frac{m\pi y}{3b} + \frac{\sqrt{3}}{4} \sin \frac{m\pi y}{b} - \frac{3\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b}, \\
U_m(y) = & - \left[(-1)^{\frac{m}{2}} \coth \alpha_m - \frac{1}{\sinh \alpha_m} \right] \frac{\alpha_m \sinh \frac{m\pi y}{a}}{\sinh \alpha_m} - \\
& - \frac{m\pi y}{a} \frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi y}{a}}{\sinh \alpha_m} - \\
& - \frac{3}{4} \cos \frac{m\pi y}{3b} + \frac{3}{4} \cos \frac{m\pi y}{b} + \frac{3}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi y}{3b},
\end{aligned}$$

$$V_m(y) = -\frac{(-1)^{\frac{m}{2}} (\alpha_m \coth \alpha_m - 1) \cosh \frac{m\pi y}{a} + \cosh \frac{m\pi}{2a} (2y-b)}{\sinh \alpha_m} + \alpha_m \frac{\cosh \frac{m\pi y}{a}}{\sinh^2 \alpha_m} -$$

$$-\frac{m\pi y}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sin \frac{m\pi y}{a}}{\sinh \alpha_m} +$$

$$+\frac{\sqrt{3}}{4} \sin \frac{m\pi y}{3b} - \frac{3}{4} \sqrt{3} \sin \frac{m\pi y}{b} + \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi y}{3b}.$$

(II) $\frac{b}{2} \leq y \leq b$

$$w = \frac{9q b^4}{D \pi^5} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m^*(y)}{m^5} \sin \frac{m\pi x}{a} + \right.$$

$$\left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m^*(y)}{m^5} \sin \frac{m\pi x}{a} + \frac{4}{9} \sum_{m=1,3,5,\dots}^{\infty} \frac{E_m(x)}{m^5} \sin \frac{m\pi y}{b} \right], \quad (58b)$$

$$\frac{\partial w}{\partial x} = \frac{3\sqrt{3}q b^3}{D \pi^4} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m^*(y)}{m^4} \cos \frac{m\pi x}{a} + \right.$$

$$\left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m^*(y)}{m^4} \cos \frac{m\pi x}{a} + \frac{4\sqrt{3}}{9} \sum_{m=1,3,5,\dots}^{\infty} \frac{F_m(x)}{m^4} \sin \frac{m\pi y}{b} \right],$$

$$\frac{\partial w}{\partial y} = \frac{3\sqrt{3}q b^3}{D \pi^4} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{L_m^*(y)}{m^4} \sin \frac{m\pi x}{a} + \right.$$

$$\left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{T_m^*(y)}{m^4} \sin \frac{m\pi x}{a} + \frac{4\sqrt{3}}{9} \sum_{m=1,3,5,\dots}^{\infty} \frac{E_m(x)}{m^4} \cos \frac{m\pi y}{b} \right],$$

$$\frac{\partial^2 w}{\partial x^2} = -\frac{3q b^2}{D \pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{K_m^*(y)}{m^3} \sin \frac{m\pi x}{a} + \right.$$

$$\left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{S_m^*(y)}{m^3} \sin \frac{m\pi x}{a} - \frac{4}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{G_m(x)}{m^3} \sin \frac{m\pi y}{b} \right],$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{3q b^2}{D \pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{L_m^*(y)}{m^3} \cos \frac{m\pi x}{a} + \right.$$

$$\left. + \sum_{m=2,4,6,\dots}^{\infty} \frac{T_m^*(y)}{m^3} \cos \frac{m\pi x}{a} + \frac{4}{3} \sum_{m=1,3,5,\dots}^{\infty} \frac{F_m(x)}{m^3} \cos \frac{m\pi y}{b} \right],$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{3q b^2}{D \pi^3} \left[\sum_{m=1,3,5,\dots}^{\infty} \frac{M_m^*(y)}{m^3} \sin \frac{m\pi x}{a} + \right]$$

$$\begin{aligned}
& + \sum_{m=2, 4, 6 \dots}^{\infty} \frac{U_m^*(y)}{m^3} \sin \frac{m\pi x}{a} - \frac{4}{3} \sum_{m=1, 3, 5 \dots}^{\infty} \frac{E_m(x)}{m^3} \sin \frac{m\pi y}{b} \Big], \\
\frac{\partial^3 w}{\partial x^3} = & - \frac{\sqrt{3} q b}{D \pi^2} \left[\sum_{m=1, 3, 5 \dots}^{\infty} \frac{K_m^*(y)}{m^2} \cos \frac{m\pi x}{a} + \right. \\
& \left. + \sum_{m=2, 4, 6 \dots}^{\infty} \frac{S_m^*(y)}{m^2} \cos \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1, 3, 5 \dots}^{\infty} \frac{H_m(x)}{m^2} \sin \frac{m\pi y}{b} \right], \\
\frac{\partial^3 w}{\partial x^2 \partial y} = & - \frac{\sqrt{3} q b}{D \pi^2} \left[\sum_{m=1, 3, 5 \dots}^{\infty} \frac{L_m^*(y)}{m^2} \sin \frac{m\pi x}{a} + \right. \\
& \left. + \sum_{m=2, 4, 6 \dots}^{\infty} \frac{T_m^*(y)}{m^2} \sin \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1, 3, 5 \dots}^{\infty} \frac{G_m(x)}{m^2} \cos \frac{m\pi y}{b} \right], \\
\frac{\partial^3 w}{\partial x \partial y^2} = & \frac{\sqrt{3} q b}{D \pi^2} \left[\sum_{m=1, 3, 5 \dots}^{\infty} \frac{M_m^*(y)}{m^2} \cos \frac{m\pi x}{a} + \right. \\
& \left. + \sum_{m=2, 4, 6 \dots}^{\infty} \frac{U_m^*(y)}{m^2} \cos \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1, 3, 5 \dots}^{\infty} \frac{F_m(x)}{m^2} \sin \frac{m\pi y}{b} \right], \\
\frac{\partial^3 w}{\partial y^3} = & \frac{\sqrt{3} q b}{D \pi^2} \left[\sum_{m=1, 3, 5 \dots}^{\infty} \frac{N_m^*(y)}{m^2} \sin \frac{m\pi x}{a} + \right. \\
& \left. + \sum_{m=2, 4, 6 \dots}^{\infty} \frac{V_m^*(y)}{m^2} \sin \frac{m\pi x}{a} - \frac{4\sqrt{3}}{3} \sum_{m=1, 3, 5 \dots}^{\infty} \frac{E_m(x)}{m^2} \cos \frac{m\pi y}{b} \right];
\end{aligned}$$

其中

$$\begin{aligned}
K_m^*(y) = & \frac{2 \cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m-1}{2}} \sqrt{3} (2 + \alpha_m \tanh \alpha_m) \sinh \frac{m\pi(y-b)}{a}}{\cosh \alpha_m} - \\
& - \alpha_m \frac{\sinh \frac{m\pi(y-b)}{a}}{\cosh^2 \alpha_m} + \frac{m\pi(y-b)}{a} \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi(y-b)}{a} - \sinh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} - \\
& - \frac{9}{4} \cos \frac{m\pi(y-b)}{3b} - \frac{1}{4} \cos \frac{m\pi y}{b} + \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi(y-b)}{3b}, \\
L_m^*(y) = & \frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m-1}{2}} \sqrt{3} (1 + \alpha_m \tanh \alpha_m) \cosh \frac{m\pi(y-b)}{a}}{\cosh \alpha_m} - \\
& - \alpha_m \frac{\cosh \frac{m\pi(y-b)}{a}}{\cosh^2 \alpha_m} + \frac{m\pi(y-b)}{a} \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} \sinh \frac{m\pi(y-b)}{a} - \cosh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} +
\end{aligned}$$

$$+ \frac{3\sqrt{3}}{4} \sin \frac{m\pi(y-b)}{3b} + \frac{\sqrt{3}}{4} \sin \frac{m\pi y}{b} + \frac{3\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi(y-b)}{3b},$$

$$M_m^*(y) = - \left[(-1)^{\frac{m-1}{2}} \sqrt{3} \tanh \alpha_m + \frac{1}{\cosh \alpha_m} \right] \frac{\alpha_m \sinh \frac{m\pi(y-b)}{a}}{\cosh \alpha_m} + \\ + \frac{m\pi(y-b)}{a} \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} \cosh \frac{m\pi(y-b)}{a} - \sinh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} + \\ + \frac{3}{4} \cos \frac{m\pi(y-b)}{3b} + \frac{3}{4} \cos \frac{m\pi y}{b} - \frac{3}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi(y-b)}{3b},$$

$$N_m^*(y) = \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} (1 - \alpha_m \tanh \alpha_m) \cosh \frac{m\pi(y-b)}{a} - \sinh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} - \\ - \alpha_m \frac{\cosh \frac{m\pi(y-b)}{a}}{\cosh^2 \alpha_m} + \frac{m\pi(y-b)}{a} \frac{(-1)^{\frac{m-1}{2}} \sqrt{3} \sinh \frac{m\pi(y-b)}{a} - \cosh \frac{m\pi}{2a} (2y-b)}{\cosh \alpha_m} - \\ - \frac{\sqrt{3}}{4} \sin \frac{m\pi(y-b)}{3b} - \frac{3\sqrt{3}}{4} \sin \frac{m\pi y}{b} - \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi(y-b)}{3b},$$

$$S_m^*(y) = \frac{2 \sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} (2 + \alpha_m \coth \alpha_m) \sinh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} +$$

$$+ \alpha_m \frac{\sinh \frac{m\pi(y-b)}{a}}{\sinh^2 \alpha_m} - \frac{m\pi(y-b)}{a} \frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} - \\ - \frac{9}{4} \cos \frac{m\pi(y-b)}{3b} + \frac{1}{4} \cos \frac{m\pi y}{b} - \frac{9}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi(y-b)}{3b},$$

$$T_m^*(y) = \frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} (1 + \alpha_m \coth \alpha_m) \cosh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} +$$

$$+ \alpha_m \frac{\cosh \frac{m\pi(y-b)}{a}}{\sinh^2 \alpha_m} - \frac{m\pi(y-b)}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} + \\ + \frac{3\sqrt{3}}{4} \sin \frac{m\pi(y-b)}{3b} - \frac{\sqrt{3}}{4} \sin \frac{m\pi y}{b} - \frac{3\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi(y-b)}{3b},$$

$$U_m^*(y) = - \left[(-1)^{\frac{m}{2}} \coth \alpha_m - \frac{1}{\sinh \alpha_m} \right] \frac{\alpha_m \sinh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} - \\ - \frac{m\pi(y-b)}{a} \frac{\cosh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \cosh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} + \\ + \frac{3}{4} \cos \frac{m\pi(y-b)}{3b} - \frac{3}{4} \cos \frac{m\pi y}{b} + \frac{3}{2} \sin \frac{m\pi}{3} \sin \frac{m\pi(y-b)}{3b},$$

$$V_m^*(y) = \frac{(-1)^{\frac{m}{2}} (1 - \alpha_m \coth \alpha_m) \cosh \frac{m\pi(y-b)}{a} - \cosh \frac{m\pi}{2a} (2y-b)}{\sinh \alpha_m} + \\ + \alpha_m \frac{\cosh \frac{m\pi(y-b)}{a}}{\sinh^2 \alpha_m} - \frac{m\pi(y-b)}{a} \frac{\sinh \frac{m\pi}{2a} (2y-b) - (-1)^{\frac{m}{2}} \sinh \frac{m\pi(y-b)}{a}}{\sinh \alpha_m} - \\ - \frac{\sqrt{3}}{4} \sin \frac{m\pi(y-b)}{3b} + \frac{3\sqrt{3}}{4} \sin \frac{m\pi y}{b} + \frac{\sqrt{3}}{2} \sin \frac{m\pi}{3} \cos \frac{m\pi(y-b)}{3b}.$$

利用以上這些式子，我們可導得三角板的剪力，撓矩和扭矩的式子，於是，我們可以得到正應力和剪應力的式子；我們還可以求出板邊支持的反作用力及其頂點的集中反作用力。所有這些公式都可以在鐵木辛哥所著書 [22] 中查到，因此這裏不再把它們羅列出來。

今將三角板撓度 w 表成如下的形式：

$$w = \alpha \frac{qb^4}{Eh^3},$$

其中 α 與板料的泊松係數 ν 有關，是 x, y 的函數。我們計算了斜邊中線上係數 α 當 $\nu = 0.3$ 時的數值（見表 1），並將它在圖 5 中用曲線表示出來。當 ν 為其他值時，係數 α 的數值很容易從表列數值或圖 5 中的曲線換算得到。

表 1. 邊緣簡支 $30^\circ - 60^\circ - 90^\circ$ 三角板斜邊中線上的 α 值， $\nu = 0.3$

	$x=0, y=0$	$x=\frac{\sqrt{3}}{40}b, y=\frac{3}{40}b$	$x=\frac{\sqrt{3}}{20}b, y=\frac{3}{20}b$	$x=\frac{3\sqrt{3}}{40}b, y=\frac{9}{40}b$	$x=\frac{\sqrt{3}}{10}b, y=\frac{3}{10}b$	$x=\frac{\sqrt{3}}{8}b, y=\frac{3}{8}b$	$x=\frac{\sqrt{3}}{6}b, y=\frac{b}{2}$
α	0	0.00032	0.00108	0.00170	0.00194	0.00162	0

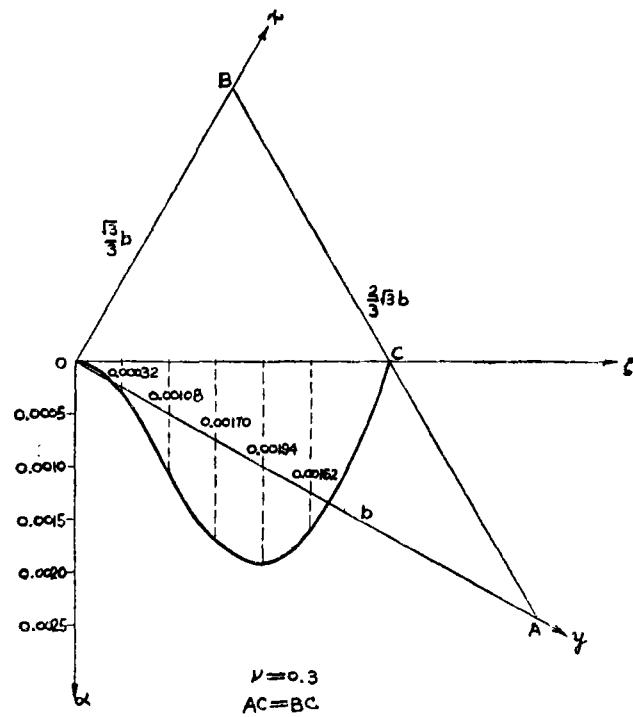


圖 5

四. 集中載荷下三角板的彎曲問題——格林函數

設有一集中力 P 作用在三角板上一點，其坐標為 (ξ, η) ，(圖 6)。今須求其撓度式子 w ，在板內除集中力作用點以外的區域應滿足方程

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0, \quad (59)$$

在板的邊緣應滿足條件

$$\left. \begin{aligned} w &= 0, \\ \frac{\partial^2 w}{\partial n^2} &= 0. \end{aligned} \right\} \quad (60)$$

這就是本節要討論的問題。

和前面兩節所說的理由及所採用的方法一樣，我們只須求出一塊矩形板的撓度式子就能導致本節問題的解決。這塊矩形板的邊緣也是簡

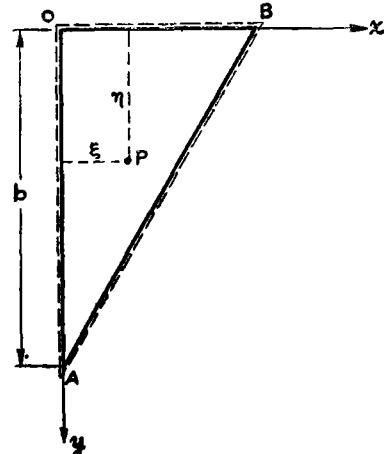


圖 6

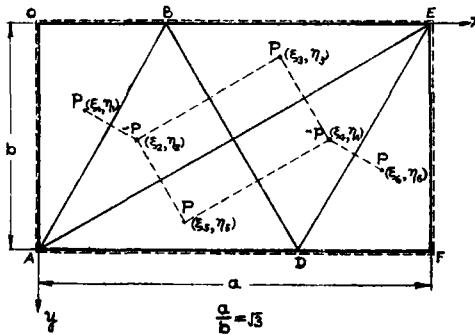


圖 7

支的，其上承担如圖 7 所示的集中載荷，各集中力作用點的坐標是：

$$\xi_1 = \xi, \quad \eta_1 = \eta; \quad (61a)$$

$$\xi_2 = \frac{1}{2}(a - \xi - \sqrt{3}\eta), \quad \eta_2 = \frac{1}{2}(b - \sqrt{3}\xi + \eta); \quad (61b)$$

$$\xi_3 = \frac{1}{2}(a - \xi + \sqrt{3}\eta), \quad \eta_3 = \frac{1}{2}(b - \sqrt{3}\xi - \eta); \quad (61c)$$

$$\xi_4 = \frac{1}{2}(a + \xi + \sqrt{3}\eta), \quad \eta_4 = \frac{1}{2}(b + \sqrt{3}\xi - \eta); \quad (61d)$$

$$\xi_5 = \frac{1}{2}(a + \xi - \sqrt{3}\eta), \quad \eta_5 = \frac{1}{2}(b + \sqrt{3}\xi + \eta); \quad (61e)$$

$$\xi_6 = a - \xi, \quad \eta_6 = b - \eta. \quad (61f)$$

根據線性理論的疊加法並利用邊緣簡支的矩形板在其上任意一點 (ξ, η) 有集中力 P 作用時的撓度式子¹⁾，即

$$w = \frac{4P}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (62)$$

則上述矩形板的撓度不難寫出為

$$w = \frac{4P}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \left(\sin \frac{m\pi\xi_1}{a} \sin \frac{n\pi\eta_1}{b} - \sin \frac{m\pi\xi_2}{a} \sin \frac{n\pi\eta_2}{b} + \right.$$

1) 見書 [22] 第 122 頁。

$$\begin{aligned}
 & + \sin \frac{m\pi\xi_3}{a} \sin \frac{n\pi\eta_3}{b} - \sin \frac{m\pi\xi_4}{a} \sin \frac{n\pi\eta_4}{b} + \sin \frac{m\pi\xi_5}{a} \sin \frac{n\pi\eta_5}{b} + \\
 & + \sin \frac{m\pi\xi_6}{a} \sin \frac{n\pi\eta_6}{b} \Big) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} .
 \end{aligned} \quad (63)$$

將各集中力作用點的坐標 (61) 式代入 (63) 式，稍事運算後，我們就得到了本節問題的解答，即上述三角板的撓度式子：

$$w = \frac{8P}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}(\xi, \eta)}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (64a)$$

這裏的函數 $\alpha_{mn}(\xi, \eta)$ 隨它指標 m, n 的數值不同而具有不同的形式，今歸納之列表如下：

(I) 當 $m \neq n$ 及 $m \neq 3n$ 時

(A) 若 m, n 皆為奇數，並且適合

$$(1) \quad \begin{cases} m=4\alpha+1 \\ n=4\beta+1 \end{cases} \text{ 或 } \begin{cases} m=4\alpha+3 \\ n=4\beta+3 \end{cases}, \quad (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\begin{aligned}
 \alpha_{mn} = & \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} - \cos \frac{m\pi}{2a} (\xi + \sqrt{3}\eta) \cos \frac{n\pi}{2b} (\sqrt{3}\xi - \eta) + \\
 & + \cos \frac{m\pi}{2a} (\xi - \sqrt{3}\eta) \cos \frac{n\pi}{2b} (\sqrt{3}\xi + \eta);
 \end{aligned}$$

$$(2) \quad \begin{cases} m=4\alpha+1 \\ n=4\beta+3 \end{cases} \text{ 或 } \begin{cases} m=4\alpha+3 \\ n=4\beta+1 \end{cases}, \quad (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\begin{aligned}
 \alpha_{mn} = & \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} + \cos \frac{m\pi}{2a} (\xi + \sqrt{3}\eta) \cos \frac{n\pi}{2b} (\sqrt{3}\xi - \eta) - \\
 & - \cos \frac{m\pi}{2a} (\xi - \sqrt{3}\eta) \cos \frac{n\pi}{2b} (\sqrt{3}\xi + \eta).
 \end{aligned}$$

(B) 若 m, n 皆為偶數，並且適合

$$(1) \quad \begin{cases} m=4\alpha \\ n=4\beta \end{cases} \text{ 或 } \begin{cases} m=4\alpha+2 \\ n=4\beta+2 \end{cases}, \quad (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\begin{aligned}
 \alpha_{mn} = & \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} - \sin \frac{m\pi}{2a} (\xi + \sqrt{3}\eta) \sin \frac{n\pi}{2b} (\sqrt{3}\xi - \eta) + \\
 & + \sin \frac{m\pi}{2a} (\xi - \sqrt{3}\eta) \sin \frac{n\pi}{2b} (\sqrt{3}\xi + \eta);
 \end{aligned}$$

$$(2) \quad \begin{cases} m=4\alpha+2 \\ n=4\beta \end{cases} \text{ 或 } \begin{cases} m=4\alpha \\ n=4\beta+2 \end{cases}, (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\alpha_{mn} = \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} + \sin \frac{m\pi}{2a} (\xi + \sqrt{3}\eta) \sin \frac{n\pi}{2b} (\sqrt{3}\xi - \eta) - \sin \frac{m\pi}{2a} (\xi - \sqrt{3}\eta) \sin \frac{n\pi}{2b} (\sqrt{3}\xi + \eta).$$

(II) 當 m, n 屬於其他情形時,

$$\alpha_{mn} = 0.$$

在 (64a) 式中令 $P = 1$, 並以 $G(\xi, \eta, x, y)$ 命名, 那末式子

$$G(\xi, \eta, x, y) = \frac{8}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}(\xi, \eta)}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (65)$$

就給出了在橫向載荷作用下三角板彎曲時的格林函數。利用這個函數我們可以計算三角板在任意載荷下的撓度。舉個例來說，如果板承担均勻分佈的載荷 q , 那末, 它的撓度為

$$w = q \int_0^b d\eta \int_0^{\frac{\sqrt{3}}{3}(b-y)} G(\xi, \eta, x, y) d\xi. \quad (66)$$

將 (65) 式及 (64b) 代入並經積分運算後, 我們有

$$w = \frac{16q}{D\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \quad (67a)$$

這裏的係數 α_{mn} 和它指標 m, n 的關係具有各種不同的情形, 今歸納之列表如下:

(I) 當 $m \neq n$ 及 $m \neq 3n$ 時

(A) 若 m, n 皆為奇數, 並且適合

$$(1) \quad \begin{cases} m=4\alpha+1 \\ n=4\beta+1 \end{cases} \text{ 或 } \begin{cases} m=4\alpha+3 \\ n=4\beta+3 \end{cases}, (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\alpha_{mn} = \frac{1}{mn} \cdot \frac{(3n+m)(n-m)}{(3n-m)(n+m)}.$$

$$(2) \quad \begin{cases} m=4\alpha+1 \\ n=4\beta+3 \end{cases} \text{ 或 } \begin{cases} m=4\alpha+3 \\ n=4\beta+1 \end{cases}, (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$\alpha_{mn} = \frac{1}{mn} \cdot \frac{(3n-m)(n+m)}{(3n+m)(n-m)}.$$

(67b)

(B) 若 m, n 皆為偶數，並且適合

$$\left. \begin{array}{l} m=4\alpha+2 \\ n=4\beta \end{array} \right\} \text{或} \left. \begin{array}{l} m=4\alpha \\ n=4\beta+2 \end{array} \right\}, (\alpha, \beta=0, 1, 2, 3, \dots),$$

則

$$a_{mn} = -\frac{16mn}{(n^2-m^2)(9n^2-m^2)}.$$

(II) 當 m, n 屬於其他情形時

$$a_{mn} = 0.$$

於是，我們又用重三角級數把第三節問題的解答表示出來了，但是我們知道級數 (67a) 的收斂性却沒有 (58a) 及 (58b) 兩級數的好。

五. 三角板的彎曲與拉壓聯合問題

我們先來討論三角板承受均勻側向力 N 及板內任意一點 (ξ, η) 上有橫向集中力 P 同時作用着的情形（圖 8）。這時板的撓度 w 在板內除集中力作用點以外的區域須滿足方程

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N}{D} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0. \quad (68)$$

在板的邊緣應滿足條件：

$$\left. \begin{array}{l} w = 0, \\ \frac{\partial^2 w}{\partial n^2} = 0. \end{array} \right\} \quad (69)$$

其中，當側向力為壓力時 $N > 0$ ；當側向力為拉力時 $N < 0$ 。

和第四節的步驟相同，根據線性理論的疊加法並利用承受均勻側向力 N 的簡支矩形板在其一點 (ξ, η) 有橫向集中力 P 作用時的撓度式子¹⁾

$$w = \frac{4P}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{N}{\pi^2 D} \right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (70)$$

我們可以得到如圖 9 所示的矩形板的撓度，圖中各集中力作用點的坐標應與 (61) 式相同。今將其撓度式子寫出如下：

1) 見書 [22] 第 310 頁。

$$\omega = \frac{8P}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}(\xi, \eta)}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{N}{\pi^2 D}\right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (71)$$

其中函數 $\alpha_{mn}(\xi, \eta)$ 的形式與 (64b) 相同。

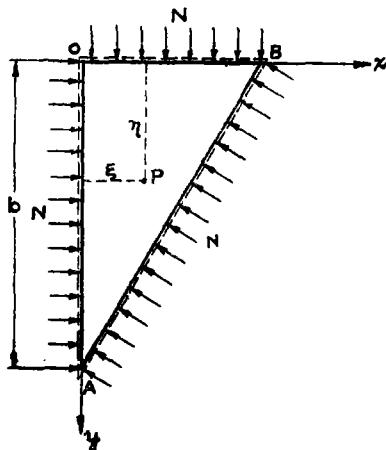


圖 8

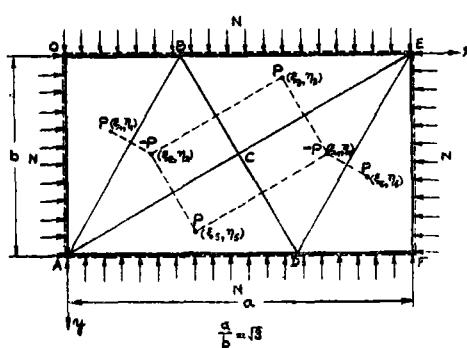


圖 9

這個 (71) 式在定義域 $\triangle OAB$ 內也就是我們所需的三角形板在均勻側向力 N 及板內任意一點 (ξ, η) 上的橫向集中力 P 同時作用下的撓度式子。比較 (71) 式及 (64a) 式，我們知道它們的差別只是級數中每項的分母。只須把 (64a) 式中各項分母 $\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$ 換以 $\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{N}{\pi^2 D}\right)$ ，我們就立即得到 (71) 式。這樣的替換法在 (65) 及 (67a) 式中仍然適用。因此，在三角板彎曲與拉壓聯合問題中，橫向載荷的格林函數可寫出為

$$G(\xi, \eta, x, y) =$$

$$= \frac{8}{D\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}(\xi, \eta)}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{N}{\pi^2 D}\right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (72)$$

其中函數 $\alpha_{mn}(\xi, \eta)$ 與 (64b) 相同。

利用格林函數，我們便可以計算三角板在任意橫向載荷與均勻側向力聯合作用下的撓度。

三角板在均佈載荷 q 及均勻側向力 N 聯合作用下的撓度可以從 (67a) 式並利用上述的替換法得到，即

$$w = \frac{16q}{D\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{a_{mn}}{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \right) \left(\frac{a_{mn}}{\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{N}{\pi^2 D}} \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \quad (73)$$

其中係數 a_{mn} 與 (67b) 相同。

六. 三角形板的穩定與振動問題

彈性系統自然振動下的基本固有頻率在實用上很重要，因此本節僅討論自然振動的問題。簡支三角板在均勻側向力 N 作用下自然振動時的撓度須滿足方程

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N}{D} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\rho}{D} \frac{\partial^2 w}{\partial t^2} = 0, \quad (74)$$

及邊界條件：

$$\left. \begin{array}{l} \text{在邊界上} \\ \quad w = 0, \\ \quad \frac{\partial^2 w}{\partial n^2} = 0; \end{array} \right\} \quad (75)$$

其中 ρ 是三角板單位面積內的質量。

假定振動是簡諧的，那末振動的基本固有型式很容易得到，即

$$w = A \left(\sin \frac{\pi x}{a} \sin \frac{3\pi y}{b} + \sin \frac{4\pi x}{a} \sin \frac{2\pi y}{b} + \sin \frac{5\pi x}{a} \sin \frac{\pi y}{b} \right) \cos \omega t, \quad (76)$$

其中 A 是任意常數。

(76) 式滿足邊界條件 (75) 的證明很簡單，這裏不再寫出來。今將 (76) 式代入方程 (74)，我們便得到基本固有頻率 ω 與側向力 N 的關係式：

$$\omega^2 = \frac{28\pi^2}{3\rho b^2} \left[\frac{28}{3} - \frac{D\pi^2}{b^2} - N \right]. \quad (77)$$

若 $\omega = 0$ ，那末 (77) 式就給出了三角形板穩定問題中屈曲的基本型式，即

$$w = A \left(\sin \frac{\pi x}{a} \sin \frac{3\pi y}{b} + \sin \frac{4\pi x}{a} \sin \frac{2\pi y}{b} + \sin \frac{5\pi x}{a} \sin \frac{\pi y}{b} \right), \quad (78)$$

(77) 式也就給出了三角形板穩定問題中最小的臨界壓力，即

$$N_{cr} = \frac{28}{3} - \frac{D\pi^2}{b^2}. \quad (79)$$

本文在胡海昌同志指導下完成，作者在這裏謹向他致衷心的謝意。

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EQUILIBRIUM, BUCKLING AND VIBRATION OF A $30^\circ-60^\circ-90^\circ$ -TRIANGULAR PLATE SIMPLY SUPPORTED AT THE EDGES

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ABSTRACT

In this paper a triangular plate is considered as a portion of a rectangular plate by using the method of images in a special manner. Thus the expression for the deflection of this triangular plate is the same as that for the deflection of the rectangular plate, in spite of the difference between the regions in which the expressions are defined. Since the solutions of the problems of a simply supported rectangular plate can be obtained without difficulty, the bending problem, as well as the problem of bending combining with tension or compression, of a $30^\circ-60^\circ-90^\circ$ -triangular plate is solved at once. Their results are expressed in the forms of trigonometric series, single or double, in this paper.

The buckling problem and the vibration problem of such a triangular plate are also discussed. The main results are as follows:

The smallest critical value of the compressive force per unit length is

$$N_{cr} = \frac{28}{3} \frac{D\pi^2}{b^2}$$

where D is the flexural rigidity of the plate and b is the length of the side opposite the 60° angle of the triangular plate.

The relation between the fundamental natural frequency ω and the tensile ($N < 0$), or compressive ($N > 0$) forces per unit length acting along the boundaries of the plate is

$$\omega^2 = \frac{28}{3} \frac{\pi^2}{\rho b^2} \left[\frac{28}{3} \frac{D\pi^2}{b^2} - N \right]$$

where ρ is the mass per unit area of the plate.

The method of images applied, in the same manner as mentioned above, to the torsion problem of a prismatical bar with $30^\circ-60^\circ-90^\circ$ -triangular cross section gives more practical results than those obtained by other authors. The torsional rigidity numerically calculated coincides with that obtained by G. E. Hay in 1939.