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任意橫向載荷下彈性圓形及圓環形薄板的彎曲*

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一. 前言

波蘭科學院通訊院士 W. 諾瓦茨基教授不久以前在北京所作的關於他在彈性薄板方面的工作的介紹報告，引起我們的興趣與注意. 在討論的過程中，本文著者曾試從有限變換式（finite transform）的角度上來說明諾瓦茨基教授從物理觀點導出的一個普遍的方法[1].

下面我們將討論邊緣爲夾住的或自由支承的彈性圓及圓環形薄板在任意橫向載荷分佈下的彎曲問題，說明有限漢克爾變換式（finite Hankel transform）的應用.

可以這樣地定義一函數 $f(x)$ 在 $0 \leqslant x \leqslant 1$ 間隔上的有限漢克爾變換式:

$$\bar{f}(\xi_{m,i}) = \int_0^1 x\, f(x)\, J_m(\xi_{m,i}\, x)\, dx\,;$$

此處 J_m 代表 m 階第一種貝塞爾函數，$\xi_{m,i}$ 代表

$$J_m(x) = 0$$

的第 i 個根. 可以證明[2]，

$$f(x) = 2 \sum_{i=1}^{\infty} \bar{f}(\xi_{m,i})\, \frac{J_m(\xi_{m,i}\, x)}{[J'_m(\xi_{m,i})]^2}\,.$$

彈性圓薄板的非對稱型彎曲的個別情形曾由 W. 弗留格[3]，H. 雷斯納爾[4]，H. 許密特[5]，A. И. 盧爾葉[6] 等人討論過，本文所提出的方法使問題的解答用很簡潔的方式給出1).

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1) B. 森在 [7] 中曾企圖用類似的方法來解決本文第二節所提的問題，但他的結果是有錯誤的，同時他也沒有考慮到各種具體的情況.

二． 邊緣夾住的圓薄板在任意橫向載荷下的彎曲

衆所周知，厚度爲 h 的彈性圓薄板在任意橫向載荷 $p(r,\theta)$ 作用之下撓度 W 所滿足的微分方程是

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 w=\frac{p}{D}; \tag{1}$$

此處 $D=\dfrac{Eh^3}{12(1-\sigma^2)}$，稱爲抗彎剛度，$E$ 是楊氏模數，σ 是泊桑係數．如果在邊緣上 $r=a$（a 爲板的半徑）板是夾住的，則有邊界條件

$$w\Big|_{r=a}=0,\qquad \frac{\partial w}{\partial r}\Big|_{r=a}=0. \tag{2}$$

引進無量綱的量

$$W=\frac{w}{a},\qquad x=\frac{r}{a},\qquad q=\frac{pa^3}{D};$$

則微分方程（1）及邊界條件（2）變爲

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)^2 W=q; \tag{3}$$

$$W\Big|_{x=1}=0,\qquad \frac{\partial W}{\partial x}\Big|_{x=1}=0. \tag{4}$$

首先將 $q(x,\theta)$ 展開爲

$$q=Q_0(x)+\sum_{n=1}^{\infty}Q_m(x)\cos m\theta+\sum_{n=1}^{\infty}R_m(x)\sin m\theta, \tag{5}$$

此處

$$\left.\begin{aligned}Q_0(x)&=\frac{1}{2\pi}\int_0^{2\pi}q\,d\theta,\\ Q_m(x)&=\frac{1}{\pi}\int_0^{2\pi}q\cos m\theta\,d\theta,\\ R_m(x)&=\frac{1}{\pi}\int_0^{2\pi}q\sin m\theta\,d\theta.\end{aligned}\right\} \tag{6}$$

今亦將 W 寫作

$$W=U_0(x)+\sum_{m=1}^{\infty}U_m(x)\cos m\theta+\sum_{m=1}^{\infty}V_m(x)\sin m\theta. \tag{7}$$

對於 U_0, U_m 及 V_m 有如下類型的邊值問題：

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m=P_m; \tag{8}$$

$$X_m\Big|_{x=1}=0, \qquad \frac{dX_m}{dx}\Big|_{x=1}=0. \tag{9}$$

現在就使用有限漢克爾變換的方法,將 (8) 式左右都乘以 $x\,J_m(\xi_{m,i}\,x)$, 再對 x 由 0 到 1 積分.

今有

$$\int_0^1 x\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)J_m(\xi_{m,i}\,x)\,dx=$$

$$=\left[x\frac{dX_m}{dx}J_m(\xi_{m,i}\,x)\right]_0^1-\int_0^1\frac{dX_m}{dx}\,x\,\xi_{m,i}\,J'_m(\xi_{m,i})\,dx-\int_0^1 X_m\left(\frac{m^2J_m(\xi_{m,i}\,x)}{x}\right)dx=$$

$$=\left[x\frac{dX_m}{dx}J_m(\xi_{m,i}\,x)-\xi_{m,i}\,x\,X_m\,J'_m(\xi_{m,i}\,x)\right]_0^1-\xi_{m,i}^2\int_0^1 x\,X_m\,J_m(\xi_{m,i}\,x)\,dx$$

或

$$\int_0^1 x\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)dx=$$

$$=\left[x\frac{dX_m}{dx}J_m(\xi_{m,i}\,x)-\xi_{m,i}\,x\,X_m\,J'_m(\xi_{m,i}\,x)\right]_0^1-\xi_{m,i}^2\,\overline{X}_m(\xi_{m,i}), \tag{10}$$

此處用有限漢克爾變換符號

$$\overline{X}_m(\xi_{m,i})=\int_0^1 x\,X_m\,J_m(\xi_{m,i}\,x)\,dx. \tag{11}$$

同理,

$$\int_0^1 x\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)J_m(\xi_{m,i}\,x)\,dx=$$

$$=\left[x\frac{d}{dx}\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)J_m(\xi_{m,i}\,x)-\right.$$

$$\left.-\xi_{m,i}\,x\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)J'_m(\xi_{m,i}\,x)\right]_0^1-$$

$$-\xi_{m,i}^2\left[x\frac{dX_m}{dx}J_m(\xi_{m,i}\,x)-\xi_{m,i}\,x\,X_m\,J'_m(\xi_{m,i}\,x)\right]_0^1+\xi_{m,i}^4\,\overline{X}_m(\xi_{m,i}). \tag{12}$$

如果已取 $\xi_{m,i}$ 爲方程

$$J_m(x)=0$$

的第 i 個根,則利用邊界條件 (9) 可以得到

$$\int_0^1 x\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m\,J_m(\xi_{m,i}\,x)\,dx=$$

$$=-\xi_{m,i}\,A_m\,J'_m(\xi_{m,i})+\xi_{m,i}^4\,\overline{X}_m(\xi_{m,i}),$$

此處

$$A_m=\left(\frac{d^2 X_m}{dx^2}\right)_{x=1} \tag{13}$$

是待定的一些量.

如果寫

$$\bar{P}_m(\xi_{m,i})=\int_0^1 x\, P_m(x)\, J_m(\xi_{m,i}\, x)\, dx\,, \tag{14}$$

則方程（8）的上述積分結果給出

$$\xi_{m,i}^4\, \bar{X}_m(\xi_{m,i})-\xi_{m,i}\, A_m\, J_m'(\xi_{m,i})=\bar{P}_m(\xi_{m,i})\,,$$

於是得到

$$\bar{X}_m(\xi_{m,i})=\frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}^4}+\frac{J_m'(\xi_{m,i})}{\xi_{m,i}^3}A_m\,; \tag{15}$$

而按有限漢克爾變換的逆繪式,得到

$$X_m(x)=2\sum_{i=1}^{\infty}\frac{\bar{P}_m(\xi_{m,i})\, J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\,[J_m'(\xi_{m,i})]^2}+2\,A_m\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}\,. \tag{16}$$

顯然 $X_m\Big|_{x=1}=0$, 而要滿足另一個條件 $\dfrac{dX_m}{dx}\Big|_{x=1}=0$、就必須有

$$2\sum_{i=1}^{\infty}\frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}+2\,A_m\sum\frac{1}{\xi_{m,i}^2}=0\,.$$

於是得出

$$A_m=-\frac{\displaystyle\sum_{i=1}^{\infty}\frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty}\frac{1}{\xi_{m,i}^2}}\,. \tag{17}$$

於是,最後得出板的無量綱撓度的表式

$$\begin{aligned}W(x,\theta)=&\;2\sum_{i=1}^{\infty}\frac{\bar{Q}_0(\xi_{0,i})\, J_0(\xi_{0,i}\, x)}{\xi_{0,i}^4\,[J_0'(\xi_{0,i})]^2}+2\,A_0\sum_{i=1}^{\infty}\frac{J_0(\xi_{0,i}\, x)}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}+\\&+2\sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\,[J_m'(\xi_{m,i})]^2}\,[\bar{Q}_m(\xi_{m,i})\cos m\theta+\bar{R}_m(\xi_{m,i})\sin m\theta]+\\&+2\sum_{m=1}^{\infty}(A_m\cos m\theta+B_m\sin m\theta)\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}\,.\end{aligned} \tag{18}$$

此處

$$\left.\begin{aligned}
\bar{Q}_0(\xi_{0,i}) &= \int_0^1 x\, Q_0\, J_0(\xi_{0,i}\, x)\, dx\,,\\
\bar{Q}_m(\xi_{m,i}) &= \int_0^1 x\, Q_m\, J_m(\xi_{m,i}\, x)\, dx\,,\\
\bar{R}_m(\xi_{m,i}) &= \int_0^1 x\, R_m\, J_m(\xi_{m,i}\, x)\, dx\,.
\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}
A_0 &= -\frac{\displaystyle\sum_{i=1}^{\infty} \frac{\bar{Q}_0(\xi_{0,i})}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}^{1)}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{0,i}^2}}\,,\\
A_m &= -\frac{\displaystyle\sum_{i=1}^{\infty} \frac{\bar{Q}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,,\\
B_m &= -\frac{\displaystyle\sum_{i=1}^{\infty} \frac{\bar{R}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,.
\end{aligned}\right\} \tag{20}$$

在圓板中心（$x = 0$）有

$$W(0) = 2 \sum_{i=1}^{\infty} \frac{\bar{Q}(\xi_{0,i})}{\xi_{0,i}^4\, [J_0'(\xi_{0,i})]^2} + 2\, A_0 \sum_{i=1}^{\infty} \frac{1}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}\,. \tag{21}$$

現在來考慮一個重要的特殊載荷情形． 設在 $x = c$（$0 \leqslant c < 1$），$\theta = 0$ 有一個"單位"（在無量綱量表示內）集中載荷． 於是可以用 $\delta-$ 函數的形式：

$$q = \frac{\delta\,(x-c)\;\delta\,(\theta-0)}{x}\,, \tag{22}$$

於是

$$\left.\begin{aligned}
Q_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} q\, d\theta = \frac{\delta\,(x-c)}{2\pi\, x}\,,\\
Q_m(x) &= \frac{1}{\pi} \int_0^{2\pi} q \cos m\theta\, d\theta = \frac{\delta\,(x-c)}{\pi x}\,,\\
R_m(x) &= \frac{1}{\pi} \int_0^{2\pi} q \sin m\theta\, d\theta = 0\,.
\end{aligned}\right\} \tag{23}$$

而

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1) 可以證明 $\sum\limits_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2} = \frac{1}{4(m+1)}$，例如 $\sum\limits_{i=1}^{\infty} \frac{1}{\xi_{0,i}} = \frac{1}{4}$.

$$\left.\begin{aligned}
\bar{Q}_0(\xi_{0,i}) &= \int_0^1 x\, Q_v\, J_0(\xi_{0,i}\, x)\, dx = \frac{J_0(\xi_{0,i}\, c)}{2\pi}\,,\\
\bar{Q}_m(\xi_{m,i}) &= \int_0^1 x\, Q_m\, J_m(\xi_{m,i}\, x)\, dx = \frac{J_m(\xi_{m,i}\, c)}{\pi}\,,\\
\bar{R}_m(\xi_{m,i}) &= \int_0^1 x\, R_m\, J_m(\xi_{m,i}\, x)\, dx = 0\,.
\end{aligned}\right\} \tag{24}$$

於是

$$\left.\begin{aligned}
A_0 &= -\frac{\displaystyle\sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, c)}{\xi_{0,i}^3\, J_0(\xi_{0,i})}}{\displaystyle 2\pi \sum \frac{1}{\xi_{0,i}^2}}\,,\\
A_m &= -\frac{\displaystyle\sum_{i=2}^{\infty} \frac{J_m(\xi_{m,i}\, c)}{\xi_{m\ i}^3\, J_m(\xi_{m,i})}}{\displaystyle \pi \sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,,\\
B_m &= 0\,.
\end{aligned}\right\} \tag{25}$$

於是有

$$\begin{aligned}
W(x,\theta) = {}& \frac{1}{\pi} \sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, c)\, J_0(\xi_{0,i}\, x)}{\xi_{0,i}^4\, [J_0'(\xi_{0,i})]^2} - \frac{1}{\pi} \frac{\left(\displaystyle\sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, c)}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}\right)}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{0,i}^2}} \sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, x)}{\xi_{0,i}^3\, J_0'(\xi_{0,i})} + {}\\
& + \frac{2}{\pi} \sum_{m=1}^{\infty} \cos m\theta \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, c)\, J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\, [J_m'(\xi_{m,i})]^2} - {}\\
& - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\left(\displaystyle\sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, c)}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}\right)}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}} \cos m\theta \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}\,,
\end{aligned} \tag{26}$$

而在圓板中心（$x = 0$）有

$$W(0) = \frac{1}{\pi} \sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, c)}{\xi_{0,i}^4\, [J_0'(\xi_{0,i})]^2} - \frac{1}{\pi} \left(\frac{\displaystyle\sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, c)}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}}{\displaystyle\sum \frac{1}{\xi_{0,i}^2}} \right) \sum_{i=1}^{\infty} \frac{1}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}\,. \tag{27}$$

如果沿 $\theta = 0$ 一線有總強度等於 1 的載荷，則

$$q = \frac{\delta(\theta - 0)}{x}\,, \tag{28}$$

而

$$Q_0=\frac{1}{2\pi\, x}, \qquad Q_m=\frac{1}{\pi x}, \qquad R_m=0\,; \tag{29}$$

而

$$\left.\begin{aligned}\bar{Q}_0(\xi_{0,i})&=\frac{1}{2\pi}\int_0^1 J_0(\xi_{0,i}\,x)\,dx\,,\\ \bar{Q}_m(\xi_{m,i})&=\frac{1}{\pi}\int_0^1 J_m(\xi_{m,i}\,x)\,dx\,,\\ \bar{R}_m(\xi_{m,i})&=0\,.\end{aligned}\right\} \tag{30}$$

代入 (18) 卽可得到解答,這當然也可以由 (26) 用對 c 由 0 到 1 積分的辦法得到.

還不難討論更一般的載荷情况,例如

$$q=\frac{g(x)\,\delta\,(\theta-0)}{x},$$

這相等於沿 $\theta=0$ 一線有變異的載荷的情形,等等.

三. 中面內有張力的邊緣夾住的圓薄板在任意橫向載荷下的彎曲

用前節中所提出的方法還可以解中面內有張力 T 的、邊緣夾住的圓薄板在任意橫向載荷下的彎曲問題. 這問題曾由 W. G. 比克來[8] 考慮過,但他只討論了均佈載荷及在平板上任意點的集中載荷的情況,並且用的是另一種展開方法.

在這情況下有方程:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)^2 W-\tau\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)W=q\,, \tag{31}$$

此處 $\tau=\frac{Ta^2}{D}$ 是一個無量綱的數量.

和以前一樣,將 q 寫爲

$$q=Q_0(x)+\sum_{m=1}^{\infty}Q_m(x)\cos m\theta+\sum_{m=1}^{\infty}R_m(x)\sin m\theta, \tag{32}$$

而將 W 展開爲

$$W=U_0(x)+\sum_{m=1}^{\infty}U_m(x)\cos m\theta+\sum_{m=1}^{\infty}V_m(x)\sin m\theta\,, \tag{33}$$

於是對於 $U_0(x)$, $U_m(x)$, $V_m(x)$ 有如下類型的邊值問題:

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m-\tau\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)X_m=P_m\,, \tag{34}$$

$$X_m\Big|_{x=1}=0\,, \qquad \frac{dX_m}{dx}\Big|_{x=1}=0\,. \tag{35}$$

以 $x\,J_m\,(\xi_{m,i}\,x)$ 乘 (34) 雙方,再對 x 由 0 到 1 積分, 利用 (10)、(12) 及邊界條

件（35）可以得到

$$\xi_{m,i}^{4}\,\bar{X}_m(\xi_{m,i})+\tau\,\xi_{m,i}^{2}\,\bar{X}_m(\xi_{m,i})-\xi_{m,i}\,A_m(\tau)\,J'_m(\xi_{m,i})=\bar{P}_m(\xi_{m,i})\,; \tag{36}$$

此處 $\xi_{m,i}$ 是方程

$$J_m(x)=0$$

的第 i 個根，而

$$A_m(\tau)=\left(\frac{d^2X_m}{dx^2}\right)_{x=1} \tag{37}$$

是待定數量，於是

$$\bar{X}_m(\xi_{m,i})=\frac{\bar{P}_m(\zeta_{m,i})}{\xi_{m,i}^{4}+\tau\,\xi_{m,i}^{2}}+\frac{J'_m(\xi_{m,i})}{\xi_{m,i}^{3}+\tau\,\xi_{m,i}}A_m(\tau) \tag{38}$$

而

$$X_m=2\sum_{i=1}^{\infty}\frac{\bar{P}_m(\xi_{m,i})\,J_m(\xi_{m,i}\,x)}{(\xi_{m,i}^{4}+\tau\,\xi_{m,i}^{2})\,[J'_m(\xi_{m,i})]^2}+2\,A_m(\tau)\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\,x)}{(\xi_{m,i}^{3}+\tau\,\xi_{m,i})\,J'_m(\xi_{m,i})}\,. \tag{39}$$

顯然，$X_m\Big|_{x=1}=0$. 而爲了滿足條件 $\dfrac{dX_m}{dx}\Big|_{x=1}=0$ 就必須有

$$A_m(\tau)=\frac{-\displaystyle\sum_{i=1}^{\infty}\frac{\bar{P}_m(\xi_{m,i})}{(\xi_{m,i}^{3}+\tau\,\xi_{m,i})\,J'_m(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty}\frac{1}{\xi_{m,i}^{2}+\tau}}\,, \tag{40}$$

而最後有

$$\begin{aligned}W(x,\theta)=&\,2\sum_{i=1}^{\infty}\frac{\bar{Q}_0(\xi_{0,i})\,J_0(\xi_{0,i}\,x)}{(\xi_{0,i}^{4}+\tau\,\xi_{0,i}^{2})\,[J'_0(\xi_{0,i})]^2}+2\,A_0(\tau)\sum_{i=1}^{\infty}\frac{J_0(\xi_{0,i}\,x)}{(\xi_{0,i}^{3}+\tau\,\xi_{0,i})\,J'_0(\xi_{0,i})}+\\&+2\sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\,x)}{(\xi_{m,i}^{4}+\tau\,\xi_{m,i}^{2})\,[J'_m(\xi_{0,i})]^2}\,[\bar{Q}_m(\xi_{m,i})\cos m\theta+\bar{R}_m(\xi_{m,i})\sin m\theta]+\\&+2\sum_{m=1}^{\infty}[A_m(\tau)\cos m\theta+B_m(\tau)\sin m\theta]\sum_{i=1}^{\infty}\frac{J_m(\xi_{m,i}\,x)}{(\xi_{m,i}^{3}+\tau\,\xi_{m,i})\,J'_m(\xi_{m,i})}\,;\end{aligned} \tag{41}$$

此處

$$\left.\begin{aligned}\bar{Q}_0(\xi_{0,i})&=\int_0^1 x\,Q_0\,J_0(\xi_{0,i}\,x)\,dx\,,\\ \bar{Q}_m(\xi_{m,i})&=\int_0^1 x\,Q_m\,J_m(\xi_{m,i}\,x)\,dx\,,\\ \bar{R}_m(\xi_{m,i})&=\int_0^1 x\,R_m\,J_m(\xi_{m,i}\,x)\,dx\,.\end{aligned}\right\} \tag{42}$$

而

$$\left.\begin{aligned}
A_0(\tau) &= -\frac{\displaystyle\sum_{i=1}^{\infty}\frac{\bar{Q}_0(\xi_{0,i})}{(\xi_{0,i}^3+\tau\,\xi_{0,i})\,J_0'(\xi_{0,i})}}{\displaystyle\sum_{i=1}^{\infty}\frac{1}{\xi_{0,i}^2+\tau}}\,,\\
A_m(\tau) &= -\frac{\displaystyle\sum_{i=1}^{\infty}\frac{\bar{Q}_m(\xi_{m,i})}{(\xi_{m,i}^3+\tau\,\xi_{m,i})\,J_m'(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty}\frac{1}{\xi_{m,i}^2+\tau}}\,,\\
B_m(\tau) &= -\frac{\displaystyle\sum_{i=1}^{\infty}\frac{\bar{R}_m(\xi_{m,i})}{(\xi_{m,i}^3+\tau\,\xi_{m,i})\,J_m'(\xi_{m,i})}}{\displaystyle\sum_{i=1}^{\infty}\frac{1}{\xi_{m,i}^2+\tau}}\,.
\end{aligned}\right\}\qquad (43)$$

四. 邊緣簡支的圓薄板在任意載荷下的彎曲

現在來考慮邊緣簡支的情況．不再重復論據，只需提出此處我們有如次類型的邊值問題：

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m = P_m\,, \qquad (44)$$

$$X_m\big|_{x=1}\,, \qquad (45)$$

$$\left[\frac{d^2X_m}{dx^2}+\frac{\sigma}{x}\frac{dX_m}{dx}\right]_{x=1}=0\,. \qquad (46)$$

(46) 式表示邊緣撓矩等於零．

設 $\xi_{m,i}$ 是方程

$$J_m(x)=0$$

的第 i 個根，則有（參見（12））

$$\int_0^1 x\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m\,J_m(\xi_{m,i}x)\,dx =$$

$$=\left[-\xi_{m,i}\,x\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)J_m'(\xi_{m,i}x)\right]_0^1+$$

$$+\xi_{m,i}^3\,x\,X_m\,J_m'(\xi_{m,i}x)\Big|_0^1+\xi_{m,i}^4\,\bar{X}_m(\xi_{m,i})\,;$$

再利用邊界條件（45）、（46）得到

$$\int_0^1 x\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m\,J_m(\xi_{m,i}x)\,dx =$$

$$=-\xi_{m,i}(1-\sigma)\,C_m\,J_m'(\xi_{m,i})+\xi_{m,i}^4\,\bar{X}_m(\xi_{m,i})\,, \qquad (47)$$

此處

$$C_m = \left(\frac{dX_m}{dx} \right)_{x=1} \tag{48}$$

是待定的數量.

於是，由方程（44）可以得到

$$\overline{X}_m = \frac{\overline{P}_m(\xi_{m,i})}{\xi_{m,i}^4} + (1-\sigma)\, C_m \frac{J'_m(\xi_{m,i})}{\xi_{m,i}^3}, \tag{49}$$

而

$$X_m = 2 \sum_{i=1}^{\infty} \frac{\overline{P}_m(\xi_{m,i})}{\xi_{m,i}^4} \frac{J_m(\xi_{m,i}\, x)}{[J'_m(\xi_{m,i})]^2} + 2\,(1-\sigma)\, C_m \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J'_m(\xi_{m,i})}, \tag{50}$$

常數 C_m 是由下面的方程決定的：

$$C_m = \left(\frac{dX_m}{dx} \right)_{x=1} = 2 \sum_{i=1}^{\infty} \frac{\overline{P}_m(\xi_{m,i})}{\xi_{m,i}^3\, J'_m(\xi_{m,i})} + 2\,(1-\sigma)\, C_m \sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2},$$

或

$$C_m = \frac{2 \sum \dfrac{\overline{P}_m(\xi_{m,i})}{\xi_{m,i}^3\, J'_m(\xi_{m,i})}}{1 - 2\,(1-\sigma) \displaystyle\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}. \tag{51}$$

最後得到板的無量綱撓度 W 表式如下：

$$\begin{aligned} W(x,\theta) = {} & 2 \sum_{i=1}^{\infty} \frac{\overline{Q}_0(\xi_{0,i})\, J_0(\xi_{0,i}\, x)}{\xi_{0,i}^4\, [J'_0(\xi_{0,i})]^2} + 2\, C_0 \sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, x)}{\xi_{0,i}^3\, J'_0(\xi_{0,i})} + \\ & + 2 \sum_{m=1}^{\infty} \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\, (J'_m(\xi_{m,i})]^2} [\overline{Q}_m(\xi_{m,i}) \cos m\theta + \overline{R}_m(\xi_{m,i}) \sin m\theta] + \\ & + 2\,(1-\sigma) \sum_{m=1}^{\infty} \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J'_m(\xi_{m,i})} [C_m \cos m\theta + D_m \sin m\theta]; \end{aligned} \tag{52}$$

此處

$$\left. \begin{aligned} \overline{Q}_0(\xi_{0,i}) &= \int_0^1 x\, Q_0\, J_0(\xi_{0,i}\, x)\, dx, \\ \overline{Q}_m(\xi_{m,i}) &= \int_0^1 x\, Q_m\, J_m(\xi_{m,i}\, x)\, dx, \\ \overline{R}_m(\xi_{m,i}) &= \int_0^1 x\, R_m\, J_m(\xi_{m,i}\, x)\, dx. \end{aligned} \right\} \tag{53}$$

而

$$\left.\begin{aligned} C_0 &= \frac{2\sum\limits_{i=1}^{\infty} \dfrac{\bar{Q}_0(\xi_{0,i})}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}}{1-2\,(1-\sigma)\sum\limits_{i=1}^{\infty}\dfrac{1}{\xi_{0,i}^2}}\,, \\ C_m &= \frac{2\sum\limits_{i=1}^{\infty} \dfrac{\bar{Q}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{1-2\,(1-\sigma)\sum\limits_{i=1}^{\infty}\dfrac{1}{\xi_{m\cdot i}^2}}\,, \\ D_m &= \frac{2\sum\limits_{i=1}^{\infty} \dfrac{\bar{R}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{1-2\,(1-\sigma)\sum\limits_{i=1}^{\infty}\dfrac{1}{\xi_{m\cdot i}^2}}\,. \end{aligned}\right\} \tag{54}$$

五. 環形板在任意橫向載荷下的彎曲

設環形的邊界由同心圓 $x=1$ 及 $x=\lambda$ ($\lambda>1$) 所組成. 在這種情況下我們可以用下面的另一種類型的有限漢克爾變換式, 對於在 $1\leqslant x\leqslant\lambda$ 內定義的函數 $f(x)$, 引進變換式

$$\bar{f}(\xi_{m,i}) = \int_1^{\lambda} x\, f(x)\,[J_m(\xi_{m,i}\,x)\,Y_m(\xi_{m,i}\,\lambda) - Y_m(\xi_{m,i}\,x)\,J_m(\xi_{m,i}\,\lambda)]\,dx\,, \tag{55}$$

其中 Y_m 是 m 階第二種貝塞爾函數,而 $\xi_{m,i}$ 是方程

$$J_m(x)\,Y_m(\lambda x) = Y_m(x)\,J_m(\lambda x) \tag{56}$$

的第 i 個根,於是可以證明[2]

$$f(x) = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\xi_{m,i}^2\,J_m^2(\xi_{m,i}\,\lambda)\,\bar{f}(\xi_{m,i})}{J_m^2(\xi_{m,i})-J_m^2(\xi_{m,i}\,\lambda)}\,[J_m(\xi_{m,i}\,x)\,Y_m(\xi_{m,i}\,\lambda)-Y_m(\xi_{m,i}\,x)\,J_m(\xi_{m,i}\,\lambda)]. \tag{57}$$

首先考慮內外兩邊夾住的情況,此時有如下類型的邊值問題:

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\,\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m = P_m\,, \tag{58}$$

$$X_m = 0\,,\qquad \frac{dX_m}{dx} = 0 \qquad 在\ x=1\,, \tag{59}$$

$$X_m = 0\,,\qquad \frac{dX_m}{dx} = 0 \qquad 在\ x=\lambda\,. \tag{60}$$

如果爲簡寫起見,可用符號

$$S_m(\xi_{m,i}\,x;\lambda) = J_m(\xi_{m,i}\,x)\,Y_m(\xi_{m,i}\,\lambda) - Y_m(\xi_{m,i}\,x)\,J_m(\xi_{m,i}\,\lambda)\,. \tag{61}$$

將 (58) 式左右方乘以 $x\,S_m\,(\xi_{m,i}\,x;\ \lambda)$,並對 x 從 1 到 λ 積分,得到 (參考 (12))

$$\int_1^\lambda x\left(\frac{d^2}{dx^2}+\frac{1}{x}\,\frac{d}{dx}-\frac{m^2}{x^2}\right)^2 X_m\, S_m(\xi_{m,i}\,x;\lambda)\,dx=$$

$$=\left[x\,\frac{d}{dx}\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\,\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)S_m(\xi_{m,i}\,x;\lambda)-\right.$$

$$\left.-\xi_{m,i}\,x\left(\frac{d^2X_m}{dx^2}+\frac{1}{x}\,\frac{dX_m}{dx}-\frac{m^2X_m}{x^2}\right)S'_m(\xi_{m,i}\,x;\lambda)\right]_1^\lambda-$$

$$-\xi^2_{m,i}\left[x\,\frac{dX_m}{dx}\,S_m(\xi_{m,i}\,x;\lambda)-\xi_{m,i}\,x\,X_m\,S'_m(\xi_{m,i}\,x;\lambda)\right]_1^\lambda+$$

$$+\xi^4_{m,i}\,X_m(\xi_{m,i})\,.$$

利用邊界條件（59）及（60），得到

$$-\xi_{m,i}\,\lambda\left(\frac{d^2X_m}{dx^2}\right)_{x=\lambda}S'_m(\xi_{m,i}\,x;\lambda)+\xi_{m,i}\left(\frac{d^2X_m}{dx^2}\right)_{x=1}S'_m(\xi_{m,i}\,;\lambda)+$$

$$+\xi^4_{m,i}\,X_m(\xi_{m,i})=\bar{P}_m(\xi_{m,i})\,.$$

由之得到

$$\bar{X}_m(\xi_{m,i})=\frac{\bar{P}_m(\xi_{m,i})}{\xi^4_{m,i}}+\frac{\lambda\left(\dfrac{d^2X_m}{dx^2}\right)_{x=\lambda}S'_m(\xi_{m,i}\,\lambda;\lambda)-\left(\dfrac{d^2X_m}{dx^2}\right)_{x=1}S'_m(\xi_{m,i}\,;\lambda)}{\xi^3_{m,i}}\,;\qquad(62)$$

此處

$$\bar{X}_m(\xi_{m,i})=\int_1^\lambda x\,X_m\,S_m(\xi_{m,i}\,x;\lambda)\,dx\,,\qquad(63)$$

$$\bar{P}_m(\xi_{m,i})=\int_1^\lambda x\,P_m\,S_m(\xi_{m,i}\,x;\lambda)\,dx\,.\qquad(64)$$

參考（57）得到

$$X_m=\frac{\pi^2}{2}\left\{\sum_{i=1}^\infty\frac{J^2_m(\xi_{m,i}\,\lambda)}{[J^2_m(\xi_{m,i})-J^2_m(\xi_{m,i}\,\lambda)]}\,\frac{\bar{P}_m(\xi_{m,i})}{\xi^2_{m,i}}\,S_m(\xi_{m,i}\,x;\lambda)+\right.$$

$$+\lambda\left(\frac{d^2X_m}{dx^2}\right)_{x=\lambda}\sum_{i=1}^\infty\frac{J^2_m(\xi_{m,i}\,\lambda)}{[J^2_n(\xi_{m,i})-J^2_m(\xi_{m,i}\,\lambda)]}\,\frac{S_m(\xi_{m,i}\,x;\lambda)}{\xi_{m,i}}-$$

$$\left.-\left(\frac{d^2X_m}{dx^2}\right)_{x=1}\sum_{i=1}^\infty\frac{J^2_m(\xi_{m,i}\,\lambda)}{[J^2_m(\xi_{m,i})-J^2_m(\xi_{m,i}\,\lambda)]}\,\frac{S_m(\xi_{m,i}\,x;\lambda)}{\xi_{m,i}}\right\}.\qquad(65)$$

常數$\left(\dfrac{d^2X_m}{dx^2}\right)_{x=\lambda}$及$\left(\dfrac{d^2X_m}{dx^2}\right)_{x=1}$可以由條件（59）及（60）來決定，卽由方程組

$$\left(\frac{d^2X_m}{dx^2}\right)_{x=1}\sum_{i=1}^\infty\frac{J^2_m(\xi_{m,i}\,\lambda)}{[J^2_m(\xi_{m,i})-J^2_m(\xi_{m,i}\,\lambda)]}\,S'_m(\xi_{m,i}\,;\lambda)-$$

$$-\lambda\left(\frac{d^2X_m}{dx^2}\right)_{x=\lambda}\sum_{i=1}^\infty\frac{J^2_m(\xi_{m,i}\,\lambda)}{[J^2_m(\xi_{m,i})-J^2_m(\xi_{m,i}\,\lambda)]}\,S'_m(\xi_{m,i}\,\lambda;\lambda)=$$

$$= \sum_{i=1}^{\infty} \frac{J_m^2(\xi_{m,i}\,\lambda)}{(J_m^2(\xi_{m,i}) - J_m^2(\xi_{m,i}\,\lambda)]} \frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}} S'_m(\xi_{m,i};\lambda)\,, \tag{66}$$

$$\left(\frac{d^2 X_m}{dx^2}\right)_{x=1} \sum_{i=1}^{\infty} \frac{J_m^2(\xi_{m,i}\,\lambda)}{[J_m^2(\xi_{m,i}) - J_m^2(\xi_{m,i}\,\lambda)]} S'_m(\xi_{m,i}\,\lambda;\lambda) -$$

$$- \lambda\left(\frac{d^2 X_m}{dx^2}\right)_{x=\lambda} \sum_{i=1}^{\infty} \frac{J_m^2(\xi_{m,i}\,\lambda)}{[J_m^2(\xi_{m,i}) - J_m^2(\xi_{m,i}\,\lambda)]} S'_m(\xi_{m,i}\,\lambda;\lambda) =$$

$$= \sum_{i=1}^{\infty} \frac{J_m^2(\xi_{m,i}\,\lambda)}{[J_m^2(\xi_{m,i}) - J_m^2(\xi_{m,i}\,\lambda)]} \frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}} S'_m(\xi_{m,i};\lambda) \tag{67}$$

決定．解出這些方程以後，我們可以着手去寫出上面所提出的問題的全部的解．

不難將此處提出的方法應用到內外均爲簡支以及內(外)邊簡支外(內)邊夾住的情況．我們不去詳細討論了．

應該指出，有限漢克爾變換式方法的優點卽在將邊緣夾住或簡支的問題化爲依一定步驟來處理的問題．它所可能有的缺陷是收斂性（特別是在環形板的情况）較差．

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BENDING OF CIRCULAR AND RING-SHAPED ELASTIC THIN PLATE UNDER ARBITRARY LATERAL LOAD

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ABSTRACT

The present paper has the purpose of illustrating the usefulness of finite transform method to certain problems in the theory of elastic thin plates and shells. This method supplements from the mathematical side the idea of Prof. W. Nowacki in his treatment of problems of this type[1].

In particular, the finite transforms of a function $f(x)$ over the interval $0 \leqslant x \leqslant 1$ are defined as

$$\bar{f}(\xi_{m,i}) = \int_0^1 x\, f(x)\, J_m(\xi_{m,i}\, x)\, dx\,,$$

where J_m represents Bessel function of the first kind and of m-th order, $\xi_{m,i}$ denoting the i-th root of the equation

$$J_m(x) = 0\,.$$

Then it may be proved that[2]

$$f(x) = 2 \sum_{i=1}^{\infty} \bar{f}(\xi_{m,i}) \frac{J_m(\xi_{m,i}\, x)}{[J'_m(\xi_{m,i})]^2}\,.$$

Using this method, we can treat the problem of a circular plate under arbitrary lateral load with clamped edge or simply supported edge.

The problem of a circular clamped plate of flexural rigidity D and radius a under arbitrary lateral load $p(r, \theta)$ reduces to a boundary-value problem of the following type: to find a dimensionless deflection function $W(x, \theta)$ satisfying the differential equation

$$\left(\frac{\partial^2}{dx^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)^2 W = q\,,$$

and the boundary conditions at $x=1$

$$W = 0\,,$$

$$\frac{\partial W}{\partial x} = 0\,,$$

where, w denoting the deflection,

$$W = \frac{w}{a}\,, \qquad x = \frac{r}{a}\,, \qquad q = \frac{pa^3}{D}\,.$$

We expand $q(x, \theta)$ into series of the form

$$q = Q_0(x) + \sum_{m=1}^{\infty} Q_m(x) \cos m\theta + \sum_{m=1}^{\infty} R_m(x) \sin m\theta\,,$$

and write out $W(x, \theta)$ also in the form

$$W = U_0(x) + \sum_{m=1}^{\infty} U_m(x) \cos mx + \sum_{m=1}^{\infty} V_m(x) \sin mx\,.$$

We have boundary-value problems of the type:

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{x^2}\right)^2 X_m = P_m\,;$$

$$X_m\Big|_{x=1} = 0\,, \qquad \frac{dX_m}{dx}\Big|_{x=1} = 0\,.$$

Applying the method of finite Hankel transform, we find that

$$\bar{X}_m(\xi_{m,i}) = \frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}^4} + \frac{J'_m(\xi_{m,i})}{\xi_{m,i}^3} A_m\,,$$

where

$$\bar{X}_m(\xi_{m,i}) = \int_0^1 x\, X_m\, J_m(\xi_{m,i}\, x)\, dx\,,$$

$$\bar{P}_m(\xi_{m,i}) = \int_0^1 x\, P_m\, J_m(\xi_{m,i}\, x)\, dx\,.$$

and $A_m = \left(\frac{d^2 X_m}{dx^2}\right)_{x=1}$ are constants to be determined. By inversion, we obtain

$$X_m(x) = 2\sum_{i=1}^{\infty} \frac{\bar{P}_m(\xi_{m,i})\, J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\, [J'_m(\xi_{m,i})]^2} + 2\, A_m \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J'_m(\xi_{m,i})}\,.$$

Obviously the condition $X_m\Big|_{x=1} = 0$ is satisfied and to satisfy the condition $\frac{dX_m}{dx}\Big|_{x=1} = 0$ implies

$$A_m = -\frac{\sum_{i=1}^{\infty} \frac{\bar{P}_m(\xi_{m,i})}{\xi_{m,i}^3\, J'_m(\xi_{m,i})}}{\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,.$$

Then, we can express the dimensionless deflection $W(x, \theta)$ in the form

$$\begin{aligned} W(x,\theta) = {} & 2\sum_{i=1}^{\infty} \frac{\bar{Q}_0(\xi_{0,i})\, J_0(\xi_{0,i}\, x)}{\xi_{0,i}^4\, [J'_0(\xi_{0,i})]^2} + 2\, A_0 \sum_{i=1}^{\infty} \frac{J_0(\xi_{0,i}\, x)}{\xi_{0,i}^3\, J'_0(\xi_{0,i})} + \\ & + 2\sum_{m=1}^{\infty}\sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^4\, [J'_m(\xi_{m,i})]^2} [\bar{Q}_m(\xi_{m,i}) \cos m\theta + \bar{R}_m(\xi_{m,i}) \sin m\theta] + \\ & + 2\sum_{m=1}^{\infty} (A_m \cos m\theta + B_m \sin m\theta) \sum_{i=1}^{\infty} \frac{J_m(\xi_{m,i}\, x)}{\xi_{m,i}^3\, J'_m(\xi_{m,i})}\,. \end{aligned}$$

where

$$\bar{Q}_0(\xi_{0,i}) = \int_0^1 x\, Q_0\, J_0(\xi_{0,i}\, x)\, dx\,,$$

$$\bar{Q}_m(\xi_{m,i}) = \int_0^1 x\, Q_m\, J_m(\xi_{m,i}\, x)\, dx\,.$$

$$\bar{R}_m(\xi_{m,i}) = \int_0^1 x\, R_m\, J_m(\xi_{m,i}\, x)\, dx\,.$$

and

$$A_0 = -\frac{\sum_{i=1}^{\infty} \frac{\bar{Q}_0(\xi_{0,i})}{\xi_{0,i}^3\, J_0'(\xi_{0,i})}}{\sum_{i=1}^{\infty} \frac{1}{\xi_{0,i}^2}}\,,$$

$$A_m = -\frac{\sum_{i=1}^{\infty} \frac{\bar{Q}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,.$$

$$B_m = -\frac{\sum_{i=1}^{\infty} \frac{\bar{R}_m(\xi_{m,i})}{\xi_{m,i}^3\, J_m'(\xi_{m,i})}}{\sum_{i=1}^{\infty} \frac{1}{\xi_{m,i}^2}}\,.$$

By specifying the load, many problems of interest can be solved, including those considered by H. Reissner and others.

By the same procedure, the problem of a clamped circular plate under tension and arbitrary lateral load and that of a simply-supported circular plate can be solved without difficulty.

While considering the problem of a ring-shaped plate under arbitrary lateral load, we use a different type of finite Hankel transform. If $f(x)$ is defined in the interval $1 \leqslant x \leqslant \lambda$ · $(\lambda > 1)$, the finite Hankel transforms are given by the formulas:

$$\bar{f}(\xi_{m,i}) = \int_1^{\lambda} x\, f(x)\, S_m(\xi_{m,i}\, x; \lambda)\, dx\,,$$

where

$$S_m(\xi_{m,i}\, x; \lambda) = J_m(\xi_{m,i}\, x)\, Y_m(\xi_{m,i}\, \lambda) - Y_m(\xi_{m,i}\, x)\, J_m(\xi_{m,i}\, \lambda)\,,$$

Y_m denoting Bessel function of the second kind and of m-th order, $\xi_{m,i}$ being the i-th root of the equation

$$J_m(x)\, Y_m(\lambda x) = Y_m(x)\, J_m(\lambda x)\,.$$

The application of this type of finite Hankel transform to ring-shaped plate under arbitrary lateral load is analogous to the case of circular plate, in that some of the boundary conditions are used after the inversion.