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旋轉對稱電子光學系統中的五級橫向像差*

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摘 要

本文用軌跡法與作用函數法求得了在旋轉對稱系統中的五級橫向像差方程式，從而推出了在高斯像平面中的五級横向像差，且定出了五級特殊形式像差的像差係數，也討論了五級像差的散射圖形，最後研究了五級畸變的影響.

一. 緒 言

(1) 在電子光學系統中，至今只考慮到三級橫向像差，與普通光學中時常需考慮到高級像差不同，原因在於在電子光學的一般系統中，孔徑角小，物面小，因此給定軌跡的平面矢量 ($\mathbf{u}_a$, $\mathbf{u}_a'$) 的五次方項就很小，以致在很多情況下三級像差在全部像差中佔了百分之九十幾，因而關於五級像差問題的研究尚未開展. 但是隨着電子光學的發展以及從目前的一些儀器來看，五級像差還是有必要提出. 這是因爲:

(i) 在有些儀器中，如像變換器，視場較大，因此五級畸變對像的影響可能很顯著，有時可能會很大. 因爲五級畸變與 u_a^5 成正比，而畸變會使像變形，當要作定量測量時需考慮之.

(ii) 在研究畸變時，發現理論與實驗有時有不符情況，於是提出了是否應更仔細地來研究五級畸變，因爲可能與它有關.

(iii) 在三級像差消除的系統中，五級像差會顯現，它將決定像的品質.

(iv) 在電子光學中，因一般系統的視場與孔徑角較小，於是籠統地認爲五級像差很小而不予考慮，但是某些特殊形式的五級像差在各種情況下究竟影響有多大，尚不清楚.

從以上這些原因看來，在電子光學中研究和決定五級像差以及考慮其影響等問題就成爲必要了.

*1957 年 2 月 6 日收到.

(2) 關於五級像差理論的文獻上有惟一的一篇，見文獻 [1]. 作者用了特徵函數的微擾理論得出了五級橫向像差方程式，其優點在於較普遍，可同時適用於彎曲軸系統；但是他沒有進而推導出在旋轉對稱情況下的方程式以及像差係數，且所得表示式沒有對給定電子束的矢量參量 $\mathbf{u}_a$, $\mathbf{u}_a'$ 展開.

(3) 本文與求三級橫向像差同樣，先求出普遍的五級橫向像差方程式，由此出發決定旋轉對稱系統中的五級橫向像差以及定出特殊像差係數，且討論五級特殊像差的散射圖形，了解其對像的影響，最後研究五級畸變的影響，使得到一些定量的概念.

二. 旋轉對稱系統中的五級橫向像差方程式

在軸對稱情況下，實際軌跡可表爲下式:

$$\mathbf{u} = \mathbf{u}_g + \triangle \mathbf{u}_3 + \triangle \mathbf{u}_5 . \tag{1}$$

$\mathbf{u}$, $\mathbf{u}_g$ 爲實際軌跡與高斯軌跡的徑向平面矢量， $\triangle\mathbf{u}_3$, $\triangle\mathbf{u}_5$ 是相應於給定軌跡的平面矢量的 3, 5 次方項，稱爲三級、五級橫向像差.

在電子光學中，帶電粒子的實際軌跡滿足最小作用原理，在旋轉對稱條件下，可以表示爲:

$$\delta \int_{z_o}^{z_i} \underset{\sim}{\mu} (z, \boldsymbol{\tau} \boldsymbol{\tau}') \, dz = 0 \, ; \tag{2}$$

其中

$$\underset{\sim}{\mu} = \{\varphi_2 (1+\boldsymbol{\tau}'^2)\}^{1/2} - \frac{\mathcal{A}_v}{\tau} (\boldsymbol{\tau} \times \boldsymbol{\tau}') \, , \tag{3}$$

$\underset{\sim}{\mu}$ 爲旋轉對稱場下的介質折射率，φ_2 爲相對論的電位，$\mathcal{A}_v$ 爲磁矢位在方位角方向的分量，其中 $' = \frac{d}{dz}$，因爲 $\boldsymbol{\tau}$, $\boldsymbol{\tau}'$ 是在垂直於 z 軸的平面中的平面矢量，因此 $(\boldsymbol{\tau} \times \boldsymbol{\tau}')$ 在空間固定地向着 z 軸方向，可以看作一個標量.

今引入新的旋轉坐標系統，其徑向平面矢量 $\mathbf{u}$ 與 $\boldsymbol{\tau}$ 有如下關係:

$$\mathbf{u} = \boldsymbol{\tau} \cos \chi - \underset{\sim}{\boldsymbol{\tau}} \sin \chi \, , \qquad \text{而} \quad \chi' = \frac{\mathscr{H}}{2V^{1/2}} . \tag{4}$$

在新的旋轉坐標系中，則最小作用原理爲:

$$\delta \int_{z_o}^{z_i} \mu_* (z, \mathbf{u}, \mathbf{u}') \, dz = 0 \, , \qquad \text{其中} \quad \underset{\sim}{\mu} (z, \boldsymbol{\tau}, \boldsymbol{\tau}') = \mu_* (z, \mathbf{u}, \mathbf{u}') . \tag{5}$$

這樣得到了實際軌跡所滿足的歐勒方程式爲:

$$\frac{d}{dz} \nabla_{\mathbf{u}'} \mu_* - \nabla_{\mathbf{u}} \mu_* = 0 . \tag{6}$$

其中

$$\nabla_{\mathbf{u}'} = \frac{\partial}{\partial \mathbf{u}'}, \qquad \nabla_{\mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}.$$

在旋轉對稱場中，μ_* 可以按 $\mathbf{u}$ 的偶次方展開，則有：

$$\mu_* = \mu_{*0} + \mu_{*2} + \mu_{*4} + \mu_{*6} + \cdots \tag{7}$$

由［3］中知道，

$$\mu_{*0} = V_2^{1/2}, \qquad \mu_{*2} = N\,\mathbf{u}'^2 + M\,\mathbf{u}^2,$$

$$\mu_{*4} = -\frac{1}{4}L\,\mathbf{u}^4 - \frac{1}{2}M\,\mathbf{u}^2\,\mathbf{u}'^2 - \frac{1}{4}N\,\mathbf{u}'^4 - k\,V_2\,(\mathbf{u}\times\mathbf{u}')^2 - $$
$$- V_2^{1/2}\,(P\,\mathbf{u}^2 + Q\,\mathbf{u}'^2)\,(\mathbf{u}\times\mathbf{u}'). \tag{8a}$$

而 N, M 是與軸上場分佈有關的函數，如：

$$N = \frac{1}{2}V_2^{1/2}, \qquad M = \frac{1}{8}\,\frac{V''\,\delta + \mathscr{H}^2}{V_2^{1/2}}. \tag{8b}$$

今把（6）式中 μ_* 展開，且把二次方的項分出，移到式子左邊，可得下式：

$$\frac{d}{dz}\,\nabla_{\mathbf{u}'}\,\mu_{*2} - \nabla_{\mathbf{u}}\,\mu_{*2} = \mathbf{F}_{3+5+7+\cdots}; \tag{9}$$

而此處：

$$\mathbf{F}_{3+5+\cdots} = \nabla_{\mathbf{u}}\,\mu_{*4+6+\cdots} - \frac{d}{dz}\,\nabla_{\mathbf{u}'}\,\mu_{*4+6+\cdots}. \tag{10}$$

利用（8a），則（9）式可以寫成：

$$2\,(N\,\mathbf{u}')' + 2\,M\,\mathbf{u} = \mathbf{F}_{3+5+\cdots}, \tag{11}$$

今把（1）式代入（11）式，得到一系列依着像差次序排列的方程式：

$$2\,(N\,\mathbf{u}_g')' + 2\,M\,\mathbf{u}_g = 0, \tag{12a}$$

$$2\,(N\,\Delta\mathbf{u}_3')' + 2\,M\,\Delta\mathbf{u}_3 = \mathbf{f}_3, \tag{12b}$$

$$2\,(N\,\Delta\mathbf{u}_5')' + 2\,M\,\Delta\mathbf{u}_5 = \mathbf{f}_5. \tag{12c}$$

此處第一式爲傍軸軌跡方程式，第二式爲三級橫向像差方程式，而第三式爲五級橫向像差方程式．方程式右面 $\mathbf{f}_3$, $\mathbf{f}_5$ 是相應於給定軌跡平面矢量的 3, 5 次方項．（12b），（12c）爲二個非齊次線性方程式，右面部分比之高斯軌跡可看作一微擾，因此當確定 $\mathbf{f}_3$、$\mathbf{f}_5$ 以後，可用任意常數變分法來解．

因爲 $\mathbf{F}_3$ 是相應於 $\mathbf{u}, \mathbf{u}'$ 的三次方項，則

$$\mathbf{f}_3 = \mathbf{F}_{3g}, \tag{13}$$

$\mathbf{F}_{3g}$ 表示 $\mathbf{F}_3$ 中的 $\mathbf{u}$ 以高斯軌跡來代替，即

$$\mathbf{F}_{3g} = \mathbf{F}_3(\mathbf{u}, \mathbf{u}')_{\mathbf{u}=\mathbf{u}_g}.$$

再看 $\mathbf{f}_5$,

$$\mathbf{f}_5 = \mathbf{f}_{5g} + (\Delta\mathbf{u}_3 \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3' \nabla_{\mathbf{u}'}) \mathbf{F}_{3g}, \tag{14}$$

於是由（12）式可以看出,當已經確定了傍軸軌跡,可以依次得解. 因爲由（12a）確定了傍軸軌跡後,在（12b）中，$\mathbf{f}_3$ 通過 $\mathbf{u}_g$ 表示出來,於是可用任意常數變分法來求出三級橫向像差 $\Delta\mathbf{u}_3$，其次用已知的 $\mathbf{u}_g$, $\Delta\mathbf{u}_3$，則 $\mathbf{f}_5$ 也可知道,再由（12c），用任意常數變分法求得五級橫向像差 $\Delta\mathbf{u}_5$.

由（12a）式得到解爲:

$$\mathbf{u}_g = \boldsymbol{\pi}_1 \tau_1(z) + \boldsymbol{\pi}_2 \tau_2(z), \tag{15}$$

此處 $\boldsymbol{\pi}_1$，$\boldsymbol{\pi}_2$ 爲給定軌跡的二個平面矢量，$\tau_1(z)$，$\tau_2(z)$ 爲線性齊次微分方程式（12a）的二個線性無關的特解.

由（12b）式得到:

$$\Delta\mathbf{u}_3 = \tau_1(z_i)\left\{\mathbf{a}_3 + \frac{1}{c}\int_{z_\sigma}^{z_i} \tau_2 \mathbf{F}_{3g}\, dz\right\} - \tau_2(z_i)\left\{\mathbf{b}_3 + \frac{1}{c}\int_{z_\sigma}^{z_i} \tau_1 \mathbf{F}_{3g}\, dz\right\}; \tag{16}$$

其中 $c = V_2(z)(\tau_1'\tau_2 - \tau_2'\tau_1) =$ 常數,常數 $\mathbf{a}_3, \mathbf{b}_3$ 由初條件決定,設起始平面中三級橫向像差及其微商爲 $\Delta\mathbf{u}_{3\sigma}$，$\Delta\mathbf{u}_{3\sigma}'$，則可以定出:

$$\mathbf{a}_3 = \left\{\frac{V_2^{1/2}}{c}(\Delta\mathbf{u}_3'\tau_2 - \Delta\mathbf{u}_3\tau_2')\right\}_\sigma, \tag{17a}$$

$$\mathbf{b}_3 = \left\{\frac{V_2^{1/2}}{c}(\Delta\mathbf{u}_3'\tau_1 - \Delta\mathbf{u}_3\tau_1')\right\}_\sigma. \tag{17b}$$

同理可解（12c）式,得到:

$$\Delta\mathbf{u}_5 = \tau_1(z_i)\left\{\mathbf{a}_5 + \frac{1}{c}\int_{z_\sigma}^{z_i} \mathbf{f}_5 \tau_2\, dz\right\} - \tau_2(z_i)\left\{\mathbf{b}_5 + \frac{1}{c}\int_{z_\sigma}^{z_i} \mathbf{f}_5 \tau_1\, dz\right\}. \tag{18}$$

今由初條件確定常數 $\mathbf{a}_5$, $\mathbf{b}_5$，設在起始平面 z_σ 中，五級橫向像差及其微商爲 $\Delta\mathbf{u}_{5\sigma}$，$\Delta\mathbf{u}_{5\sigma}'$，則得:

$$\mathbf{a}_5 = \left\{\frac{V_2^{1/2}}{c}(\Delta\mathbf{u}_5'\tau_2 - \Delta\mathbf{u}_5\tau_2')\right\}_\sigma, \tag{19a}$$

$$\mathbf{b}_5 = \left\{\frac{V_2^{1/2}}{c}(\Delta\mathbf{u}_5'\tau_1 - \Delta\mathbf{u}_5\tau_1')\right\}_\sigma. \tag{19b}$$

以下變換（18）式中積分：因爲,

$$\mathbf{f}_5 = \mathbf{F}_{5g} + (\Delta\mathbf{u}_3 \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3' \nabla_{\mathbf{u}'}) \mathbf{F}_{3g},$$

於是得:

$$\int_{z_\sigma}^{z_i} \mathbf{F}_{5g}\,\tau_2\,dz = \int_{z_\sigma}^{z_i} \left(\nabla_{\mathbf{u}}\,\mu_{*6} - \frac{d}{dz}\,\nabla_{\mathbf{u}'}\,\mu_{*6}\right)_g \tau_2\,dz =$$

$$= \int_{z_\sigma}^{z_i} (\tau_2\ \nabla_{\mathbf{u}}\,\mu_{*6} + \tau_2'\ \nabla_{\mathbf{u}'}\,\mu_{*6})_g\,dz - \left|\tau_2\ \boldsymbol{\eta}_{5g}\right|_{z_\sigma}^{z_i}, \tag{20a}$$

同理得

$$\int_{z_\sigma}^{z_i} \mathbf{F}_{5g}\,\tau_1\,dz = \int_{z_\sigma}^{z_i} (\tau_1\ \nabla_{\mathbf{u}}\,\mu_{*6} + \tau_1'\ \nabla_{\mathbf{u}'}\,\mu_{*6})_g\,dz - \left|\tau_1\ \boldsymbol{\eta}_{5g}\right|_{z_\sigma}^{z_i}, \tag{20b}$$

今再計算，得到：

$$\int_{z_\sigma}^{z_i} (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\,\mathbf{F}_{3g}\,\tau_j\,dz = \int_{z_\sigma}^{z_i} (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\nabla_{\mathbf{u}'})\,(\tau_j\ \nabla_{\mathbf{u}}\,\mu_{*4} + \tau_j'\,\nabla_{\mathbf{u}'}\,\mu_{*4})_g\,dz -$$

$$- \left|\tau_j\,(\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ \boldsymbol{\eta}_{3g}\right|_{z_\sigma}^{z_i}, \tag{21, a, b}$$

其中 $j = 1,\ 2$，因此上式卽爲相似之二式。

今把（20a, b）及（21a, b）代入（18）式，且合併一些項，如：

$$\mathbf{a}_5 + \frac{1}{c}\left\{\tau_2\,[\boldsymbol{\eta}_{5g} + (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ \boldsymbol{\eta}_{3g}]\right\}_\sigma = \frac{1}{c}\,(\tau_2\ \Delta\boldsymbol{\eta}_5 - \tau_2'\ V_2^{1/2}\ \Delta\mathbf{u}_5)_\sigma; \tag{22a}$$

同樣，

$$\mathbf{b}_5 + \frac{1}{c}\,\{\tau_1\,[\boldsymbol{\eta}_{5g} + (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ \boldsymbol{\eta}_{3g}]\}_\sigma = \frac{1}{c}\,(\tau_1\ \Delta\boldsymbol{\eta}_5 - \tau_1'\ V_2^{1/2}\ \Delta\mathbf{u}_5)_\sigma, \tag{22b}$$

其中

$$\Delta\boldsymbol{\eta}_5 = \boldsymbol{\eta}_{5g} + (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ \boldsymbol{\eta}_{3g} + (\Delta\mathbf{u}_5\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_5'\ \nabla_{\mathbf{u}'})\ \boldsymbol{\eta}_{1g}. \tag{23}$$

於是最後得到在任意邊界面，任意初條件下的五級橫向普遍方程式爲：

$$\Delta\mathbf{u}_5 = \frac{\tau_1(z_i)}{c}\left\{(\tau_2\ \Delta\boldsymbol{\eta}_5 - \tau_2'\ V_2^{1/2}\ \Delta\mathbf{u}_5)_\sigma + \int_{z_\sigma}^{z_i} (\tau_2\ \nabla_{\mathbf{u}}\,\mu_{*6} + \tau_2'\ \nabla_{\mathbf{u}'}\,\mu_{*6})_g\,dz + \right.$$

$$\left. + \int_{z_\sigma}^{z_i} (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ (\tau_2\ \nabla_{\mathbf{u}}\,\mu_{*4} + \tau_2'\ \nabla_{\mathbf{u}'}\,\mu_{*4})_g\,dz\right\} -$$

$$- \frac{\tau_2(z_i)}{c}\left\{(\tau_1\ \Delta\boldsymbol{\eta}_5 - \tau_1'\ V_2^{1/2}\ \Delta\mathbf{u}_5)_\sigma + \int_{z_\sigma}^{z_i} (\tau_1\ \nabla_{\mathbf{u}}\,\mu_{*6} + \tau_1'\ \nabla_{\mathbf{u}'}\,\mu_{*6})_g\,dz + \right.$$

$$\left. + \int_{z_\sigma}^{z_i} (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ (\tau_1\ \nabla_{\mathbf{u}}\,\mu_{*4} + \tau_1'\ \nabla_{\mathbf{u}'}\,\mu_{*4})_g\,dz\right\}, \tag{24a}$$

由（23）式得到在任意邊界面任意初條件下的法線矢量的五級橫向像差爲：

$$\Delta\boldsymbol{\eta}_5 = \frac{\tau_1'(z_i)\ V_2^{1/2}}{c}\left\{(\tau_2\ \Delta\boldsymbol{\eta}_5 - \tau_2'\ V_2^{1/2}\ \Delta\mathbf{u}_5)_\sigma + \int_{z_\sigma}^{z_i} (\tau_2\ \nabla_{\mathbf{u}}\,\mu_{*6} + \tau_2'\ \nabla_{\mathbf{u}'}\,\mu_{*6})_g\,dz + \right.$$

$$\left. + \int_{z_\sigma}^{z_i} (\Delta\mathbf{u}_3\ \nabla_{\mathbf{u}} + \Delta\mathbf{u}_3'\ \nabla_{\mathbf{u}'})\ (\tau_2\ \nabla_{\mathbf{u}}\,\mu_{*4} + \tau_2'\ \nabla_{\mathbf{u}'}\,\mu_{*4})_g\,dz\right\} -$$

$$-\frac{\tau_2'(z_i)\,V_2^{1/2}}{c}\left\{(\tau_1\,\triangle\boldsymbol{\eta}_5-\tau_1'\,V_2^{1/2}\,\triangle\mathbf{u}_5)_\sigma+\int_{z_\sigma}^{z_i}(\tau_1\,\triangle_{\mathbf{u}}\mu_{*6}+\tau_1'\,\nabla_{\mathbf{u}'}\mu_{*6})_g\,dz+\right.$$

$$\left.+\int_{z_\sigma}^{z_i}(\triangle\mathbf{u}_3\,\nabla_{\mathbf{u}}+\triangle\mathbf{u}_3'\,\nabla_{\mathbf{u}'})\,(\tau_1\,\nabla_{\mathbf{u}}\mu_{*4}+\tau_2\,\nabla_{\mathbf{u}'}\mu_{*4})_g\,dz\right\}. \qquad (24\text{b})$$

如果我們把 (24a, b) 對給定軌跡的平面矢量 $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ 展開，又經過下式變換，如：

$$\nabla_{\pi_2}=\tau_2\,\nabla_{\mathbf{u}}+\tau_2'\,\nabla_{\mathbf{u}'},\qquad \nabla_{\pi_1}=\tau_1\,\nabla_{\mathbf{u}}+\tau_1'\,\nabla_{\mathbf{u}'}. \qquad (25)$$

則在 (24a, b) 中，有

$$\int_{z_\sigma}^{z_i}(\tau_{1,2}\,\nabla_{\mathbf{u}}\mu_{*6}+\tau_{1,2}'\,\nabla_{\mathbf{u}'}\mu_{*6})_g\,dz=\int_{z_\sigma}^{z_i}\nabla_{\pi_{1,2}}\mu_{*6g}\,dz=\nabla_{\pi_{1,2}}\varepsilon_{6g}; \qquad (26)$$

另外，

$$\triangle\mathbf{u}_3\,\nabla_{\mathbf{u}}+\triangle\mathbf{u}_3'\,\nabla_{\mathbf{u}'}=\frac{V_2^{1/2}}{c}\,(-\triangle\mathbf{u}_3\,\tau_2'+\triangle\mathbf{u}_3'\,\tau_2)\,\nabla_{\pi_1}+$$

$$+\frac{V_2^{1/2}}{c}\,(\triangle\mathbf{u}_3\,\tau_1'-\triangle\mathbf{u}_3'\,\tau_1)\,\nabla_{\pi_2}, \qquad (27)$$

而

$$\frac{V_2^{1/2}}{c}\,(\triangle\mathbf{u}_3'\,\tau_2-\triangle\mathbf{u}_3\,\tau_2')=\mathbf{a}_3+\frac{1}{c}\int_{z_\sigma}^{z_i}\mathbf{F}_{3g}\,\tau_2\,dz=$$

$$=\left\{\frac{V_2^{1/2}}{c}\,(\triangle\mathbf{u}_3'\,\tau_2-\triangle\mathbf{u}_3\,\tau_2')\right\}_\sigma+\frac{1}{c}\left\{\nabla_{\pi_2}\varepsilon_{4g}-\left|\boldsymbol{\eta}_{3g}\,\tau_2\right|_{z_\sigma}^{z}\right\}, \qquad (28\text{a})$$

同樣：

$$\frac{V_2^{1/2}}{c}\,(\triangle\mathbf{u}_3'\,\tau_1-\triangle\mathbf{u}_3\,\tau_1')=\mathbf{b}_3+\frac{1}{c}\int_{z_\sigma}^{z_i}\mathbf{F}_{3g}\,\tau_1\,dz=$$

$$=\left\{\frac{V_2^{1/2}}{c}\,(\triangle\mathbf{u}_3'\,\tau_1-\triangle\mathbf{u}_3\,\tau_1')\right\}_\sigma+\frac{1}{c}\left\{\nabla_{\pi_1}\varepsilon_{4g}-\left|\boldsymbol{\eta}_{3g}\,\tau_1\right|_{z_\sigma}^{z}\right\}. \qquad (28\text{b})$$

於是把 (26), (28a, b) 代入 (24a, b) 中，於是得到了在任意邊界面任意初條件下的對 $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ 展開的五級橫向像差方程式爲：

$$\triangle\mathbf{u}_5=\frac{\tau_1(z_i)}{c}\left\{\nabla_{\pi_2}\varepsilon_{6g}+[\triangle\boldsymbol{\eta}_5\,\tau_2-\triangle\mathbf{u}_5\,\tau_2'\,V_2^{1/2}]_{z=z_\sigma}\right\}+$$

$$+\frac{\tau_1(z_i)}{c}\int_{z_\sigma}^{z_i}\frac{1}{c}\left[\left\{\nabla_{\pi_2}\varepsilon_{4g}+[V_2^{1/2}\,(\triangle\mathbf{u}_3'\,\tau_2-\triangle\mathbf{u}_3\,\tau_2')]_\sigma-\left|\boldsymbol{\eta}_{3g}\,\tau_2\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_1}-\right.$$

$$\left.-\left\{\nabla_{\pi_1}\varepsilon_{4g}+[V_2^{1/2}\,(\triangle\mathbf{u}_3'\,\tau_1-\triangle\mathbf{u}_3\,\tau_1')]_\sigma-\left|\boldsymbol{\eta}_{3g}\,\tau_1\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_2}\right]\nabla_{\pi_2}\mu_{*4g}\,dz-$$

$$-\frac{\tau_2(z_i)}{c}\left\{\nabla_{\pi_1}\varepsilon_{6g}+[\triangle\boldsymbol{\eta}_5\,\tau_1-\triangle\mathbf{u}_5\,\tau_1'\,V_2^{1/2}]_{z=z_\sigma}\right\}-$$

$$-\frac{\tau_2(z_i)}{c}\int_{z_\sigma}^{z_i}\frac{1}{c}\left[\left\{\nabla_{\pi_2}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_2-\Delta\mathbf{u}_3\tau_2')\right]_\sigma-\left|\boldsymbol{\eta}_{3g}\tau_2\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_1}-\right.$$

$$\left.-\left\{\nabla_{\pi_1}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_1-\Delta\mathbf{u}_3\tau_1')\right]_\sigma-\left|\boldsymbol{\eta}_{3g}\tau_1\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_2}\right]\nabla_{\pi_1}\mu_{*4g}\,dz,\quad(29a)$$

而相應的對 $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$ 展開的法線矢量的五級橫向像差爲：

$$\Delta\boldsymbol{\eta}_5=\frac{\tau_1'(z_i)V_2^{1/2}}{c}\left\{\nabla_{\pi_2}\varepsilon_{6g}+\left[\Delta\boldsymbol{\eta}_5\tau_2-\Delta\mathbf{u}_5\tau_2'V_2^{1/2}\right]_{z=z_\sigma}\right\}+$$

$$+\frac{\tau_1'(z_i)V_2^{1/2}}{c}\int_{z_\sigma}^{z_i}\frac{1}{c}\left[\left\{\nabla_{\pi_2}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_2-\Delta\mathbf{u}_3\tau_2')\right]_\sigma-\left|\boldsymbol{\eta}_{3g}\tau_2\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_1}-\right.$$

$$\left.-\left\{\nabla_{\pi_1}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_1-\Delta\mathbf{u}_3\tau_1')\right]_\sigma-\left|\boldsymbol{\eta}_{3g}\tau_1\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_2}\right]\nabla_{\pi_2}\mu_{*4g}\,dz-$$

$$-\frac{\tau_2'(z_i)V_2^{1/2}}{c}\left\{\nabla_{\pi_1}\varepsilon_{6g}+\left[\Delta\boldsymbol{\eta}_5\tau_1-\Delta\mathbf{u}_5\tau_1'V_2^{1/2}\right]_{z=z_\sigma}\right\}-$$

$$-\frac{\tau_2'(z_i)V_2^{1/2}}{c}\int_{z_\sigma}^{z_i}\frac{1}{c}\left[\left\{\nabla_{\pi_2}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_2-\Delta\mathbf{u}_3\tau_2')\right]_\sigma-\left|\eta_{3g}\tau_2\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_1}-\right.$$

$$\left.-\left\{\nabla_{\pi_1}\varepsilon_{4g}+\left[V_2^{1/2}(\Delta\mathbf{u}_3'\tau_1-\Delta\mathbf{u}_3\tau_1')\right]_\sigma-\left|\eta_{3g}\tau_1\right|_{z_\sigma}^{z}\right\}\nabla_{\pi_2}\right]\nabla_{\pi_1}\mu_{*4g}\,dz.\quad(29b)$$

如果我們取起始平面爲物平面，即 $z_\sigma=z_a$，另一邊界面爲高斯像平面 ($z_i=z_b$)，且設邊界條件爲：

$$\Delta\mathbf{u}_{3a}=0,\qquad\Delta\mathbf{u}_{3a}'=0,\qquad\Delta\mathbf{u}_{5a}=0,\qquad\Delta\mathbf{u}_{5a}'=0;$$

再取參量：$\boldsymbol{\pi}_1=\mathbf{u}_a'$, $\boldsymbol{\pi}_2=\mathbf{u}_a$，則 $\tau_1=\tau_a$, $\tau_2=\tau_\beta$.

把這些量代入 (29a) 中，得到在高斯像平面中的五級橫向像差爲：

$$\Delta\mathbf{u}_{5b}=-\frac{m}{c}\nabla_{\mathbf{u}_a'}\varepsilon_{6g}-\frac{m}{c}\int_{z_a}^{z_b}\left\{\frac{1}{c}\left[\nabla_{\mathbf{u}_a}\varepsilon_{4g}(z_a,z)-\left|\boldsymbol{\eta}_{3g}\tau_\beta\right|_{z_a}^{z}\right]\nabla_{\mathbf{u}_a'}-\right.$$

$$\left.-\frac{1}{c}\left[\nabla_{\mathbf{u}_a'}\varepsilon_{4g}(z_a,z)-\left|\eta_{3g}\tau_a\right|_{z_a}^{z}\right]\nabla_{\mathbf{u}_a}\right\}\nabla_{\mathbf{u}_a'}\mu_{*4g}\,dz.\quad(30)$$

其中 m 爲系統的放大倍數，ε_{6g}, ε_{4g} 分別爲六級及四級點作用函數的高斯值，μ_{*4g} 爲折射率展開式之四次方項的高斯值。

從以上得出了在旋轉對稱系統中，在任意初條件，任意邊界面下的五級橫向普遍方程式，也進而推出了在高斯像平面下的五級橫向像差，由此出發可定出五級橫向像差的特殊像差係數。

三．在旋轉對稱系統中的五級像差係數

由五級横向像差方程式出發，可以決定五級横向像差，而全部的横向像差又可按其

與軌跡參量 u_a, u'_a 不同方次的關係，分成各種特殊形式的像差，而各項前面的係數稱爲五級特殊形式的像差係數。這樣，我們不僅可以決定系統中總的五級横向像差，且可研究各類特殊像差對像的影響。但因五級像差係數的公式複雜冗長，故不一一列出，此處只較詳細地寫出五級球差及畸變係數在複合場下的公式。最後從這部分工作的適用範圍來看，因爲導出結果時，假定了電子束的起始斜率幾近於零，因此不適用於浸沒物鏡系統，但是在一般系統中，利用此式定五級像差可以做到絕對精確。

首先從五級横向像差方程式出發，定出全部的五級横向像差，然後分別定出各類特殊像差的像差係數。由 (30)式我們把它分成二部分來計算：

(1) 計算第一部分"$\frac{m}{c}\nabla_{\mathbf{u}_a'}\varepsilon_{6g}$"。

由前知：

$$\varepsilon_{6g}=\int_{z_o}^{z_i}\mu_{*6}\,dz \qquad (\mathbf{u}=\mathbf{u}_g)\,, \tag{31}$$

因此首先把折射率 μ_* 展開，從而求出 μ_{*6g}。

由 [3] 知道，在軸對稱的電磁場中，介質的折射率爲：

$$\mu=\{\varphi_*(1+\tau'^2)\}^{1/2}-\frac{A_9}{\tau}(\boldsymbol{\tau}\times\boldsymbol{\tau}')\,, \tag{32}$$

其中 φ_*, A_9 分別爲空間電位及磁矢位在方位角方向的分量；如無空間電荷及電流存在，則 φ_*, A_9 可以軸上場分佈表示之：

$$\varphi_*=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\tau}{2}\right)^{2k}V^{2k}(z)=V_*-\frac{1}{4}V''\boldsymbol{\tau}^2+\frac{1}{64}V^{\mathrm{IV}}\boldsymbol{\tau}^4-\frac{1}{36\times 64}\boldsymbol{\tau}^6+\cdots, \tag{33a}$$

$$\frac{A_9}{\tau}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)\,(k+1)!}\left(\frac{\tau}{2}\right)^{2k}\mathscr{H}^{(2k)}(z)\,. \tag{33b}$$

今把 (33a), (33b) 代入 (32) 式，且展開到六次方項，則最後得到：

$$\begin{aligned}\mu={}&V_*^{1/2}-\frac{V''}{8V_*^{1/2}}\boldsymbol{\tau}^2+\frac{V_*^{1/2}}{2}\boldsymbol{\tau}'^2-\frac{\mathscr{H}}{2}(\boldsymbol{\tau}\times\boldsymbol{\tau}')+\frac{1}{128V_*^{1/2}}\left[V^{\mathrm{IV}}-\frac{V''^2}{V_*}\right]\boldsymbol{\tau}^4-\\&-\frac{V''}{16V_*^{1/2}}\boldsymbol{\tau}^2\boldsymbol{\tau}'^2-\frac{V_*^{1/2}}{8}\boldsymbol{\tau}'^4+\frac{\mathscr{H}''}{16}\boldsymbol{\tau}^2(\boldsymbol{\tau}\times\boldsymbol{\tau}')-\frac{1}{2\times 64\times 36V_*^{1/2}}\left[V^{\mathrm{VI}}-\frac{9}{2}\frac{V''V^{\mathrm{IV}}}{V_*}+\right.\\&\left.+\frac{9}{2}\frac{V''^3}{V_*^2}\right]\boldsymbol{\tau}^6+\frac{1}{256V_*^{1/2}}\left[V^{\mathrm{IV}}-\frac{V''^2}{V_*}\right]\boldsymbol{\tau}^4\boldsymbol{\tau}'^2+\frac{V''}{64V_*^{1/2}}\boldsymbol{\tau}^2\boldsymbol{\tau}'^4+\frac{V_*^{1/2}}{16}\boldsymbol{\tau}'^6-\\&-\frac{\mathscr{H}^{\mathrm{IV}}}{6\times 64}\boldsymbol{\tau}^4(\boldsymbol{\tau}\times\boldsymbol{\tau}')+\cdots=\mu_0+\mu_2+\mu_4+\mu_6+\cdots;\end{aligned} \tag{34}$$

其中 μ_0, μ_2 μ_4, μ_6, … 各相應於 τ, τ' 的零次，二次，四次，六次方等項。當有磁場存在時，

引入旋轉坐標系，其中 $\mathbf{u}$ 與 $\boldsymbol{\tau}$ 的關係見 (4) 式，而新的坐標系中的折射率 μ_* 對 $\mathbf{u}$, $\mathbf{u}'$ 展開爲：

$$\mu_* = \mu_{*0} + \mu_{*2} + \mu_{*4} + \mu_{*6} + \cdots, \tag{35}$$

於是利用 (4) 式，由 (34) 式可以求出 μ_{*6} 如下：

$$\begin{aligned}\mu_{*6} = {} & \frac{A}{64}\mathbf{u}_6 + \frac{B}{64}\mathbf{u}^4\mathbf{u}'^2 + \frac{C}{64}\mathbf{u}'^4\mathbf{u}^2 + \frac{V_*^{1/2}}{16}\mathbf{u}'^6 + \frac{D}{64}\mathbf{u}^4(\mathbf{u}\times\mathbf{u}') + \\ & + \frac{E}{32}\mathbf{u}^2\mathbf{u}'^2(\mathbf{u}\times\mathbf{u}') + \frac{F}{16}\mathbf{u}'^4(\mathbf{u}\times\mathbf{u}') + \frac{G}{64}\mathbf{u}^2(\mathbf{u}\times\mathbf{u}')^2 + \frac{H}{16}\mathbf{u}'^2(\mathbf{u}\times\mathbf{u}')^2 + \\ & + \frac{I}{16}(\mathbf{u}\times\mathbf{u}')^3;\end{aligned} \tag{36a}$$

其中

$$\left.\begin{aligned}
A &= \frac{1}{16}\left[\frac{\mathscr{H}^6+\mathscr{H}^4V''-V''^2\mathscr{H}^2-V''^3}{V_*^{5/2}} + \frac{\mathscr{H}^2V^{\mathrm{IV}}+V''V^{\mathrm{IV}}}{V_*^{3/2}} - \right.\\
&\qquad\qquad\left. - \frac{4/3\,\mathscr{H}\,\mathscr{H}^{\mathrm{IV}}+2/9\,V^{\mathrm{VI}}}{V_*^{1/2}}\right],\\
B &= \frac{1}{4}\left[\frac{3\mathscr{H}^4+2\mathscr{H}^2V''-V''^2}{V_*^{3/2}}+\frac{V^{\mathrm{IV}}}{V_*^{1/2}}\right], \qquad C = \frac{3\mathscr{H}^2+V''}{V_*^{1/2}},\\
D &= \frac{1}{4}\left[\frac{3\mathscr{H}^5+2\mathscr{H}^3V''-\mathscr{H}V''^2}{V_*^2}+\frac{V^{\mathrm{IV}}\mathscr{H}}{V_*}-\frac{2}{3}\mathscr{H}^{\mathrm{IV}}\right],\\
E &= \frac{3\mathscr{H}^3+V''\mathscr{H}}{V_*}, \qquad F = 3\mathscr{H}, \qquad G = \frac{3\mathscr{H}^4+V''\mathscr{H}^2}{V_*^{3/2}},\\
H &= \frac{3\mathscr{H}^2}{V_*^{1/2}}, \qquad I = \frac{\mathscr{H}^3}{V_*}.
\end{aligned}\right\} \tag{36b}$$

於是，在 (36a) 中，令 $\mathbf{u}=\mathbf{u}_g$，且以 $\mathbf{u}_g = \mathbf{u}_a\tau_\beta + \mathbf{u}_a'\tau_a$ 代入則可得 μ_{*6g} 對 $\mathbf{u}_a$, $\mathbf{u}_a'$ 展開式，再代入 (31) 式，可得 ε_{6g}. 於是最後可以得到：

$$\begin{aligned}\frac{m}{c}\nabla_{\mathbf{u}_a'}\varepsilon_{6g} = \frac{m}{c}\int_{z_a}^{z_b} & \{B'\mathbf{u}_a^5+2C'\mathbf{u}_a^4\mathbf{u}_a'+2D'\mathbf{u}_a^3(\mathbf{u}_a\cdot\mathbf{u}_a')+2E'\mathbf{u}_a^2\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a') + \\ & + E'\mathbf{u}_a^3\mathbf{u}_a'^2+3F'(\mathbf{u}_a\cdot\mathbf{u}_a')^2\mathbf{u}_a+4G'\mathbf{u}_a^2\mathbf{u}_a'^3+2H'\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')^2+2H'\mathbf{u}_a'^2\mathbf{u}_a(\mathbf{u}_a\cdot\mathbf{u}_a') + \\ & + 4I'\mathbf{u}_a'^3(\mathbf{u}_a\cdot\mathbf{u}_a')+I'\mathbf{u}_a'^4\mathbf{u}_a+6J'\mathbf{u}_a'^5+L'\mathbf{u}_a^4{}_a\underset{\sim}{\mathbf{u}}+M'\mathbf{u}_a^3(\mathbf{u}_a\times\mathbf{u}_a')+M'\mathbf{u}_a^2(\mathbf{u}_a\cdot\mathbf{u}_a')\underset{\sim}{\mathbf{u}}_a+ \\ & + 2N'\mathbf{u}_a(\mathbf{u}_a\cdot\mathbf{u}_a')(\mathbf{u}_a\times\mathbf{u}_a')+N'(\mathbf{u}_a\cdot\mathbf{u}_a')^2\underset{\sim}{\mathbf{u}}_a+2O'\mathbf{u}_a^2\mathbf{u}_a'(\mathbf{u}_a\times\mathbf{u}_a')+O'\mathbf{u}_a^2\mathbf{u}_a'^2\underset{\sim}{\mathbf{u}}_a + \\ & + 2P'\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')(\mathbf{u}_a\times\mathbf{u}_a')+P'\mathbf{u}_a'^2\mathbf{u}_a(\mathbf{u}_a\times\mathbf{u}_a')+P'\mathbf{u}_a'^2\underset{\sim}{\mathbf{u}}_a(\mathbf{u}_a\cdot\mathbf{u}_a)+Q'\mathbf{u}_a'^4\underset{\sim}{\mathbf{u}}_a + \\ & + 4Q'\mathbf{u}_a'^3(\mathbf{u}_a\times\mathbf{u}_a')+2R'\mathbf{u}_a^2\underset{\sim}{\mathbf{u}}_a(\mathbf{u}_a\times\mathbf{u}_a')+2S'\mathbf{u}_a'(\mathbf{u}_a\times\mathbf{u}_a')^2+2S'\mathbf{u}_a'^2\underset{\sim}{\mathbf{u}}_a(\mathbf{u}_a\times\mathbf{u}_a') +\end{aligned}$$

$$+T'\,\mathbf{u}_a\,(\mathbf{u}_a\times\mathbf{u}_a')^2+2T'\,(\mathbf{u}_a\cdot\mathbf{u}_a')\,(\mathbf{u}_a\times\mathbf{u}_a')\,\underset{\sim}{\mathbf{u}}_a+3u'\,(\mathbf{u}_a\times\mathbf{u}_a')^2\,\underset{\sim}{\mathbf{u}}_a\}\,dz\,; \tag{37}$$

其中

$$\left.\begin{aligned}
B' &= \left[\frac{3}{32}A\,\tau_\beta^5\,\tau_a+\frac{B}{32}(\tau_\beta^4\,\tau_\beta'\,\tau_a'+2\,\tau_\beta^3\,\tau_\beta'^2\,\tau_a)+\frac{C}{32}(\tau_\beta'^4\,\tau_\beta\,\tau_a+\right.\\
&\qquad\left.+2\,\tau_\beta'^3\,\tau_\beta^2\,\tau_a')+\frac{3\,V_*^{1/2}}{8}\,\tau_\beta'^5\,\tau_a'\right],\\
C' &= \left[\frac{3}{64}A\,\tau_\beta^4\,\tau_a^2+\frac{B}{64}(\tau_\beta^4\,\tau_a'^2+2\,\tau_\beta^2\,\tau_\beta'^2\,\tau_a^2)+\frac{C}{64}(\tau_\beta'^4\,\tau_a^2+\right.\\
&\qquad\left.+2\,\tau_\beta'^2\,\tau_\beta^2\,\tau_a'^2)+\frac{3\,V_*^{1/2}}{16}\,\tau_\beta'^4\,\tau_a'^2\right],\\
D' &= \left[\frac{3}{16}A\,\tau_\beta^4\,\tau_a^2+\frac{B}{16}(\tau_\beta^2\,\tau_\beta'^2\,\tau_a^2+2\,\tau_\beta^3\,\tau_\beta'\,\tau_a\,\tau_a')+\frac{C}{16}(\tau_\beta^2\,\tau_\beta'^2\,\tau_a'^2+\right.\\
&\qquad\left.+2\,\tau_\beta'^3\,\tau_\beta\,\tau_a'\,\tau_a)+\frac{3\,V_*^{1/2}}{4}\,\tau_\beta'^4\,\tau_a'^2\right],\\
E' &= \left[\frac{3A}{16}\,\tau_a^3\,\tau_\beta^3+\frac{B}{16}(\tau_\beta\,\tau_\beta'^2\,\tau_a^3+\tau_\beta^3\,\tau_a'^2\,\tau_a+\tau_\beta^2\,\tau_\beta'\,\tau_a^2\,\tau_a')+\frac{C}{16}(\tau_a'^3\,\tau_\beta'\,\tau_\beta^2+\right.\\
&\qquad\left.+\tau_\beta'^3\,\tau_a'\,\tau_a^2+\tau_\beta'^2\,\tau_\beta\,\tau_a'^2\,\tau_a)+\frac{3}{4}\,V_*^{1/2}\,\tau_a'^3\,\tau_\beta'^3\right],\\
F' &= \left[\frac{A}{8}\,\tau_a^3\,\tau_\beta^3+\frac{B}{8}\,\tau_\beta^2\,\tau_\beta'\,\tau_a^2\,\tau_a'+\frac{C}{8}\,\tau_\beta'^2\,\tau_\beta\,\tau_a'^2\,\tau_a+\frac{V_*^{1/2}}{2}\,\tau_a'^3\,\tau_\beta'^3\right],\\
G' &= \left[\frac{3}{64}A\,\tau_\beta^2\,\tau_a^4+\frac{B}{64}(\tau_a^4\,\tau_\beta'^2+2\,\tau_\beta^2\,\tau_a^2\,\tau_a'^2)+\frac{C}{64}(\tau_a'^4\,\tau_\beta^2+2\,\tau_\beta'^2\,\tau_a'^2\,\tau_a^2)+\right.\\
&\qquad\left.+\frac{3\,V_*^{1/2}}{16}\,\tau_\beta'^2\,\tau_a'^4\right],\\
H' &= \left[\frac{3A}{16}\,\tau_\beta^2\,\tau_a^4+\frac{B}{16}(\tau_\beta^2\,\tau_a^2\,\tau_a'^2+2\,\tau_\beta\,\tau_\beta'\,\tau_a^3\,\tau_a')+\frac{C}{16}(\tau_\beta'^2\,\tau_a'^2\,\tau_a^2+\right.\\
&\qquad\left.+2\,\tau_a'^3\,\tau_a\,\tau_\beta\,\tau_\beta')+\frac{3\,V_*^{1/2}}{4}\,\tau_\beta'^2\,\tau_a'^4\right],\\
I' &= \left[\frac{3A}{32}\,\tau_a^5\,\tau_\beta+\frac{B}{32}(\tau_a^4\,\tau_a'\,\tau_\beta'+2\,\tau_\beta\,\tau_a^3\,\tau_a'^2)+\frac{C}{32}(\tau_\beta\,\tau_a\,\tau_a'^4+2\,\tau_a'^3\,\tau_\beta'\,\tau_a^2)+\right.\\
&\qquad\left.+\frac{3\,V_*^{1/2}}{8}\,\tau_a'^5\,\tau_\beta'\right],\\
J' &= \left[\frac{A}{64}\,\tau_a^6+\frac{B}{64}\,\tau_a^2\,\tau_a'^2+\frac{C}{64}\,\tau_a'^4\,\tau_a^2+\frac{V_*^{1/2}}{16}\,\tau_a'^6\right],\\
L' &= \frac{V_{*a}^{1/2}}{V_*^{1/2}}\left[\frac{D}{64}\,\tau_\beta^4+\frac{E}{32}\,\tau_\beta^2\,\tau_\beta'^2+\frac{F}{16}\,\tau_\beta'^4\right],\\
M' &= \frac{V_{*a}^{1/2}}{V_*^{1/2}}\left[\frac{D}{16}\,\tau_\beta^3\,\tau_a+\frac{E}{16}(\tau_\beta\,\tau_a\,\tau_\beta'^2+\tau_\beta'\,\tau_a'\,\tau_\beta^2)+\frac{F}{4}\,\tau_\beta^3\,\tau_a'\right].
\end{aligned}\right\} \tag{38}$$

$$N' = \frac{V_{*a}^{1/2}}{V_*^{1/2}} \left[\frac{D}{16}\tau_\beta^2\tau_a^2 + \frac{E}{8}\tau_\beta\tau_a\tau'_\beta\tau'_a + \frac{F}{4}\tau_\beta'^2\tau_a'^2\right],$$

$$O' = \frac{V_{*a}^{1/2}}{V_*^{1/2}} \left[\frac{D}{32}\tau_\beta^2\tau_a^2 + \frac{E}{32}(\tau_a^2\tau_\beta'^2+\tau_\beta^2\tau_a'^2) + \frac{F}{8}\tau_\beta'^2\tau_a'^2\right],$$

$$P' = \frac{V_{*a}^{1/2}}{V_*^{1/2}} \left[\frac{D}{16}\tau_\beta\tau_a^3 + \frac{E}{16}(\tau_\beta\tau_a\tau_a'^2+\tau'_\beta\tau'_a\tau_a^2) + \frac{F}{4}\tau_a'^3\tau'_\beta\right],$$

$$Q' = \frac{V_{*a}^{1/2}}{V_*^{1/2}} \left[\frac{D}{64}\tau_a^4 + \frac{E}{32}\tau_a^2\tau_a'^2 + \frac{F}{16}\tau_a'^4\right],$$

$$R' = \frac{V_{*a}}{V_*}\left[\frac{G}{64}\tau_\beta^2 + \frac{H}{16}\tau_\beta'^2\right],$$

$$S' = \frac{V_{*a}}{V_*}\left[\frac{G}{64}\tau_a^2 + \frac{H}{16}\tau_a'^2\right],$$

$$T' = \frac{V_{*a}}{V_*}\left[\frac{G}{32}\tau_\beta\tau_a + \frac{H}{8}\tau'_\beta\tau'_a\right],$$

$$U' = \frac{I}{16}\frac{V_{*a}^{3/2}}{V_*^{3/2}}.$$

(2) 今求第二部分：

$$“\int_{z_a}^{z_b}\left\{\frac{1}{c}\left[\nabla_{\mathbf{u}_a}\varepsilon_{4g} - \left|\eta_{3g}\tau_\beta\right|_{z_a}^{z}\right]\nabla_{\mathbf{u}'_a} - \frac{1}{c}\left[\nabla_{\mathbf{u}'_a}\varepsilon_{4g} - \left|\eta_{3g}\tau_a\right|_{z_a}^{z}\right]\nabla_{\mathbf{u}_a}\right\}\nabla_{u'_a}\mu_{*4g}\,dz.”$$

按以上算符逐步運算，由 [3] 知

$$-\frac{1}{c}\varepsilon_{4g} = Q_{00}(\mathbf{u}_a\times\mathbf{u}'_a)^2 + (Q_{10}\mathbf{u}_a'^2 - 2Q_{11}\mathbf{u}_a\cdot\mathbf{u}'_a + Q_{12}\mathbf{u}_a^2)(\mathbf{u}_a\times\mathbf{u}'_a) + \frac{1}{4}Q_{20}\mathbf{u}_a'^4 -$$

$$- Q_{21}\mathbf{u}_a'^3\mathbf{u}_a + \frac{1}{2}Q_{22}[\mathbf{u}_a^2\mathbf{u}_a'^2 + 2(\mathbf{u}_a\cdot\mathbf{u}'_a)^2] - Q_{23}\mathbf{u}'_a\mathbf{u}_a^3 + \frac{1}{4}Q_{24}\mathbf{u}_a^4; \tag{39}$$

其中 $Q_{00}, Q_{10}, Q_{11}, \cdots$ 爲四級點作用函數展開式中各項的係數，詳細表示式可參閱 [3]．

(39) 式對 $\mathbf{u}_a$ 求偏陡度，則有：

$$-\frac{1}{c}\nabla_{\mathbf{u}_a}\varepsilon_{4g} = Q_{10}\mathbf{u}_a'^2\underset{\sim}{\mathbf{u}}'_a + Q_{21}\mathbf{u}_a'^3 + 2Q_{00}(\mathbf{u}_a\times\mathbf{u}'_a)\underset{\sim}{\mathbf{u}}'_a - Q_{22}[\mathbf{u}_a\mathbf{u}_a'^2 - 2(\mathbf{u}_a\cdot\mathbf{u}'_a)\mathbf{u}'_a] +$$

$$+ 2Q_{11}[\mathbf{u}'_a(\mathbf{u}_a\times\mathbf{u}'_a) - (\mathbf{u}_a\cdot\mathbf{u}'_a)\underset{\sim}{\mathbf{u}}'_a] + Q_{12}[\mathbf{u}_a^2\underset{\sim}{\mathbf{u}}'_a - 2\mathbf{u}_a(\mathbf{u}_a\times\mathbf{u}'_a)] +$$

$$+ Q_{23}[\mathbf{u}_a^2\mathbf{u}'_a + 2\mathbf{u}_a(\mathbf{u}_a\cdot\mathbf{u}'_a)] - Q_{24}\mathbf{u}_a^3, \tag{40a}$$

同理可得：

$$\frac{1}{c}\nabla_{\mathbf{u}'_a}\varepsilon_{4g} = -Q_{20}\mathbf{u}_a'^3 - Q_{10}[2(\mathbf{u}_a\times\mathbf{u}'_a)\mathbf{u}'_a + \mathbf{u}_a'^2\underset{\sim}{\mathbf{u}}_a] + Q_{21}[2\mathbf{u}'_a(\mathbf{u}_a\cdot\mathbf{u}'_a) + \mathbf{u}_a'^2\mathbf{u}_a] -$$

$$- 2Q_{00}(\mathbf{u}_a\times\mathbf{u}'_a)\underset{\sim}{\mathbf{u}}_a + 2Q_{11}[(\mathbf{u}_a\cdot\mathbf{u}'_a)\underset{\sim}{\mathbf{u}}_a + (\mathbf{u}_a\times\mathbf{u}'_a)\mathbf{u}_a] -$$

$$- Q_{22} \left[\mathbf{u}_a^2 \mathbf{u}_a' + 2 (\mathbf{u}_a \cdot \mathbf{u}_a') \mathbf{u}_a\right] + Q_{23} \mathbf{u}_a^3 - Q_{12} \mathbf{u}_a^2 \underset{\smile}{\mathbf{u}}_a . \tag{40b}$$

另外由 [3] 知:

$$\mu_{*4} = -\frac{1}{4} L \mathbf{u}^4 - \frac{1}{2} M \mathbf{u}^2 \mathbf{u}'^2 - \frac{1}{4} N \mathbf{u}'^4 - k V_* (\mathbf{u} \times \mathbf{u}')^2 -$$

$$- V_*^{1/2} (P \mathbf{u}^2 + Q \mathbf{u}'^2) (\mathbf{u} \times \mathbf{u}') , \tag{41}$$

今以 $u = \mathbf{u}_g$ 代入,則得 μ_{*4g} 對 $\mathbf{u}_a$, $\mathbf{u}_a'$ 的展開式,再把此式對 $\mathbf{u}_a'$ 求偏陡度,最後得到:

$$\nabla_{\mathbf{u}_a'} \mu_{*4g} = \left[a_{11} \mathbf{u}_a^2 + a_{12} (\mathbf{u}_a \cdot \mathbf{u}_a') + a_{13} \mathbf{u}_a'^2 + a_{14} (\mathbf{u}_a \times \mathbf{u}_a')\right] \mathbf{u}_a +$$

$$+ \left[a_{21} \mathbf{u}_a^2 + a_{22} (\mathbf{u}_a \cdot \mathbf{u}_a') + a_{23} \mathbf{u}_a'^2 + a_{24} (\mathbf{u}_a \times \mathbf{u}_a')\right] \underset{\smile}{\mathbf{u}}_a +$$

$$+ \left[a_{31} \mathbf{u}_a^2 + a_{32} (\mathbf{u}_a \cdot \mathbf{u}_a') + a_{33} \mathbf{u}_a'^2 + a_{34} (\mathbf{u}_a \times \mathbf{u}_a')\right] \mathbf{u}_a' =$$

$$= A_1 \mathbf{u}_a + A_2 \underset{\smile}{\mathbf{u}}_a + A_3 \mathbf{u}_a' ; \tag{42a}$$

其中

$$a_{11} = \left[-L \tau_\beta^3 \tau_a - M (\tau_\beta \tau_\beta'^2 \tau_a + \tau_\beta' \tau_\beta^2 \tau_a') - N \tau_\beta'^3 \tau_a'\right] ,$$

$$a_{12} = 2 \left[-L \tau_\beta^2 \tau_a^2 - 2M \tau_\beta \tau_a \tau_\beta' \tau_a' - N \tau_\beta'^2 \tau_a'^2\right] ,$$

$$a_{13} = \left[-L \tau_\beta \tau_a^3 - M (\tau_\beta \tau_a \tau_a'^2 + \tau_\beta' \tau_a' \tau_a^2) - N \tau_\beta' \tau_a'^3\right] ,$$

$$a_{14} = - 2 V_{*a}^{1/2} \left[P \tau_\beta \tau_a + Q \tau_\beta' \tau_a'\right] , \qquad a_{21} = - V_{*a}^{1/2} \left[P \tau_\beta^2 + Q \tau_\beta'^2\right] ,$$

$$a_{22} = - 2 V_{*a}^{1/2} \left[P \tau_\beta \tau_a + Q \tau_\beta' \tau_a'\right] , \qquad a_{23} = - V_{*a}^{1/2} \left[P \tau_a^2 + Q \tau_a'^2\right] ,$$

$$a_{24} = - 2k V_{*a} , \qquad a_{31} = \left[-L \tau_\beta^2 \tau_a^2 - M (\tau_a^2 \tau_\beta'^2 + \tau_\beta^2 \tau_a'^2) - N \tau_\beta'^2 \tau_a'^2\right] ,$$

$$a_{32} = 2\left[- L\tau_\beta \tau_a^3 - M(\tau_\beta \tau_a \tau_a'^2 + \tau_\beta' \tau_a' \tau_a^2) - N\tau_\beta' \tau_a'^3\right] ,$$

$$a_{33} = \left[- L\tau_a^4 - 2M\tau_a^2 \tau_a'^2 - N\tau_a'^4\right], \qquad a_{34} = - 2V_{*a}^{1/2}\left[P\tau_a^2 + Q\tau_a'^2\right] .$$

由 [3] 知:

$$\boldsymbol{\eta}_3 = \nabla_{\mathbf{u}'} \mu_{*4} = - M\mathbf{u}^2 \mathbf{u}' - N\mathbf{u}'^3 - 2KV_*(\mathbf{u} \times \mathbf{u}')\underset{\smile}{\mathbf{u}} - V_*^{1/2} P\mathbf{u}^2 \underset{\smile}{\mathbf{u}} -$$

$$- V_*^{1/2} Q\left[2\mathbf{u}'(\mathbf{u} \times \mathbf{u}') + \mathbf{u}'^2 \underset{\smile}{\mathbf{u}}\right] ,$$

介 $\mathbf{u} = \mathbf{u}_g$, 最後得:

$$\boldsymbol{\eta}_{3g} = - (M\tau_\beta^2 \tau_\beta' + N\tau_\beta'^3) \mathbf{u}_a^3 - V_*^{1/2}(P\tau_\beta^3 + Q\tau_\beta'^2 \tau_\beta) \mathbf{u}_a^2 \underset{\smile}{\mathbf{u}}_a - (M\tau_\beta^2 \tau_a' + N\tau_\beta'^2 \tau_a') \mathbf{u}_a^2 \mathbf{u}_a' -$$

$$- V_*^{1/2}(P\tau_\beta^2 \tau_a + Q\tau_\beta'^2 \tau_a) \mathbf{u}_a^2 \underset{\smile}{\mathbf{u}}_a' - 2(M\tau_\beta \tau_\beta' \tau_a + N\tau_\beta'^2 \tau_a') \mathbf{u}_a(\mathbf{u}_a \cdot \mathbf{u}_a') -$$

$$- 2V_*^{1/2}(P\tau_\beta^2 \tau_a + Q\tau_\beta' \tau_a' \tau_\beta) (\mathbf{u}_a \cdot \mathbf{u}_a') \underset{\smile}{\mathbf{u}}_a - (M\tau_a^2 \tau_\beta' + N\tau_a'^2 \tau_\beta') \mathbf{u}_a'^2 \mathbf{u}_a -$$

$$
\begin{aligned}
&-V_*^{1/2}(P\tau_a^2\tau_\beta+Q\tau_a'^2\tau_\beta)\,\mathbf{u}_a'^2\,\underset{\smile}{\mathbf{u}}_a-2(M\tau_\beta\tau_a\tau_a'+N\tau_\beta'\tau_a'^2)\,\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')-\\
&-2V_*^{1/2}(P\tau_\beta\tau_a^2+Q\tau_\beta'\tau_a'\tau_a)\,\underset{\smile}{\mathbf{u}}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')-(M\tau_a^2\tau_a'+N\tau_a'^3)\,\mathbf{u}_a'^3-\\
&-V_*^{1/2}(P\tau_a^3+Q\tau_a'^2\tau_a)\,\mathbf{u}_a'^2\,\underset{\smile}{\mathbf{u}}_a'-2KV_*^{1/2}V_{*a}^{1/2}\tau_\beta(\mathbf{u}_a\times\mathbf{u}_a')\,\underset{\smile}{\mathbf{u}}_a-\\
&\qquad\quad-2k\,V_*^{1/2}V_{*a}^{1/2}\tau_a(\mathbf{u}_a\times\mathbf{u}_a')\,\underset{\smile}{\mathbf{u}}_a'-\\
&-2V_{*a}^{1/2}Q\tau_\beta'\,\mathbf{u}_a(\mathbf{u}_a\times\mathbf{u}_a')-2Q\,V_{*a}^{1/2}\tau_a'\,\mathbf{u}_a'(\mathbf{u}_a\times\mathbf{u}_a')\,. \qquad (43)
\end{aligned}
$$

於是最後由（40a）及（43）式得到：

$$
\begin{aligned}
\Big[\frac{1}{c}\nabla\mathbf{u}_a\,\varepsilon_{4g}-\frac{1}{V_{*a}^{1/2}}\,\big|\boldsymbol{\eta}_{3g}\,\tau_\beta\big|_{z_a}^{z}\Big]\cdot\nabla\mathbf{u}_a' = \big[&\text{①}\,\mathbf{u}_a^2+\text{⑤}(\mathbf{u}_a\cdot\mathbf{u}_a')+\text{⑦}\,\mathbf{u}_a'^2+\\
&+\text{⑮}(\mathbf{u}_a\times\mathbf{u}_a')\big]\,\mathbf{u}_a'\cdot\nabla_{\mathbf{u}_a'}+\big[\text{②}\mathbf{u}_a^2+\text{⑥}(\mathbf{u}_a\cdot\mathbf{u}_a')+\text{⑧}\,\mathbf{u}_a'^2+\\
&+\text{⑬}(\mathbf{u}_a\times\mathbf{u}_a')\big]\,\underset{\smile}{\mathbf{u}}_a\cdot\nabla_{\mathbf{u}_a'}+\big[\text{③}\mathbf{u}_a^2+\text{⑨}(\mathbf{u}_a\cdot\mathbf{u}_a')+\text{⑪}\,\mathbf{u}_a'^2+\\
&+\text{⑯}(\mathbf{u}_a\times\mathbf{u}_a')\big]\,\mathbf{u}_a'\cdot\nabla_{\mathbf{u}_a'}+\big[\text{④}\mathbf{u}_a^2+\text{⑩}(\mathbf{u}_a\cdot\mathbf{u}_a')+\text{⑫}\,\mathbf{u}_a'^2+\\
&+\text{⑭}(\mathbf{u}_a\times\mathbf{u}_a')\big]\,\underset{\smile}{\mathbf{u}}_a'\cdot\nabla_{\mathbf{u}_a'}=\mathrm{I}\,\mathbf{u}_a\cdot\nabla_{\mathbf{u}_a'}+\mathrm{II}\,\underset{\smile}{\mathbf{u}}_a\cdot\nabla_{\mathbf{u}_a'}+\\
&+\mathrm{III}\,\mathbf{u}_a'\cdot\nabla_{\mathbf{u}_a'}+\mathrm{IV}\,\underset{\smile}{\mathbf{u}}_a'\cdot\nabla_{\mathbf{u}_a'}; \qquad (44a)
\end{aligned}
$$

其中 ①, ②, ③, … 是符號,代表之數如下：

$$
\left.\begin{aligned}
\text{①} &= \frac{1}{V_{*a}^{1/2}}(M\tau_\beta^3\tau_\beta'+N\tau_\beta'^3\tau_\beta)-Q_{24},\\
\text{②} &= \frac{V_*^{1/2}}{V_{*a}^{1/2}}\Big(P\tau_\beta^4+Q\tau_\beta'^2\tau_\beta^2-\frac{V_{*a}^{1/2}}{V_*^{1/2}}P_a\Big),\\
\text{③} &= \frac{1}{V_{*a}^{1/2}}(M\tau_\beta^3\tau_a'+N\tau_\beta'^2\tau_\beta\tau_a'-M_a)+Q_{23},\\
\text{④} &= \frac{V_*^{1/2}}{V_{*a}^{1/2}}(P\tau_\beta^3\tau_a+Q\tau_\beta'^2\tau_\beta\tau_a)+Q_{12},\\
\text{⑤} &= \frac{2}{V_{*a}^{1/2}}(M\tau_\beta^2\tau_\beta'\tau_a+N\tau_\beta'^2\tau_\beta\tau_a')+2Q_{23},\\
\text{⑥} &= \frac{2V_*^{1/2}}{V_{*a}^{1/2}}(P\tau_\beta^3\tau_a+Q\tau_\beta'\tau_\beta^2\tau_a'),\\
\text{⑦} &= \frac{1}{V_{*a}^{1/2}}(M\tau_a^2\tau_\beta'\tau_\beta+N\tau_a'^2\tau_\beta'\tau_\beta)-Q_{22},\\
\text{⑧} &= \frac{V_*^{1/2}}{V_{*a}^{1/2}}\Big(P\tau_a^2\tau_\beta^2+Q\tau_a'^2\tau_\beta^2-\frac{V_{*a}^{1/2}}{V_*^{1/2}}Q_a\Big),\\
\text{⑨} &= \frac{2}{V_{*a}^{1/2}}(M\tau_\beta^2\tau_a\tau_a'+N\tau_\beta'\tau_\beta\tau_a'^2)-2Q_{22},\\
\text{⑩} &= \frac{2V_*^{1/2}}{V_{*a}^{1/2}}(P\tau_\beta^2\tau_a^2+Q\tau_\beta'\tau_a'\tau_\beta\tau_a)-2Q_{11},\\
\text{⑪} &= \frac{1}{V_{*a}^{1/2}}(M\tau_a^2\tau_a'\tau_\beta+N\tau_a'^3\tau_\beta-N_a)+Q_{21},
\end{aligned}\right\}\qquad (44b)
$$

$$\left.\begin{aligned}
⑫ &= \frac{V_*^{1/2}}{V_{*a}^{1/2}}\left[(P\tau_a^3\tau_\beta + Q\tau_a'^2\tau_a\tau_\beta) + Q_{10}\right],\\
⑬ &= 2(K V_*^{1/2}\tau_\beta^2 - K_a V_{*a}^{1/2}),\\
⑭ &= 2K V_*^{1/2}\tau_a\tau_\beta + 2Q_{00},\\
⑮ &= 2Q\tau_\beta'\tau_\beta - 2Q_{12},\\
⑯ &= [2(Q\tau_a'\tau_\beta - Q_a) + 2Q_{11}).
\end{aligned}\right\}$$

最後由 (42a) 及 (44a) 式,得到:

$$\left\{\left[\frac{1}{c}\nabla_{\mathbf{u}_a}\varepsilon_{4g} - \frac{1}{V_{*a}^{1/2}}\,|\,\eta_{3g}\tau_\beta|_{z_a}^{z}\right]\cdot\nabla_{\mathbf{u}_a'}\right\}\nabla_{\mathbf{u}_a'}\mu_{*4g} =$$

$$\begin{aligned}
&= \mathrm{I}\{[a_{12}\mathbf{u}_a^2+2a_{13}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a+[a_{22}\mathbf{u}_a^2+2a_{23}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\underline{\mathbf{u}}_a+[a_{32}\mathbf{u}_a^2+\\
&\quad+2a_{33}(\mathbf{u}_a\cdot\mathbf{u}_a')]\mathbf{u}_a'+[a_{31}\mathbf{u}_a^2+a_{32}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{33}\mathbf{u}_a'^2+a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a\}+\\
&\quad+\mathrm{II}\{[2a_{13}(\mathbf{u}_a\times\mathbf{u}_a')+a_{14}\mathbf{u}_a^2]\,\mathbf{u}_a+[2a_{23}(\mathbf{u}_a\times\mathbf{u}_a')+a_{24}\mathbf{u}_a^2]\,\underline{\mathbf{u}}_a+\\
&\quad+[2a_{33}(\mathbf{u}_a\times\mathbf{u}_a')+a_{34}\mathbf{u}_a^2]\,\mathbf{u}_a'+[a_{31}\mathbf{u}_a^2+a_{32}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{33}\mathbf{u}_a'^2+\\
&\quad+a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underline{\mathbf{u}}_a\}+\mathrm{III}\{[a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a')+2a_{13}\mathbf{u}_a'^2+a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a+\\
&\quad+[a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a')+2a_{23}\mathbf{u}_a'^2+a_{24}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underline{\mathbf{u}}_a+[a_{23}(\mathbf{u}_a\cdot\mathbf{u}_a')+2a_{33}\mathbf{u}_a'^2+\\
&\quad+a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a'+[a_{31}\mathbf{u}_a^2+a_{32}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{33}\mathbf{u}_a'^2+a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a'\}+\\
&\quad+\mathrm{IV}\{[-a_{12}(\mathbf{u}_a\times\mathbf{u}_a')+a_{14}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\mathbf{u}_a+[-a_{22}(\mathbf{u}_a\times\mathbf{u}_a')+a_{14}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\underline{\mathbf{u}}_a+\\
&\quad+[-a_{32}(\mathbf{u}_a\times\mathbf{u}_a')+a_{34}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\mathbf{u}_a'+[a_{31}\mathbf{u}_a^2+a_{32}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{33}\mathbf{u}_a'^2+\\
&\quad+a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underline{\mathbf{u}}_a'.
\end{aligned}\tag{45}$$

再由 (40b) 及 (43) 式,得到:

$$\begin{aligned}
\frac{1}{c}\,&[\nabla_{\mathbf{u}_a'}\varepsilon_{4g} - |\,\eta_{3g}\tau_a|_{z_a}^{z}]\cdot\nabla_{\mathbf{u}_a} = [①'\,\mathbf{u}_a^2+⑤'(\mathbf{u}_a\cdot\mathbf{u}_a')+⑦'\,\mathbf{u}_a'^2+\\
&+⑮'(\mathbf{u}_a\times\mathbf{u}_a')](\mathbf{u}_a\cdot\nabla_{\mathbf{u}_a})+[②'\,\mathbf{u}_a^2+⑥'(\mathbf{u}_a\cdot\mathbf{u}_a')+⑧'\,\mathbf{u}_a'^2+\\
&+⑬'(\mathbf{u}_a\times\mathbf{u}_a')]\,(\underline{\mathbf{u}}_a\cdot\nabla_{\mathbf{u}_a})+[③'\,\mathbf{u}_a^2+⑨'(\mathbf{u}_a\cdot\mathbf{u}_a')+⑪'\,\mathbf{u}_a'^2+\\
&+⑯'(\mathbf{u}_a\times\mathbf{u}_a')]\,(\mathbf{u}_a'\cdot\nabla_{\mathbf{u}_a})+[④'\,\mathbf{u}_a^2+⑩'(\mathbf{u}_a\cdot\mathbf{u}_a')+⑫'\,\mathbf{u}_a'^2+\\
&+⑭'(\mathbf{u}_a\times\mathbf{u}_a')](\underline{\mathbf{u}}_a'\cdot\nabla_{\mathbf{u}_a})=\mathrm{I}'(\mathbf{u}_a\cdot\nabla_{\mathbf{u}_a})+\mathrm{II}'(\underline{\mathbf{u}}_a\cdot\nabla_{\mathbf{u}_a})+\\
&+\mathrm{III}'(\mathbf{u}_a'\cdot\nabla_{\mathbf{u}_a})+\mathrm{IV}'(\underline{\mathbf{u}}_a'\cdot\nabla_{\mathbf{u}_a});
\end{aligned}\tag{46a}$$

其中:

$$\left.\begin{aligned}
&①' = \frac{1}{V_{*a}^{1/2}}\left(M\tau_\beta^2\tau_\beta'\tau_a + N\tau_\beta'^3\tau_a\right) + Q_{23},\\
&②' = \frac{1}{V_{*a}^{1/2}}\left(P\tau_\beta^3\tau_a + Q\tau_\beta'^2\tau_\beta\tau_a\right) - Q_{12},\\
&③' = \frac{1}{V_{*a}^{1/2}}\left(M\tau_\beta^2\tau_a'\tau_a + N\tau_\beta'^2\tau_a'\tau_a\right) - Q_{22},\\
&④' = \frac{V_*^{1/2}}{V_{*a}^{1/2}}\left(P\tau_\beta^2\tau_a^2 + Q\tau_\beta'^2\tau_a^2\right),\\
&⑤' = 2\left[\frac{1}{V_{*a}^{1/2}}\left(M\tau_\beta\tau_\beta'\tau_a^2 + N\tau_\beta'^2\tau_a'\tau_a\right) - Q_{22}\right],\\
&⑥' = 2\left[\frac{V_*^{1/2}}{V_{*a}^{1/2}}\left(P\tau_\beta^2\tau_a^2 + Q\tau_\beta'\tau_a'\tau_\beta\tau_a\right) + Q_{11}\right],\\
&⑦' = \left[\frac{1}{V_{*a}^{1/2}}\left(M\tau_a^3\tau_\beta' + N\tau_a'^2\tau_\beta'\tau_a\right) + Q_{21}\right],\\
&⑧' = \left[\frac{V_*^{1/2}}{V_{*a}^{1/2}}\left(P\tau_a^3\tau_\beta + Q\tau_a'^2\tau_a^2\tau_\beta\right) - Q_{10}\right],\\
&⑨' = \frac{2}{V_{*a}^{1/2}}\left(M\tau_\beta\tau_a^2\tau_a' + N\tau_\beta'\tau_a'^2\tau_a\right) + 2Q_{21},\\
&⑩' = \frac{2V_*^{1/2}}{V_{*a}^{1/2}}\left(P\tau_\beta\tau_a^3 + Q\tau_\beta'\tau_a'\tau_a^2\right)\\
&⑪' = \left[\frac{1}{V_{*a}^{1/2}}\left(M\tau_a^3\tau_a' + N\tau_a\tau_a'^3\right) - Q_{20}\right],\\
&⑫' = \frac{V_*^{1/2}}{V_{*a}^{1/2}}\left(P\tau_a^4 + Q\tau_a'^2\tau_a^2\right).\\
&⑬' = 2V_*^{1/2}K\tau_\beta\tau_a - 2Q_{00},\\
&⑭' = 2KV_*^{1/2}\tau_a^2,\\
&⑮' = 2Q\tau_\beta'\tau_a + 2Q_{11},\\
&⑯' = 2\left(Q\tau_a'\tau_a - Q_{10}\right).
\end{aligned}\right\}\qquad (46\text{b})$$

再由 (46a) 及 (42a) 式,得到:

$$\begin{aligned}
&\left\{\left[\nabla_{\mathbf{u}_a'}\varepsilon_{4g} - \left[\boldsymbol{\eta}_{3g}\tau_a\right|_{z_a}^{z}\right]\cdot\nabla_{\mathbf{u}_a}\right\}\nabla_{\mathbf{u}_a'}\mu_{*4g} =\\
&\quad = \mathbf{I}'\{[2a_{11}\mathbf{u}_a^2 + a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a') + a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a + [2a_{21}\mathbf{u}_a^2 + a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a') +\\
&\qquad + a_{24}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a + [2a_{31}\mathbf{u}_a^2 + a_{32}(\mathbf{u}_a\cdot\mathbf{u}_a') + a_{34}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a' +\\
&\qquad + [a_{11}\mathbf{u}_a^2 + a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a') + a_{13}\mathbf{u}_a'^2 + a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\,\mathbf{u}_a + [a_{21}\mathbf{u}_a^2 + a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a') +\\
&\qquad + a_{23}\mathbf{u}_a'^2 + a_{24}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a\} + \mathbf{I}\mathbf{I}'\{[a_{12}(\mathbf{u}_a\times\mathbf{u}_a') - a_{14}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\mathbf{u}_a +\\
&\qquad + [a_{22}(\mathbf{u}_a\times\mathbf{u}_a') - a_{24}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a + [a_{32}(\mathbf{u}_a\times\mathbf{u}_a') - a_{34}(\mathbf{u}_a\cdot\mathbf{u}_a')]\,\mathbf{u}_a' +
\end{aligned}$$

$$+[a_{11}\mathbf{u}_a^2+a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{13}\mathbf{u}_a'^2+a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a-[a_{21}\mathbf{u}_a^2+a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a')+$$
$$+a_{23}\mathbf{u}_a'^2+a_{24}(\mathbf{u}_a'\times\mathbf{u}_a')]\,\mathbf{u}_a\}+\mathrm{III}'\{[2a_{11}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{12}\mathbf{u}_a'^2]\,\mathbf{u}_a+$$
$$+[2a_{21}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{22}\mathbf{u}_a'^2]\,\underset{\sim}{\mathbf{u}}_a+[2a_{31}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{32}\mathbf{u}_a'^2]\,\mathbf{u}_a'+[a_{11}\mathbf{u}_a^2+$$
$$+a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{13}\mathbf{u}_a'^2+a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\mathbf{u}_a'+[a_{21}\mathbf{u}_a^2+a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{23}\mathbf{u}_a'^2+$$
$$+a_{24}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a'\}+\mathrm{IV}'\{[-2a_{11}(\mathbf{u}_a\times\mathbf{u}_a')-a_{14}\mathbf{u}_a'^2]\,\mathbf{u}_a+$$
$$+[-2a_{21}(\mathbf{u}_a\times\mathbf{u}_a')-a_{24}\mathbf{u}_a'^2]\,\underset{\sim}{\mathbf{u}}_a+[-2a_{31}(\mathbf{u}_a\times\mathbf{u}_a')-a_{34}\mathbf{u}_a'^2]\mathbf{u}_a'+$$
$$+[a_{11}\mathbf{u}_a^2+a_{12}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{13}\mathbf{u}_a'^2+a_{14}(\mathbf{u}_a\times\mathbf{u}_a')]\,\underset{\sim}{\mathbf{u}}_a'-[a_{21}\mathbf{u}_a^2+$$
$$+a_{22}(\mathbf{u}_a\cdot\mathbf{u}_a')+a_{23}\mathbf{u}_a'^2+a_{24}(\mathbf{u}_a\times\mathbf{u}_a')\,\mathbf{u}_a'\}\,. \tag{47}$$

因此最後把 (37), (45) 及 (47) 式代入 (30) 式中加以整理,得到:

$$\Delta\mathbf{u}_{5b}=g_5\mathbf{u}_a^5+g_5'\mathbf{u}_a^4\,\underset{\sim}{\mathbf{u}}_a+a_0\mathbf{u}_a^4\mathbf{u}_a'+a_1\mathbf{u}_a^3(\mathbf{u}_a\cdot\mathbf{u}_a')+a_0'\mathbf{u}_a^4\,\underset{\sim}{\mathbf{u}}_a'+a_1'\mathbf{u}_a^2\,\underset{\sim}{\mathbf{u}}_a(\mathbf{u}_a\cdot\mathbf{u}_a')+$$
$$+a_2'\mathbf{u}_a^3(\mathbf{u}_a\times\mathbf{u}_a')+K_0\mathbf{u}_a^3\mathbf{u}_a'^2+K_1\mathbf{u}_a^2\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')+K_2(\mathbf{u}_a\cdot\mathbf{u}_a')^2\mathbf{u}_a+K_0'\mathbf{u}_a'^2\mathbf{u}_a^2\,\underset{\sim}{\mathbf{u}}_a+$$
$$+K_1'\mathbf{u}_a^2\mathbf{u}_a'(\mathbf{u}_a\times\mathbf{u}_a')+K_2'\mathbf{u}_a^2(\mathbf{u}_a\cdot\mathbf{u}_a')\underset{\sim}{\mathbf{u}}_a'+K_3'(\mathbf{u}_a\cdot\mathbf{u}_a')^2\,\underset{\sim}{\mathbf{u}}_a+K_4'(\mathbf{u}_a\cdot\mathbf{u}_a')(\mathbf{u}_a\times\mathbf{u}_a')\,\mathbf{u}_a+$$
$$+D_0\mathbf{u}_a^2\mathbf{u}_a'^3+D_1\mathbf{u}_a'^2\mathbf{u}_a(\mathbf{u}_a\cdot\mathbf{u}_a')+D_2\mathbf{u}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')^2+D_0'\mathbf{u}_a^2\mathbf{u}_a'^2\,\underset{\sim}{\mathbf{u}}_a'+$$
$$+D_1'\mathbf{u}_a'^2\,\underset{\sim}{\mathbf{u}}_a(\mathbf{u}_a\cdot\mathbf{u}_a')+D_2'\,\underset{\sim}{\mathbf{u}}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')^2+D_3'(\mathbf{u}_a\cdot\mathbf{u}_a')(\mathbf{u}_a\times\mathbf{u}_a')\,\mathbf{u}_a'+$$
$$+D_4'\mathbf{u}_a'^2\mathbf{u}_a(\mathbf{u}_a\times\mathbf{u}_a')+C_0\mathbf{u}_a'^4\mathbf{u}_a+C_1\mathbf{u}_a'^3(\mathbf{u}_a\cdot\mathbf{u}_a')+C_0'\mathbf{u}_a'^4\,\underset{\sim}{\mathbf{u}}_a+C_1'\mathbf{u}_a'^2\,\underset{\sim}{\mathbf{u}}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')+$$
$$+C_2'\mathbf{u}_a'^3(\mathbf{u}_a\times\mathbf{u}_a')+S_5\mathbf{u}_a'^5+S_5'\mathbf{u}_a'^4\,\underset{\sim}{\mathbf{u}}_a'\,. \tag{48}$$

上式中各項前面的係數即爲各種特殊像差的像差係數,因公式複雜冗長,此處只詳細寫出在複合場下的五級球差和畸變係數.

(i) 在複合場下的五級各向同性球差係數爲:

$$S_5=-\frac{m}{c}\int_{z_a}^{z_b}\left\{\frac{3}{64\times 8}\left[\frac{\mathscr{H}^6+\mathscr{H}^4V''-V''^2\mathscr{H}^2-V''^3}{V_*^{5/2}}+\frac{\mathscr{H}^2V^{\mathrm{IV}}+V''V^{\mathrm{IV}}}{V_*^{3/2}}-\right.\right.$$
$$\left.-\frac{4/3\,\mathscr{H}\cdot\mathscr{H}^{\mathrm{IV}}+2/9\,V^{\mathrm{VI}}}{V_*^{1/2}}\right]\tau_a^6+\frac{3}{128}\left[\frac{3\mathscr{H}^4+2\mathscr{H}^2V''-V''^2}{V_*^{3/2}}\right]\tau_a^4\tau_a'^2+$$
$$+\frac{3}{32}\left[\frac{3\mathscr{H}^2+V''}{V_*^{1/2}}\right]\tau_a'^4\tau_a^2+\frac{3V_*^{1/2}}{16}\tau_a'^6+$$
$$+3\left[\frac{1}{V_{*a}^{1/2}}(M\tau_a^3\tau_a'+N\tau_a\tau_a'^3)-Q_{20}\right]\cdot[L\tau_\beta\tau_a^3+M(\tau_\beta\tau_a\tau_a'^2+\tau_\beta'\tau_a'\tau_a^2)+N\tau_\beta'\tau_a'^3]-$$
$$-3\left[\frac{1}{V_{*a}^{1/2}}(M\tau_a^2\tau_a'\tau_\beta+N\tau_a'^3\tau_\beta-N_a)+Q_{21}\right]\cdot[L\tau_a^4+2M\tau_a^2\tau_a'^2+N\tau_a'^4]-$$
$$\left.-3V_*^{1/2}(P\tau_a^4+Q\tau_a'^2\tau_a^2)\cdot[P\tau_a^2+Q\tau_a'^2]\right\}dz\,. \tag{49}$$

(ii) 在複合場下的五級各向異性球差係數爲:

$$S_5'=-\frac{m}{c}\int_{z_a}^{z_b}\left\{\left[\frac{1}{V_{*a}^{1/2}}(M\tau_a^3\tau_a'+N\tau_a\tau_a'^3)-Q_{20}\right]\cdot[P\tau_a^2+Q\tau_a'^2]+\right.$$

$$
+\left[V_*^{1/2}(P\tau_\alpha^4+Q\tau_\alpha'^2\tau_\alpha^2)\right]\cdot\left[L\tau_\beta\tau_\alpha^3+M(\tau_\beta\tau_\alpha\tau_\alpha'^2+\tau_\beta'\tau_\alpha'\tau_\alpha^2)+N\tau_\beta'\tau_\alpha'^3\right]-
$$

$$
-\left.\frac{V_*^{1/2}}{V_{*a}^{1/2}}\left[(P\tau_\alpha^3\tau_\beta+Q\tau_\alpha'^2\tau_\alpha\tau_\beta)+Q_{10}\right]\cdot\left[L\tau_\alpha^4+2M\tau_\alpha^2\tau_\alpha'^2+N\tau_\alpha'^4\right]\right\}dz. \tag{50}
$$

(iii) 在複合場下五級各向同性畸變係數爲：

$$
g_5=-\frac{m}{c}\int_{z_a}^{z_b}\left\{\frac{3}{64\times 8}\left[\frac{\mathscr{H}^6+\mathscr{H}^4V''-\mathscr{H}^2V''^2-V''^3}{V_*^{5/2}}+\frac{\mathscr{H}^2V^{\mathrm{IV}}+V''V^{\mathrm{IV}}}{V_*^{3/2}}-\right.\right.
$$

$$
\left.-\frac{(4/3)\mathscr{H}\mathscr{H}^{\mathrm{IV}}+(2/9)V^{\mathrm{VI}}}{V_*^{1/2}}\right]\tau_\beta^5\tau_\alpha+\frac{1}{128}\left[\frac{3\mathscr{H}^4+2\mathscr{H}^2V''-V''^2}{V_*^{3/2}}+\frac{V^{\mathrm{IV}}}{V_*^{1/2}}\right]\cdot
$$

$$
\cdot(\tau_\beta^4\tau_\beta'\tau_\alpha'+2\tau_\beta'\tau_\beta\tau_\beta'\tau_\alpha)+\frac{1}{32}\left[\frac{3\mathscr{H}^2+V''}{V_*^{1/2}}\right](\tau_\beta'^4\tau_\beta\tau_\alpha+2\tau_\beta'^3\tau_\beta^2\tau_\alpha')+\frac{3}{8}V_*^{1/2}\tau_\beta'^5\tau_\alpha'+
$$

$$
+3\left[\frac{1}{V_{*a}^{1/2}}(M\tau_\beta^2\tau_\beta'\tau_\alpha+N\tau_\beta'^3\tau_\alpha)+Q_{23}\right]\cdot\left[L\tau_\beta^3\tau_\alpha+M(\tau_\beta\tau_\beta'^2\tau_\alpha+\tau_\beta'\tau_\beta^2\tau_\alpha')-\right.
$$

$$
\left.-N\tau_\beta'^3\tau_\alpha'\right]-\left[\frac{1}{V_{*a}^{1/2}}(M\tau_\beta^3\tau_\beta'+N\tau_\beta'^3\tau_\beta)-Q_{24}\right]\cdot\left[3L\tau_\beta^2\tau_\alpha^2+M(\tau_\alpha^2\tau_\beta'^2+4\tau_\beta'\tau_\beta\tau_\alpha'\tau_\alpha+\right.
$$

$$
\left.+\tau_\beta^2\tau_\alpha'^2)+3N\tau_\beta'^2\tau_\alpha'^2\right]-2\left[V_*^{1/2}\left(P\tau_\beta^4+Q\tau_\beta'^2\tau_\beta^2-\frac{V_{*a}^{1/2}}{V_*^{1/2}}P_a\right)\right]\cdot\left[P\tau_\beta\tau_\alpha+Q\tau_\beta'\tau_\alpha'\right]-
$$

$$
-\left.\left[(P\tau_\beta^3\tau_\alpha+Q\tau_\beta'^2\tau_\beta\tau_\alpha)-V_{*a}^{1/2}Q_{12}\right]\cdot\left[P\tau_\beta^2+Q\tau_\beta'^2\right]\right\}dz. \tag{51}
$$

(iv) 在複合場下五級各向異性畸變係數爲：

$$
g_5'=-\frac{m}{c}\int_{z_a}^{z_b}\left\{\frac{V_{*a}^{1/2}}{V_*^{1/2}}\left[\frac{1}{64\times 4}\left(\frac{3\mathscr{H}^5+2\mathscr{H}^3V''-\mathscr{H}V''^2}{V_*^2}+\frac{V^{\mathrm{IV}}\mathscr{H}}{V_*}-\frac{2}{3}\mathscr{H}^{\mathrm{IV}}\right)\tau_\beta^4+\right.\right.
$$

$$
\left.+\frac{3\mathscr{H}^3+V''\mathscr{H}}{32V_*}\tau_\beta^2\tau_\beta'^2+\frac{3}{16}\mathscr{H}\tau_\beta'^4\right]+V_*^{1/2}(P\tau_\alpha^4+Q\tau_\alpha'^2\tau_\alpha^2)(P\tau_\alpha^2+Q\tau_\alpha'^2)+
$$

$$
+\left[\frac{1}{V_{*a}^{1/2}}(M\tau_\alpha^3\tau_\alpha'+N\tau_\alpha^2\tau_\alpha'^3)-Q_{20}\right]\cdot\left[L\tau_\beta\tau_\alpha^3+M(\tau_\beta\tau_\alpha\tau_\alpha'^2+\tau_\beta'\tau_\alpha'\tau_\alpha^2)+N\tau_\beta'\tau_\alpha'^3\right]-
$$

$$
-\left.\frac{V_*^{1/2}}{V_{*a}^{1/2}}\left[(P\tau_\alpha^3\tau_\beta+Q\tau_\alpha'^2\tau_\alpha\tau_\beta)+Q_{10}\right]\cdot\left[L\tau_\alpha^4+2M\tau_\alpha^2\tau_\alpha'^2+N\tau_\alpha'^4\right]\right\}dz. \tag{52}
$$

在以上公式中，m 爲放大倍數，$c=V_*{}^{1/2}(\tau_\alpha'\tau_\beta-\tau_\beta'\tau_\alpha)=$ 常數，Q_{20}，Q_{21}，Q_{10}，Q_{23}，… 等爲四級點作用函數展開式中各項前面的係數[3]，L, M, N, … 是軸上場分佈的函數[3].

對於在純電場或純磁場下的相應公式，可(由上式)分別令 $\mathscr{H}\equiv 0$ 或 $V=$ 常數而推得，此處從略.

所得結果繁複，且有重積分出現，因此需藉助於計算機，但是由此總可以去研究和決定軸對稱系統中的五級像差.

四. 五級特殊像差的散射圖形

此處把全部五級橫向像差按 u_a, u_a' 的不同方次的組合來分類；共可分成六類，以 $u_a^{5-p}\,u_a'^{p}$ 來表示，其中 $p=0, 1, 2, 3, 4, 5$. 然後又分别討論這些特殊形式像差的散射圖形，使我們對各類特殊形式像差對像的影響有一個較形象化的概念.

以下分别討論之.

第一類：五級畸變.

如果其他像差不存在，由於五級畸變引起的橫向像差為：

$$\Delta \mathbf{u}_{5b} = g_5\,\mathbf{u}_a^5 + g_5'\,\mathbf{u}_a^4\,\underline{\mathbf{u}}_a\,. \tag{1}$$

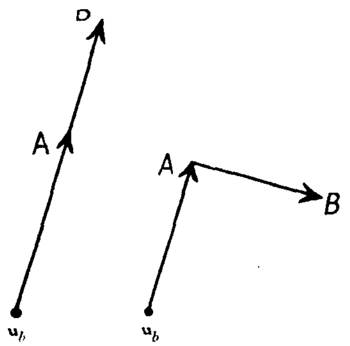

圖 1(a) 圖 1(b)

因此它使像點相對於高斯像點發生一偏離. 當只有各向同性畸變存在，使高斯像點 A 位移到 B，位移 AB 在 $\mathbf{u}_b$ 方向，其長與 u_a^5 成比例. 如只有各向異性畸變存在，位移發生在垂直於 $\mathbf{u}_b$ 的方向，其長與 u_a^5 成比例. 如二部分畸變同時存在，則像點相對於高斯像點的位移為 $r=\sqrt{g_5^2+g_5'^2}\,u_a^5$，而位移方向與 $\mathbf{u}_b$ 成 α 角，$\operatorname{tg}\alpha=\dfrac{g_5'}{g_5}$. 這二部分畸變對像的影響，可看作使像的放大倍數與旋轉角發生變化. 若僅有五級畸變存在，則成像位置為：

$$\begin{aligned}\mathbf{u}_b &= m\,\mathbf{u}_a + g_5\,\mathbf{u}_a^5 + g_5'\,\mathbf{u}_a^4\,\underline{\mathbf{u}}_a = \\ &= m\,\mathbf{u}_a\left(1+\frac{g_5}{m}\,\mathbf{u}_a^4\right) + g_5'\,\mathbf{u}_a^4\,\underline{\mathbf{u}}_a = m_{\rho 5}\,\mathbf{u}_a + m\,\underline{\mathbf{u}}_a\,\Delta\chi_{b5}\,;\end{aligned} \tag{2}$$

因此，由於各向同性畸變的存在，放大倍數變為：

$$m_{\rho 5} = m\left(1+\frac{g_5}{m}\,\mathbf{u}_a^4\right), \tag{3a}$$

則放大倍數的相對改變為：

$$\frac{\Delta m_5}{m} = \frac{g_5}{m}\,\mathbf{u}_a^4\,. \tag{3b}$$

而由於各向異性畸變的存在，引起像的旋轉為：

$$\Delta\chi_{b5} = \frac{g_5'}{m}\,\mathbf{u}_a^4\,. \tag{4}$$

第二類：與 $u_a^4\,u_a'$ 成比例的項.

當只有這種像差存在時，則得：

$$\triangle \mathbf{u}_{5b} = \cdots\cdots a_0 \mathbf{u}_a^4 \mathbf{u}_a' + a_1 \mathbf{u}_a^3(\mathbf{u}_a \cdot \mathbf{u}_a') + a_0' \mathbf{u}_a^4 \underset{\smile}{\mathbf{u}}_a' + a_1' \mathbf{u}_a^2 \underset{\smile}{\mathbf{u}}_a(\mathbf{u}_a \cdot \mathbf{u}_a') + + a_2' \mathbf{u}_a^3(\mathbf{u}_a \times \mathbf{u}_a'), \tag{5}$$

今把 $\triangle \mathbf{u}_{5b}$ 投射到 $\mathbf{u}_a$, $\underset{\smile}{\mathbf{u}}_a$ 方向上去，其分量以 ξ, η 表示之. 令 $\mathbf{u}_a$, $\mathbf{u}_a'$ 間夾角爲 γ, 則得：

$$\xi = \frac{\Delta \mathbf{u}_{5b} \cdot \mathbf{u}_a}{u_a} = u_a^4 u_a'[(a_0+a_1)\cos\gamma + (a_2'-a_0')\sin\gamma], \tag{6a}$$

$$\eta = \frac{\Delta \mathbf{u}_{5b} \cdot \underset{\smile}{\mathbf{u}}_a}{u_a} = u_a^4 u_a'[a_0 \sin\gamma + (a_0'+a_1')\cos\gamma]. \tag{6b}$$

(6a, b) 爲一橢圓方程式，此橢圓之二半軸及 $\mathbf{u}_b$ 與主軸間夾角 ϑ 可由公式計算之. 因此，總的散射圖形爲：由同一點發出之不同斜率之電子束，其成像爲一系列同中心之橢

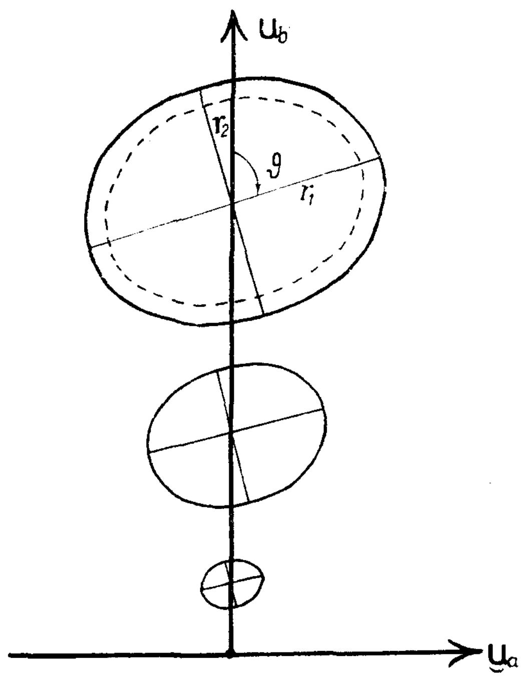

圖 2

圓，其半軸之長與 $u_a^4 u_a'$ 成比例，當從不同高度發出的但具有同一孔徑角的電子束，其散射圖形爲一系列大小不等其中心在相應的高斯像點上的橢圓.

第三類：與 $u_a^3 u_a'^2$ 成比例的項.

如果只有這類像差存在，則可得：

$$\triangle \mathbf{u}_{5b} = \cdots + K_0 \mathbf{u}_a^3 \mathbf{u}_a'^2 + K_1 \mathbf{u}_a^2 \mathbf{u}_a'(\mathbf{u}_a \cdot \mathbf{u}_a') + K_2(\mathbf{u}_a \cdot \mathbf{u}_a')^2 \mathbf{u}_a +$$

$$+ K_0' \mathbf{u}_a'^2 \mathbf{u}_a^2 \underline{\mathbf{u}}_a + K_1' \mathbf{u}_a^2 \mathbf{u}_a'(\mathbf{u}_a \times \mathbf{u}_a') + K_2' \mathbf{u}_a^2 \underline{\mathbf{u}}_a'(\mathbf{u}_a \cdot \mathbf{u}_a') + K_3'(\mathbf{u}_a \cdot \mathbf{u}_a')^2 \underline{\mathbf{u}}_a +$$

$$+ K_4'(\mathbf{u}_a \cdot \mathbf{u}_a')(\mathbf{u}_a \times \mathbf{u}_a') \mathbf{u}_a + \cdots; \tag{7}$$

同樣可得分量式爲:

$$\left\{\xi - \left[K_0 + \frac{1}{2}(K_1+K_2)\right] u_a^3 u_a'^2\right\} =$$

$$= u_a^3 u_a'^2 \left[\frac{1}{2}(K_1+K_2)\cos 2\gamma + \frac{1}{2}(K_1'-K_2'+K_4')\sin 2\gamma\right], \tag{8a}$$

$$\left\{\eta - \left[K_0' + \frac{1}{2}(K_1'+K_2'+K_3') u_a^3 u_a'^2\right\} =$$

$$= u_a^3 u_a'^2 \left[\frac{1}{2}(K_2'+K_3'-K_1')\cos 2\gamma + \frac{1}{2} K_1 \sin 2\gamma\right]. \tag{8b}$$

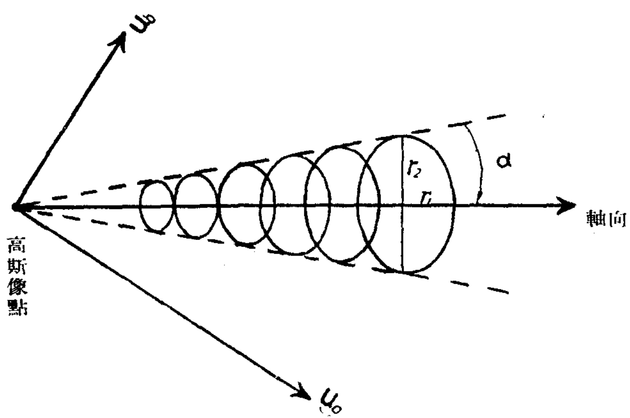

圖 3

此爲一橢圓方程式,其散射中心對高斯像點有一位移爲:

$$l = \sqrt{\left[K_0 + \frac{1}{2}(K_1+K_2)\right]^2 + \left[K_0' + \frac{1}{2}(K_1'+K_2'+K_3')\right]^2}\; u_a^3 u_a'^2. \tag{9}$$

而 ϑ 角及半軸 r_1, r_2 可由公式計算之. 總的圖形爲: 對應於同一物點發出的不同斜率的電子束,所成之像爲一系列大小不同之橢圓,其中心對高斯像點有不同之位移,而 α 角可由下式決定:

$$\operatorname{tg}\alpha = \frac{r_2}{l}. \tag{10}$$

第四類: 與 $u_a^2 u_a'^3$ 成比例的項.

如果其他像差不存在,則可得:

$$\Delta \mathbf{u}_{5b} = \cdots D_0 \mathbf{u}_a^2 \mathbf{u}_a'^3 + D_1 \mathbf{u}_a'^2 \mathbf{u}_a(\mathbf{u}_a \cdot \mathbf{u}_a') + D_2 \mathbf{u}_a'(\mathbf{u}_a \cdot \mathbf{u}_a')^2 +$$

$$+ D_0' \mathbf{u}_a^2 \mathbf{u}_a'^2 \underset{\smile}{\mathbf{u}}_a' + D_1' \mathbf{u}_a'^2 \underset{\smile}{\mathbf{u}}_a(\mathbf{u}_a\cdot\mathbf{u}_a') + D_2' \underset{\smile}{\mathbf{u}}_a'(\mathbf{u}_a\cdot\mathbf{u}_a')^2 + D_3'(\mathbf{u}_a\cdot\mathbf{u}_a')(\mathbf{u}_a\times\mathbf{u}_a')\,\mathbf{u}_a' +$$

$$+ D_4' \mathbf{u}_a'^2 \mathbf{u}_a(\mathbf{u}_a\times\mathbf{u}_a')\,; \tag{11}$$

則其分量式爲:

$$\xi = u_a^2 u_a'^3 \left\{ \left[D_0 + D_1 + \frac{3}{4} D_2 \right] \cos\gamma + \right.$$

$$\left. + \left[D_4' + \frac{1}{4}(D_3' - D_2') - D_0' \right] \sin\gamma + \frac{1}{4} D_2 \cos 3\gamma + \frac{1}{4}(D_3' - D_2') \sin 3\gamma \right\}, \tag{12a}$$

$$\eta = u_a^2 u_a'^3 \left\{ \left[D_0' + D_1' + \frac{1}{4} D_3' + \frac{3}{4} D_2' \right] \cos\gamma + \right.$$

$$\left. + \left[D_0 + \frac{1}{4} D_2 \right] \sin\gamma + \frac{1}{4}(D_2' - D_3') \cos 3\gamma + \frac{1}{4} D_2 \sin 3\gamma \right\}. \tag{12b}$$

此爲一對以 γ 爲參數之參數方程式,當係數確定時,代表一組曲線,此時整個圖形的排列是這樣的:對應於同一物點發出的不同斜率的電子束,成像爲一組以高斯像點爲中心的大小不同的曲線; 對應於不同物點發出的具有同樣斜率的電子束, 成像爲一系列各以對應之高斯像點爲散射中心的不同大小的曲線.

第五類: 與 $u_a'^4 u_a$ 成比例的項.

此時可得:

$$\Delta\mathbf{u}_{5b} = C_0 \mathbf{u}_a'^4 \mathbf{u}_a + C_1 \mathbf{u}_a'^3(\mathbf{u}_a\cdot\mathbf{u}_a') +$$

$$+ C_0' \mathbf{u}_a'^4 \underset{\smile}{\mathbf{u}}_a + C_1' \mathbf{u}_a'^2 \underset{\smile}{\mathbf{u}}_a'(\mathbf{u}_a\cdot\mathbf{u}_a') + C_2' \mathbf{u}_a'^3(\mathbf{u}_a\times\mathbf{u}_a') + \cdots; \tag{13}$$

於是分量式爲:

$$\left\{ \xi - \left[C_0 + \frac{1}{2} C_1 \right] u_a'^4 u_a \right\} = \left[\frac{1}{2} C_1 \cos 2\gamma + \frac{1}{2}(C_2' - C_1') \sin 2\gamma \right] u_a'^4 u_a, \tag{14a}$$

$$\left\{ \eta - \left[C_0' + \frac{1}{2}(C_1' + C_2') \right] u_a'^4 u_a \right\} = \left[\frac{1}{2}(C_1' - C_2') \cos 2\gamma + \frac{1}{2} C_1 \sin 2\gamma \right] u_a'^4 u_a. \tag{14b}$$

此代表一圓方程式,其中心離原點之坐標爲:

$$\xi = \left(C_0 + \frac{1}{2} C_1\right) u_a'^4 u_a, \qquad \eta = \left[C_0' + \frac{1}{2}(C_1' + C_2') \right] u_a'^4 u_a, \tag{15}$$

即散射圓心離高斯像點的距離爲:

$$l = \sqrt{\left(C_0 + \frac{1}{2} C_1\right)^2 + \left[C_0' + \frac{1}{2}(C_1' + C_2') \right]^2}\; u_a'^4 u_a, \tag{16}$$

而圓半徑爲:

$$r = \frac{1}{2}\sqrt{C_1^2+(C_2'-C_1')^2}\; u_a'^4 u_a, \tag{17}$$

$$\operatorname{tg} \alpha = \frac{r}{l} = \frac{\sqrt{C_1^2+(C_2'-C_1')^2}}{2\sqrt{\left(C_0+\frac{1}{2}C_1\right)^2+\left[C_0'+\frac{1}{2}(C_1'+C_2')\right]^2}}. \tag{18}$$

故散射圖形爲：對應於同一物點發出之不同斜率之電子束所成的像爲一系列中心對高斯像點有不同 l 的大小不同的圓，組成如彗星狀見下圖所示：

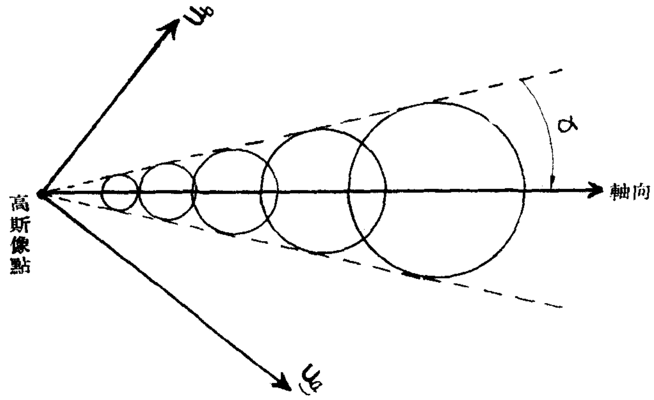

圖 4

第六類：五級球差.

如果其他像差不存在，則得：

$$\Delta \mathbf{u}_{5b} = S_5 \mathbf{u}_a'^5 + S_5' \mathbf{u}_a'^4 \underset{\sim}{\mathbf{u}}_a'; \tag{19}$$

其分量式爲：

$$\xi = [S_5 \cos\gamma - S_5' \sin\gamma]\, u_a'^5, \tag{20a}$$

$$\eta = [S_5 \sin\gamma - S_5' \cos\gamma]\, u_a'^5. \tag{20b}$$

此爲一圓方程式，半徑

$$r = u_a'^5 \sqrt{S_5^2+S_5'^2}. \tag{21}$$

五級球差中雖有各向異性部分存在，但仍爲一圓，故散射圖形爲：對應於同一物點發出之不同斜率的電子束爲一系列以高斯像點爲中心之大小不同之圓，其半徑與 $u_a'^5$ 成比例.

在以上幾類特殊像差的散射圖形討論中，五級球差畸變的散射圖形與三級中相同，而其他幾類像差則與三級中有相似之處，但也有新的圖形出現. 如第二類 $(u_a^4 u_a')$ 其圖形排列與三級中場曲像散圖形相似. 在第三類中 $(u_a^3 u_a'^2)$，其圖形排列猶似彗星，與

三級中有相似之處,但其爲許多橢圓組成,這是與三級中不同的. 其他如第四類 ($u_a^2u_a'^3$), 爲一以參數方程式代表之曲線,在三級像差中是不出現的. 又如第五類 ($u_au_a'^4$) 爲一系列圓組成之彗星形,與三級中很相似,但在三級像差中, $\alpha = 60°$, 而在五級中 α 之大小決定於像差係數,此乃不同之點.

五. 研究電子光學中五級畸變的影響

在研究五級特殊像差中, 對畸變頗感興趣, 因爲它與 u_a^5 成比例, 故在大視場系統中,影響會較大. 而在電子光學中,對五級畸變在各種具體情況下的影響尙不清楚. 而畸變會使像變形,因此當要作定量測量時,就需要精確知道畸變的影響,從這些來看,研究五級畸變對於某些儀器是重要的. 研究的方法是通過對一些不同類型的具體系統的計算,從中得出一些關於五級畸變的普遍概念. 本文計算了二個具體系統的五級畸變.

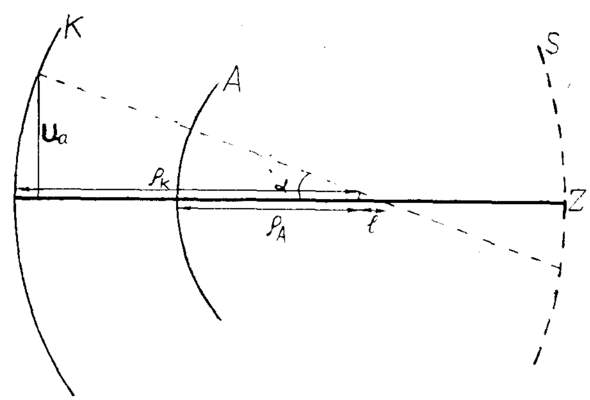

圖 5

(1) 作爲電子光學系統的球形電容器.

對此系統感興趣之原因在於場有解析形式表示, 且運用一級近似理論可詳盡研究其軌跡、光學性質以及像差; 另外也有它的實際意義,嚴格地或近似的球形電容器應用在實際儀器中,作爲實際儀器的一部分;如離子電子投射器及像變換器[4],[5],[6]. 在此系統中, u_a 爲物所在處電子束出來之高度, ρ_k, ρ_A 爲球形電容器陰極陽極之曲率半徑, l 爲球形電容器之中心與聚焦點間距離, $\dfrac{u_a}{\rho_k} = \sin\alpha$, 各量見圖所示. 在球形電容器中,在一級近似理論下,考慮到畸變存在,則成像位置爲:

$$\mathbf{u}_b = m\, u_a \frac{1}{\sqrt{1-\left(\dfrac{u_a}{\rho_k}\right)^2}},$$

把它展開可求得三級、五級、七級等畸變，今把計算得到的各級畸變列表如下：

$\frac{u_a}{\rho_k}$	α	$\frac{\Delta m}{m}(\%)$	$\frac{\Delta m_3}{m}(\%)$	$\frac{\Delta m_5}{m}(\%)$	$\frac{\Delta m_7}{m}(\%)$	$\frac{\Delta m_3}{\Delta m}(\%)$	$\frac{\Delta m_5}{\Delta m}(\%)$	$\frac{\Delta m_7}{\Delta m}(\%)$
0.9	65°10′	129.41	40.5	24.6	16.6	31.3	19.01	12.8
0.8	53°8′	66.67	32.0	15.36	8.192	48.0	23.04	12.3
0.7	44°24′	40.04	24.5	9.0	3.68	61.19	22.48	9.18
0.6	36°52′	25.0	18	4.86	1.45	72.0	19.44	5.81
0.5	30°	15.47	12.5	2.34	0.49	80.8	15.15	3.15
0.4	23°35′	9.11	8.0	0.96	0.13	87.8	10.54	1.4
0.3	17°27′	4.84	4.5	0.304	0.023	92.97	6.28	0.47
0.2	11°32′	2.06	2.0	0.06	0.002	96.9	2.9	0.097

其中 $\frac{\Delta m}{m}$ 爲全部畸變引起的放大倍數的相對改變，$\frac{\Delta m_3}{\Delta m}$，$\frac{\Delta m_5}{\Delta m}$，… 爲三級，五級等畸變引起的放大倍數的相對改變；$\frac{\Delta m_3}{\Delta m}$，$\frac{\Delta m_5}{\Delta m}$，… 爲三級，五級等畸變在全部畸變中所佔的百分比．

由上表可得如下結論：(i) 各級畸變在全部畸變中所佔百分比隨 $\frac{u_a}{\rho_k}$ 而變化，當 $\frac{u_a}{\rho_k}$ 漸趨小時，三級畸變的百分比漸趨大，當 $\frac{u_a}{\rho_k} = 0.3$ 以下時，三級畸變確佔全部畸變的百分之九十幾．(ii) 如認爲相對畸變爲 1% 時需要考慮，則在此系統中，當 $\frac{u_a}{\rho_k} = 0.4$ 以上時，需要考慮五級畸變的影響，在一般實際系統中，$\frac{u_a}{\rho_k} < 0.7$，故尙需考慮到五級畸變之影響．

(2) 靜電浸沒系統的計算[7]：

首先找到一個理想的解析場分佈，然後用數值計算法找出像平面位置、放大倍數、傍軸主軌跡及三級五級畸變，因計算繁長，此處只列出結果．

(i) 找出的解析場分佈：根據雙圓筒透鏡的邊界條件找到的解析場分佈爲：

$$V(z) = 0.0474 \int_0^x e^{2.7 \sin^2 \frac{\pi}{2\times 1.8} x}\, dx \, .$$

但此函數有週期性；當 $x = 3.6(R)$ 以後，又重覆上升，不合要求，故設在 $x = 3.6(R)$ 以後，$V(z)$ 爲常數．

(ii) 高斯像平面的位置：由 $r_a(z_b) = 0$ 的條件求得：

$$z_b = 4.7904\,(R)\,; \qquad R \text{ 爲圓筒之半徑}.$$

(iii) 在高斯像平面處傍軸主軌跡之高度：設電子從 $u_a = 0.8(R)$ 處發射出來，則：

$$r = \mathbf{u}_a r_\beta = 0.8 \times (-0.72974) = -0.5838(R)\,.$$

(iv) 在高斯像平面處之三級五級畸變及其影響.

求得:

$$\Delta\tau_3(z_b) = -0.1068(R),$$

故三級畸變引起的放大倍數改變為:

$$\frac{\Delta m_3}{m} = \frac{-0.1068}{-0.5838} = 18.29\%.$$

求得 $\Delta\tau_5(z_b) = -0.0222\,(R)$, 則 $\frac{\Delta m_5}{m} = 3.8\%$.

從以上二個具體系統的計算中,發現五級畸變之影響有時較顯著,在以前籠統地認為 u_a 小,對五級畸變不予考慮,此處使我們得出了新的概念,即需要考慮其影響.

結　束　語

以上用軌跡法與作用函數複合法得出了在旋轉對稱條件下的五級橫向像差方程式,且由此定出了五級特殊形式的像差係數,同時也討論了各類特殊像差的散射圖形,最後研究了五級畸變的影響. 在電子光學中五級像差方面的工作尚未很好開展,此處工作也只是初步,有待於進一步的研究.

上述工作是在導師 О. И. Семан 親切的指導下完成的,此處謹向專家表示衷心的感謝.

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ИССЛЕДОВАНИЕ АБЕРРАЦИЙ ПЯТОГО ПОРЯДКА В ЭЛЕКТРОННО-ОПТИЧЕСКИХ СИСТЕМАХ С СИММЕТРИЕЙ ВРАЩЕНИЯ

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Резюме

В этой работе исследована поперечная аберрация пятого порядка в электронно-оптических системах с симметрией вращения. С использованием метода траектории и эйконала, выведено обшее уравнение полной поперечной аберрации пятого порядка при любых начальных условиях аберрации и в произвольных плоскостях установки, и получена формула поперечной аберрации пятого порядка в плоскости гауссового изображения. В работе построена общая теория поперечной аберрации пятого порядка в системах с симметрией вращения, определены поперечные аберрации пятого порядка в комбинированных, чисто электростатическом и чисто магнитном полях, а также общие формулы коэффициентов различных частных видов аберраций. хотя формулы сложны они дают средство в необходимых случаях определять аберрации пятого порядка для конкретных систем с помощью метода численного вычисления.

В настоящей работе также обсуждена классификация поперечных аберраций пятого порядка и фигуры рассеяния различных частных видов аберраций. В сопоставлении с аберрацями третьего порядка существуют частные аберрации, фигуры рассеяния которых отличны, а некоторые похожи на фигуры аберраций третьего порядка, а также обнаружено, что на плоскости гауссового изображения появилась анизотропная сферическая аберрация третьего порядка. Это и являеться особенностью. Наконец, исследовано влияние дисторсии пятого порядка в двух разных конкретных системах. Эти две системы представляют собой сферический конденсатор как электронно-оптическую систему и электростатическую иммерсионную систему. Из полученных результатов можно получить представление о роли дисторсии пятого порядка.

В заключение автор выражает благодарность советскому специалисту О. И. Семану за руководство данной работой.