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一邊平夾另一邊受均佈力矩作用的環形薄板大撓度問題*

黄擇言
(北京地質勘探學院力學教研室)

一. 引言

作者曾在文獻 [1] 中指出，用攝動法來處理在均佈邊緣力矩作用之下的彈性圓薄板大撓度問題，不但遠較用數字積分法來得簡便，而且還提供了最大撓度以及應力的設計公式，使它們能直接應用到工程設計問題上去．本文則引申之，同樣成功地處理了另一些具有實用價值但一直還未被研究過的問題，卽外(內)邊平夾，而沿內(外)邊有均佈邊緣力矩作用的環形薄板大撓度問題．

二. 基本方程

我們考慮一外半徑爲 a，內半徑爲 b，厚度爲 h 的彈性環形薄板．板的一邊平夾，另一邊則受均佈力矩的作用．有兩種可能的情形：甲．情形 A：板的外邊平夾，沿內邊邊緣每單位長度上作用着均勻分佈的力矩 M_0 (圖 1-A)；乙．情形 B：板的內邊平夾，沿外邊邊緣每單位長度上作用着均勻分佈的力矩 M_0 (圖 1-B)．令板的橫向撓度爲 w (我們規定：在情形 A，板的撓度從內邊水平面量起；在情形 B，板的撓度

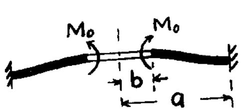

圖 1-A

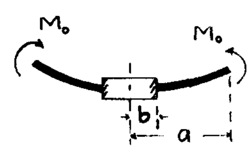

圖 1-B

從外邊水平面量起)，徑向薄膜應力爲 N_r，切向薄膜應力爲 N_t．它們都是離開中心的徑向距離 r 的函數，並且滿足下列方程：

* 1957 年 3 月 18 日收到．

$$\left.\begin{aligned} &D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}=N_r\frac{dw}{dr}, \\ &r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)+\frac{Eh}{2}\left(\frac{dw}{dr}\right)^2=0, \\ &N_t=\frac{d}{dr}(rN_r); \end{aligned}\right\} \tag{1}$$

式中 E 是板料的楊氏彈性模數，$D=\dfrac{Eh^3}{12(1-\nu^2)}$ 是板的抗撓剛度，ν 是板料的泊松比.

相應的邊界條件是:

甲. 在情形 A

當 $r=b$ 時，

$$w=0, \quad -D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)=M_0, \quad N_r=0;$$

當 $r=a$ 時，

$$\frac{dw}{dr}=0, \quad r\frac{dN_r}{dr}+(1-\nu)N_r=0. \tag{2-A}$$

乙. 在情形 B

當 $r=b$ 時，

$$\frac{dw}{dr}=0, \quad r\frac{dN_r}{dr}+(1-\nu)N_r=0;$$

當 $r=a$ 時，

$$w=0, \quad -D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)=M_0, \quad N_r=0. \tag{2-B}$$

爲了使方程 (1) 和邊界條件 (2) 簡化起見，在情形 A 和情形 B 分别引入下列的無量綱新量:

甲. 在情形 A

$$\left.\begin{aligned} &\rho=\frac{r^2}{b^2}, \quad \alpha=\frac{a^2}{b^2}, \quad W=\sqrt{3(1-\nu^2)}\,\frac{w}{h}, \\ &S=3(1-\nu^2)\frac{b^2N_r}{Eh^3}, \quad T=3(1-\nu^2)\frac{b^2N_t}{Eh^3}, \\ &M=6(1-\nu^2)\sqrt{3(1-\nu^2)}\,\frac{b^2M_0}{Eh^4}; \end{aligned}\right\} \tag{3-A}$$

乙. 在情形 B

$$\rho=\frac{r^2}{a^2}, \quad \alpha=\frac{b^2}{a^2}, \quad W=\sqrt{3(1-\nu^2)}\,\frac{w}{h},$$

$$
\left.\begin{aligned}
&S = 3(1-\nu^2)\,\frac{a^2N_r}{Eh^3}, \qquad T = 3(1-\nu^2)\,\frac{a^2N_t}{Eh^3},\\
&M = 6(1-\nu^2)\sqrt{3(1-\nu^2)}\;\frac{a^2M_0}{Eh^4}.
\end{aligned}\right\} \tag{3-B}
$$

於是，方程 (1) 變爲

$$
\left.\begin{aligned}
&\frac{d^2}{d\rho^2}\left(\rho\,\frac{dW}{d\rho}\right) = S\,\frac{dW}{d\rho},\\
&\frac{d^2}{d\rho^2}(\rho S) + \frac{1}{2}\left(\frac{dW}{d\rho}\right)^2 = 0,\\
&T = S + 2\rho\,\frac{dS}{d\rho};
\end{aligned}\right\} \tag{4}
$$

而邊界條件 (2)，不論在情形 A 或情形 B，都化爲

當 $\rho = 1$ 時，

$$
\left.\begin{aligned}
&W = 0, \quad (1+\nu)\,\frac{dW}{d\rho} + 2\rho\,\frac{d^2W}{d\rho^2} = -M, \quad S = 0;\\
&\text{當 } \rho = \alpha \text{ 時},\\
&\frac{dW}{d\rho} = 0, \quad 2\rho\,\frac{dS}{d\rho} + (1-\nu)\,S = 0.
\end{aligned}\right\} \tag{5}
$$

當然，還必須注意到：

$\rho = 1$ 在情形 A 是板的內邊，在情形 B 是板的外邊；

$\rho = \alpha$ 在情形 A 是板的外邊，在情形 B 是板的內邊.

這樣，我們的問題，無論是情形 A 或情形 B，都歸結爲在邊界條件 (5) 下求解方程 (4).

三. 方程的解法

現在，和文獻 [1] 完全一樣，我們要以板的最大撓度(在情形 A 是外邊的撓度，在情形 B 則是內邊的撓度)爲基礎，應用攝動法，在邊界條件 (5) 下來求解方程 (4). 令

$$
W_m = W_{\rho=\alpha} = \sqrt{3(1-\nu^2)}\left(\frac{w}{h}\right)_{r=k}. \tag{6}
$$

此處 $k = a$ 或 b，視在情形 A 或情形 B 而定. 顯然

$$
\left.\begin{aligned}
&M = M(W_m), \qquad W = W(W_m, \rho),\\
&S = S(W_m, \rho), \quad T = T(W_m, \rho).
\end{aligned}\right\} \tag{7}
$$

當 W_m 不大時，在可能的收斂範圍內，這些量都可以展開成爲 W_m 的升冪級數，即

$$
M = \gamma_1 W_m + \gamma_3 W_m^3 + \cdots,
$$

$$\left.\begin{aligned} W &= w_1(\rho)\, W_m + w_3(\rho)\, W_m^3 + \cdots, \\ S &= s_2(\rho)\, W_m^2 + s_4(\rho)\, W_m^4 + \cdots, \\ T &= t_2(\rho) W_m^2 + t_4(\rho)\, W_m^4 + \cdots; \end{aligned}\right\} \tag{8}$$

式中的 γ_1, γ_3, $\cdots$ 是待定常數; $w_1(\rho)$, $w_3(\rho)$, $\cdots$; $s_2(\rho)$, $s_4(\rho)$, $\cdots$ 以及 $t_2(\rho)$, $t_4(\rho)$, $\cdots$ 都是 ρ 的待定函數. 將 (8) 式代入 (4), (5), (6) 諸式,並逐一收集 W_m 的同次項,我們就可得到一系列對於 γ_1, w_1, s_2, t_2, γ_3, w_3, s_4, t_4, $\cdots$ 及其相應的邊界條件的線性微分方程.

對於 γ_1 和 w_1, 我們有下列的方程及邊界條件:

$$\frac{d^2}{d\rho^2}\left(\rho\,\frac{dw_1}{d\rho}\right) = 0; \tag{9}$$

$$\left.\begin{aligned} &w_1(\alpha) = 1, \quad w_1(1) = 0, \quad w_1'(\alpha) = 0, \\ &(1+\nu)\, w_1'(1) + 2w_1''(1) = -\gamma_1. \end{aligned}\right\} \tag{10}$$

將 (9) 式積分,並注意到 (10) 式的條件,我們可得

$$\gamma_1 = -C_1\,[1 + \nu + \alpha(1-\nu)], \tag{11}$$

$$\frac{dw_1}{d\rho} = C_1\left(1 - \frac{\alpha}{\rho}\right), \tag{12}$$

$$w_1 = C_1(\rho - \alpha \ln \rho - 1); \tag{13}$$

其中

$$C_1 = \frac{1}{\alpha - \alpha \ln \alpha - 1}. \tag{14}$$

這也就是小撓度理論的解答[2].

對於 s_2 和 t_2, 我們有下列的方程及邊界條件:

$$\left.\begin{aligned} &\frac{d^2}{d\rho^2}(\rho\, s_2) + \frac{1}{2}\left(\frac{dw_1}{d\rho}\right)^2 = 0, \\ &t_2 = s_2 + 2\rho\,\frac{ds_2}{d\rho}; \end{aligned}\right\} \tag{15}$$

$$s_2(1) = 0, \quad 2\alpha\, s_2'(\alpha) + (1-\nu)\, s_2(\alpha) = 0; \tag{16}$$

其中 $\frac{dw_1}{d\rho}$ 已由 (12) 式給出. 方程 (15) 在條件 (16) 下的解答是

$$s_2 = -\frac{C_1^2}{4}\left[\rho - 4\alpha \ln \rho + \frac{1}{\rho}\,(C_2 - 2\alpha^2 \ln \rho) - (C_2 + 1)\right], \tag{17}$$

$$t_2 = -\frac{C_1^2}{4}\left[3\rho - 4\alpha \ln \rho - \frac{1}{\rho}\,(C_2 + 4\alpha^2 - 2\alpha^2 \ln \rho) - (8\alpha + C_2 + 1)\right]; \tag{18}$$

其中

$$C_2 = -\frac{\alpha[(2-6\nu)\alpha\ln\alpha+(9+\nu)\alpha+1-\nu]}{1+\nu+\alpha(1-\nu)}. \tag{19}$$

對於 γ_3 和 w_3，我們有

$$\frac{d^2}{d\rho^2}\left(\rho\frac{dw_3}{d\rho}\right) = s_2\frac{dw_1}{d\rho}; \tag{20}$$

$$\left.\begin{aligned} &w_3(\alpha) = 0, \quad w_3(1) = 0, \quad w_3'(\alpha) = 0, \\ &(1+\nu)\, w_3'(1) + 2w_3''(1) = -\gamma_3. \end{aligned}\right\} \tag{21}$$

解答是

$$\begin{aligned} \gamma_3 = -\frac{C_1^3}{24}\,[&24\alpha^2(\alpha+1) - 18\alpha - 6C_2(4\alpha+1) + 12C_3 + 2 - \\ &-(1+\nu)\,(15\alpha - 3C_2 + 6C_3 + 6C_4 - 2)], \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{dw_3}{d\rho} = -\frac{C_1^3}{24}\Big\{&\rho^2 + 6\alpha^2(\ln\rho)^2 - 12\alpha\rho\ln\rho - 3(C_2+1-5\alpha)\,\rho + \\ &+ 6[\alpha(C_2+1) + C_2 - 2\alpha^2]\ln\rho + 6C_4 + \\ &+ \frac{6}{\rho}\,[C_3 + \alpha(C_2 - 2\alpha^2)\ln\rho - \alpha^3(\ln\rho)^2]\Big\}, \end{aligned} \tag{23}$$

$$\begin{aligned} w_3 = -\frac{C_1^3}{144}\Big\{&2\rho^3 - 36\alpha\rho^2\ln\rho + 36\alpha^2\rho(\ln\rho)^2 - 12\alpha^3(\ln\rho)^3 + \\ &+ 18\alpha(C_2 - 2\alpha^2)\,(\ln\rho)^2 - 9(C_2+1-7\alpha)\,\rho^2 + \\ &+ 36\,[\alpha(C_2+1) + C_2 - 4\alpha^2]\,\rho\ln\rho + 36C_3\ln\rho + \\ &+ 36\,[4\alpha^2 - \alpha(C_2+1) - C_2 + C_4]\,\rho + C_5\Big\}; \end{aligned} \tag{24}$$

其中

$$\begin{aligned} C_3 = -\frac{\alpha C_1}{36}\Big\{&12\alpha^3(\ln\alpha)^3 - 36\alpha^3\ln\alpha - 113\alpha^3 + 216\alpha^2\ln\alpha - \\ &- 18C_2\alpha(\ln\alpha)^2 + [27(C_2+1) + 48]\,\alpha^2 - 36\alpha\ln\alpha + \\ &+ (18C_2 + 45)\,\alpha - 72C_2\ln\alpha - 45C_2 - 7\Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} C_4 = -\frac{8}{3}\alpha^2 + \frac{1}{2}(C_2+1)\,\alpha + [&6\alpha^2 - (C_2+1)\,\alpha - \\ &- 2C_2]\ln\alpha - \frac{C_3}{\alpha}, \end{aligned} \tag{26}$$

$$C_5 = 9C_2(4\alpha+5) - 9\alpha(16\alpha+3) + 7 - 36C_4. \tag{27}$$

對於 s_4 和 t_4，我們有

$$\left.\begin{aligned} \frac{d^2}{d\rho^2}(\rho s_4) &= -\frac{dw_1}{d\rho}\frac{dw_3}{d\rho}, \\ t_4 &= s_4 + 2\rho\frac{ds_4}{d\rho}; \end{aligned}\right\} \tag{28}$$

$$s_4(1) = 0, \quad 2\alpha s_4'(\alpha) + (1-\nu)s_4(\alpha) = 0; \tag{29}$$

其中 $\frac{dw_1}{d\rho}$ 及 $\frac{dw_3}{d\rho}$ 已分别由（12）及（23）二式給出．方程（28）在條件（29）下的解答是

$$\begin{aligned} s_4 = \frac{C_1^4}{288}\Big\{&\rho^3 - 24\alpha\ \rho^2\ln\rho + 36\alpha^2\rho(\ln\rho)^2 - 48\alpha^3(\ln\rho)^3 + \\ &+ 36\alpha^2[4\alpha - (C_2+1)](\ln\rho)^2 - 36[3\alpha^2 - \alpha(C_2+1) - C_2]\rho\ln\rho + \\ &+ 2[24\alpha - 3(C_2+1)]\rho^2 + 18[2\alpha^2 - 2\alpha(C_2+1) - 3C_2 + 2C_4]\rho - \\ &- 72[4\alpha^3 - \alpha^2(C_2+1) + \alpha C_4 - C_3]\ln\rho - \\ &- 24\alpha^4\frac{(\ln\rho)^3}{\rho} + 36\alpha^2(C_2 - 4\alpha^2)\frac{(\ln\rho)^2}{\rho} - \\ &- 72\alpha(4\alpha^3 - \alpha C_2 - C_3)\frac{\ln\rho}{\rho} + \frac{C_6}{\rho} + C_7\Big\}, \end{aligned} \tag{30}$$

$$\begin{aligned} t_4 = \frac{C_1^4}{288}\Big\{&7\rho^3 - 120\alpha\ \rho^2\ln\rho + 108\alpha^2\rho(\ln\rho)^2 - 48\alpha^3(\ln\rho)^3 - \\ &- 36\alpha^2(4\alpha + C_2 + 1)(\ln\rho)^2 - 36[5\alpha^2 - 3\alpha(C_2+1) - 3C_2]\rho\ln\rho + \\ &+ 6[32\alpha - 5(C_2+1)]\rho^2 - 18[6\alpha^2 + 2\alpha(C_2+1) + 5C_2 - 6C_4]\rho + \\ &+ 72[4\alpha^3 - \alpha^2(C_2+1) - \alpha C_4 + C_3]\ln\rho + 24\alpha^4\frac{(\ln\rho)^3}{\rho} - \\ &- 36\alpha^2 C_2\frac{(\ln\rho)^2}{\rho} - 72\alpha(4\alpha^3 - \alpha C_2 + C_3)\frac{\ln\rho}{\rho} - \\ &- [144\alpha(4\alpha^3 - \alpha C_2 - C_3) + C_6]\frac{1}{\rho} - \\ &- 144[4\alpha^3 - \alpha^2(C_2+1) + \alpha C_4 - C_3] + C_7\Big\}; \end{aligned} \tag{31}$$

其中

$$C_6 = \frac{\alpha[(1-\nu)K_0 - \nu K_1 + K_2]}{1+\nu+\alpha(1-\nu)}, \tag{32}$$

$$C_7 = K_0 - C_6, \tag{33}$$

而

$$K_0 = 5 - 12\alpha(1+3\alpha-3C_2) + 60C_2 - 36C_4, \tag{34}$$

$$K_1 = \frac{288}{C_1^4}s_4(\alpha) - \frac{C_6}{\alpha} - C_7 =$$

$$= 36(1-2\ln\alpha)\,\alpha^3(\ln\alpha)^2+36(C_2+1)(3-\ln\alpha)\,\alpha^2\ln\alpha+(85-708\ln\alpha)\,\alpha^3-$$
$$-42(C_2+1)\,\alpha+36(4C_3+C_2\,\alpha\ln\alpha)\ln\alpha-18(3C_2-2C_4)(1-2\ln\alpha)\,\alpha, \tag{35}$$

$$K_2=\frac{288}{C_1^4}t_4(\alpha)+\frac{C_6}{\alpha}-C_7=$$
$$=-12[25+3\ln\alpha+2(\ln\alpha)^2]\,\alpha^3\ln\alpha+$$
$$+36(C_2+1)(1-\ln\alpha)\,\alpha^2\ln\alpha+36[C_2(5-\ln\alpha)-2C_4]\,\alpha\ln\alpha-$$
$$-1061\alpha^3+78(C_2+1)\,\alpha^2+18(3C_2-2C_4)\,\alpha+288C_3. \tag{36}$$

我們的計算到此爲止.

表 1.　不同的比值 a/b 所對應的各個常數 C 和 γ_1, γ_3 之值 ($\nu=0.3$)

常數 \ a/b		5	4	3	2	1.5	1.25
C_1	情形 A	−0.01771	−0.03406	−0.08493	−0.3929	−1.740	−7.417
	情形 B	−1.203	−1.309	−1.551	−2.479	−5.124	−13.45
C_2	情形 A	−331.5	−202.7	−104.6	−38.06	−17.21	−10.03
	情形 B	−0.03151	−0.05798	−0.1358	−0.5010	−1.313	−2.415
C_3	情形 A	−11672	−9288	−3094	−472.3	−115.7	−46.70
	情形 B	−0.009700	−0.01953	−0.06103	−0.4306	−1.893	−4.848
C_4	情形 A	35470	12615	3238	445.4	101.8	38.11
	情形 B	0.1485	0.1082	0.01558	−0.1179	0.4576	2.374
C_5	情形 A	−1680881	−617329	−167091	−25632	−6616	−2775
	情形 B	−1.121	−1.886	−4.995	−31.56	−130.0	−319.0
C_6	情形 A	−247494770	−54190169	−7556280	−460749	−62434	−17471
	情形 B	−0.1406	−0.3664	−1.786	−26.95	−206.5	−753.1
C_7	情形 A	245876790	53597686	7396501	436332	56138	14831
	情形 B	−2.680	−3.030	−4.246	−3.624	82.74	449.7
γ_1	情形 A	0.3329	0.4257	0.6454	1.611	5.004	17.75
	情形 B	1.598	1.758	2.137	3.656	8.256	23.50
γ_3	情形 A	0.06084	0.07360	0.09374	0.1270	0.1495	0.1522
	情形 B	0.1478	0.1530	0.1605	0.1708	0.1751	0.1837

在情形 B 中，如果令 $\alpha=0$，則我們現在所研究的問題便化爲文獻 [1] 中所研究的問題，卽在均佈邊緣力矩作用之下的圓薄板大撓度問題了．本文在此特殊情況下所得的結果與文獻 [1] 中所得者完全一致.

我們取泊松比 $\nu=0.3$，則在情形 A 以及情形 B 中對於各種不同的外半徑與內半徑之比 a/b 所對應的各個常數 C 和 γ_1, γ_3 之值如表 1 所示.

四. 最大撓度設計公式

綜合以上的計算，可見一邊平夾，另一邊受均佈力矩作用的環形薄板，其最大撓度（在情形 A 是外邊的撓度，在情形 B 是內邊的撓度）與彎曲力矩之間的關係式是

$$M=\gamma_1 W_m+\gamma_3 W_m^3. \tag{37}$$

因此,在情形 A 及情形 B 中,當 $\nu = 0.3$ 時,最大撓度對彎曲力矩的設計公式為:

甲. 在情形 A

i) 當 $a/b = 5$ 時,

$$M = 0.3329\, W_m + 0.06084\, W_m^3\,; \tag{38-A$_1$}$$

ii) 當 $a/b = 4$ 時,

$$M = 0.4257\, W_m + 0.07360\, W_m^3\,; \tag{38-A$_2$}$$

iii) 當 $a/b = 3$ 時,

$$M = 0.6454\, W_m + 0.09374\, W_m^3\,; \tag{38-A$_3$}$$

iv) 當 $a/b = 2$ 時,

$$M = 1.611\, W_m + 0.1270\, W_m^3\,; \tag{38-A$_4$}$$

v) 當 $a/b = 1.5$ 時,

$$M = 5.004\, W_m + 0.1495\, W_m^3\,; \tag{38-A$_5$}$$

vi) 當 $a/b = 1.25$ 時,

$$M = 17.75\, W_m + 0.1522\, W_m^3\,. \tag{38-A$_6$}$$

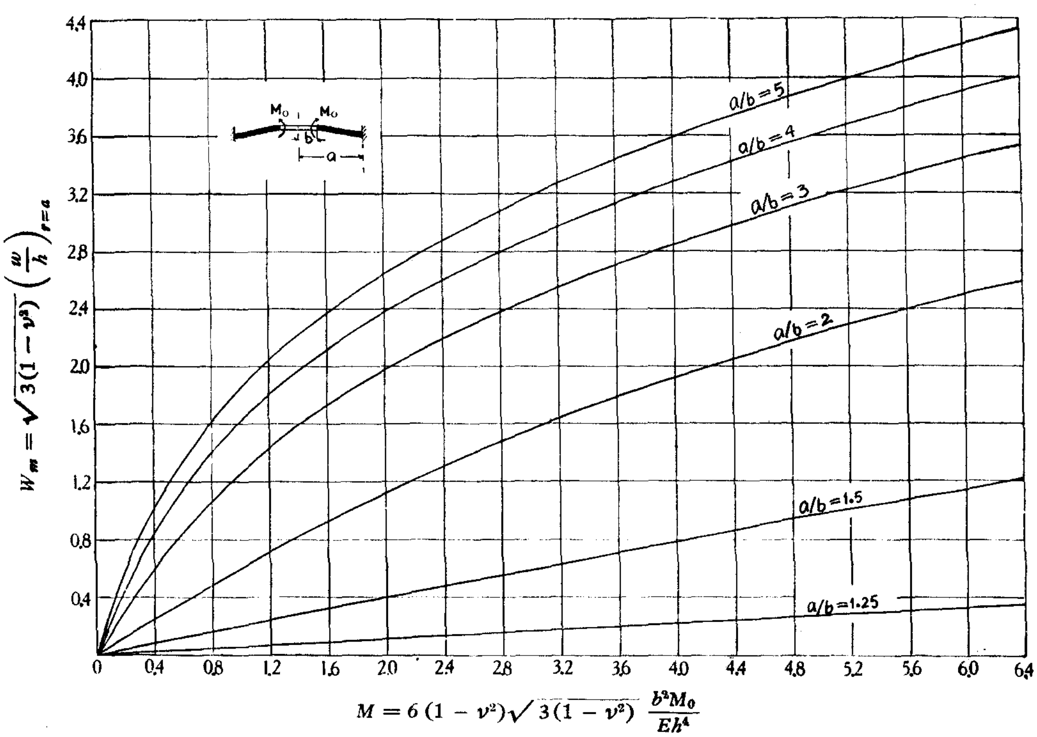

圖 2-A. 在情形 A 中,在不同的比值 a/b 下,最大撓度對彎曲力矩的曲線 ($\nu=0.3$)

乙．在情形 B

i) 當 $a/b=5$ 時，

$$M = 1.598\, W_m + 0.1478\, W_m^3\,; \tag{38-B$_1$}$$

ii) 當 $a/b=4$ 時，

$$M = 1.758\, W_m + 0.1530\, W_m^3\,; \tag{38-B$_2$}$$

iii) 當 $a/b=3$ 時，

$$M = 2.137\, W_m + 0.1605\, W_m^3\,; \tag{38-B$_3$}$$

iv) 當 $a/b=2$ 時，

$$M = 3.656\, W_m + 0.1708\, W_m^3\,; \tag{38-B$_4$}$$

v) 當 $a/b=1.5$ 時，

$$M = 8.256\, W_m + 0.1751\, W_m^3\,; \tag{38-B$_5$}$$

vi) 當 $a/b=1.25$ 時，

$$M = 23.50\, W_m + 0.1837\, W_m^3\,. \tag{38-B$_6$}$$

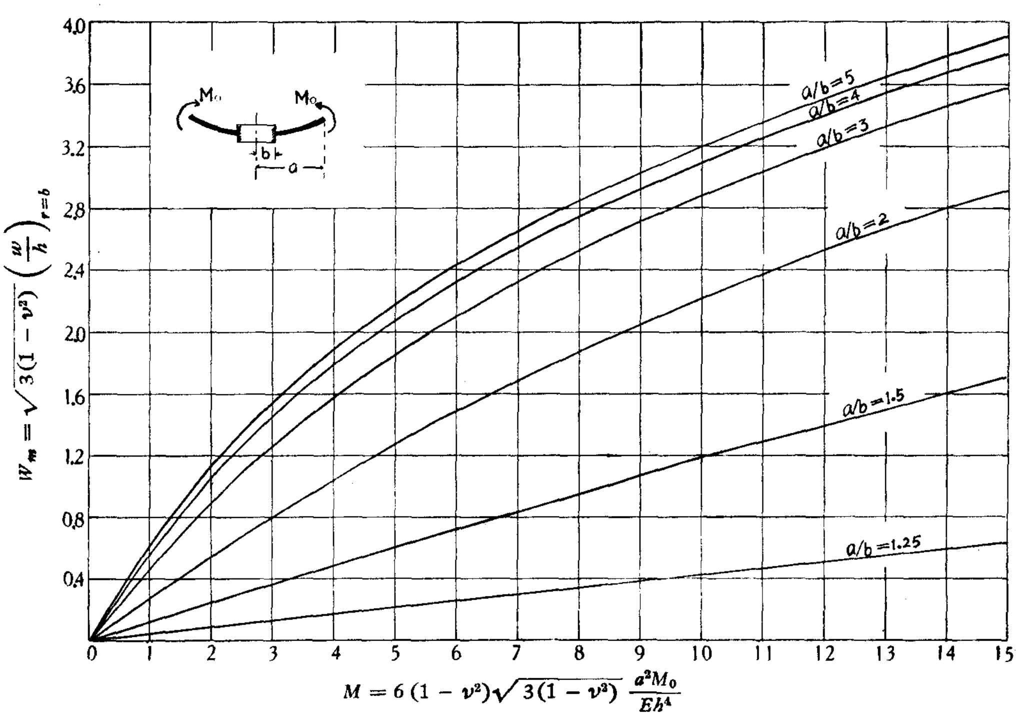

圖 2-B. 在情形 B 中，在不同的比值 a/b 下，最大撓度對彎曲力矩的曲線 ($\nu=0.3$)

圖 2-A 和 圖 2-B 分別表示在情形 A 和情形 B 中,當 $\nu=0.3$ 時,在不同的比值 a/b 下,最大撓度對彎曲力矩的曲線.

五. 應力設計公式

現在,我們再來求出板中應力的表達式. 令徑向拉伸應力 σ_r' 和切向拉伸應力 σ_t' 的無量綱形式是 $\Sigma_r'(\rho)$ 和 $\Sigma_t'(\rho)$, 徑向彎曲應力 σ_r'' 和切向彎曲應力 σ_t'' 的無量綱形式是 $\Sigma_r''(\rho)$ 和 $\Sigma_t''(\rho)$, 亦卽[1]

甲. 在情形 A

$$\left.\begin{aligned}
&\Sigma_r'(\rho)=3(1-\nu^2)\frac{b^2\sigma_r'}{Eh^2}=3(1-\nu^2)\frac{b^2N_r}{Eh^3}=S,\\
&\Sigma_t'(\rho)=3(1-\nu^2)\frac{b^2\sigma_t'}{Eh^2}=3(1-\nu^2)\frac{b^2N_t}{Eh^3}=T,\\
&\Sigma_r''(\rho)=3(1-\nu^2)\frac{b^2\sigma_r''}{Eh^2}=\mp\frac{3}{\sqrt{3(1-\nu^2)}}\left[(1+\nu)\frac{dW}{d\rho}+2\rho\frac{d^2W}{d\rho^2}\right]^{1)},\\
&\Sigma_t''(\rho)=3(1-\nu^2)\frac{b^2\sigma_t''}{Eh^2}=\mp\frac{3}{\sqrt{3(1-\nu^2)}}\left[(1+\nu)\frac{dW}{d\rho}+2\nu\rho\frac{d^2W}{d\rho^2}\right],\\
&\Sigma_r(\rho)=\text{折合徑向應力}=\Sigma_r'(\rho)+\Sigma_r''(\rho),\\
&\Sigma_t(\rho)=\text{折合切向應力}=\Sigma_t'(\rho)+\Sigma_t''(\rho);
\end{aligned}\right\}\quad(39\text{-A})$$

乙. 在情形 B

$$\left.\begin{aligned}
&\Sigma_r'(\rho)=3(1-\nu^2)\frac{a^2\sigma_r'}{Eh^2}=3(1-\nu^2)\frac{a^2N_r}{Eh^3}=S,\\
&\Sigma_t'(\rho)=3(1-\nu^2)\frac{a^2\sigma_t'}{Eh^2}=3(1-\nu^2)\frac{a^2N_t}{Eh^3}=T,\\
&\Sigma_r''(\rho)=3(1-\nu^2)\frac{a^2\sigma_r''}{Eh^2}=\mp\frac{3}{\sqrt{3(1-\nu^2)}}\left[(1+\nu)\frac{dW}{d\rho}+2\rho\frac{d^2W}{d\rho^2}\right],\\
&\Sigma_t''(\rho)=3(1-\nu^2)\frac{a^2\sigma_t''}{Eh^2}=\mp\frac{3}{\sqrt{3(1-\nu^2)}}\left[(1+\nu)\frac{dW}{d\rho}+2\nu\rho\frac{d^2W}{d\rho^2}\right],\\
&\Sigma_r(\rho)=\text{折合徑向應力}=\Sigma_r'(\rho)+\Sigma_r''(\rho),\\
&\Sigma_t(\rho)=\text{折合切向應力}=\Sigma_t'(\rho)+\Sigma_t''(\rho).
\end{aligned}\right\}\quad(39\text{-B})$$

1) 每當式中有 + 號及 − 號同時並用時,則對於板的凸面 $\left(z=+\frac{h}{2}\right)$ 取上面一符號,對於板的凹面 $\left(z=-\frac{h}{2}\right)$ 取下面一符號.

於是，我們得到下面的一些有用的結果：

當 $\rho = 1$ 時(在情形 A 是板的內邊，在情形 B 是板的外邊)，

$$
\left.
\begin{aligned}
&\sum\nolimits_r'(1) = S(1) = 0,\\
&\sum\nolimits_t'(1) = T(1) = t_2(1)\,W_m^2 + t_4(1)\,W_m^4,\\
&\sum\nolimits_r''(1) = \mp\frac{3}{\sqrt{3(1-\nu^2)}}\Big[(1+\nu)\,W'(1) + 2W''(1)\Big] =\\
&\qquad = \mp\frac{3}{\sqrt{3(1-\nu^2)}}\Big\{[(1+\nu)\,w_1'(1) + 2w_1''(1)]\,W_m +\\
&\qquad + [(1+\nu)\,w_3'(1) + 2w_3''(1)]\,W_m^3\Big\},\\
&\sum\nolimits_t''(1) = \mp\frac{3}{\sqrt{3(1-\nu^2)}}[(1+\nu)\,W'(1) + 2\nu\,W''(1)] =\\
&\qquad = \mp\frac{3}{\sqrt{3(1-\nu^2)}}\Big\{[(1+\nu)\,w_1'(1) + 2\nu w_1''(1)]\,W_m +\\
&\qquad + [(1+\nu)\,w_3'(1) + 2\nu w_3''(1)]\,W_m^3\Big\},\\
&\sum\nolimits_r(1) = \sum\nolimits_r'(1) + \sum\nolimits_r''(1) = \sum\nolimits_r''(1),\\
&\sum\nolimits_t(1) = \sum\nolimits_t'(1) + \sum\nolimits_t''(1);
\end{aligned}
\right\}\qquad (40)
$$

當 $\rho = \alpha$ 時(在情形 A 是板的外邊，在情形 B 是板的內邊)，

$$
\left.
\begin{aligned}
&\sum\nolimits_r'(\alpha) = S(\alpha) = s_2(\alpha)\,W_m^2 + s_4(\alpha)\,W_m^4,\\
&\sum\nolimits_t'(\alpha) = T(\alpha) = t_2(\alpha)\,W_m^2 + t_4(\alpha)\,W_m^4 = \nu s_2(\alpha)\,W_m^2 +\\
&\qquad + \nu s_4(\alpha)\,W_m^4 = \nu\sum\nolimits_r'(\alpha),\\
&\sum\nolimits_r''(\alpha) = \mp\frac{3}{\sqrt{3(1-\nu^2)}}[(1+\nu)\,W'(\alpha) + 2\alpha\,W''(\alpha)] =\\
&\qquad = \mp\frac{6\alpha}{\sqrt{3(1-\nu^2)}}[w_1''(\alpha)\,W_m + w_3''(\alpha)\,W_m^3],\\
&\sum\nolimits_t''(\alpha) = \mp\frac{3}{\sqrt{3(1-\nu^2)}}[(1+\nu)\,W'(\alpha) + 2\nu\alpha\,W''(\alpha)] =\\
&\qquad = \mp\frac{6\alpha\nu}{\sqrt{3(1-\nu^2)}}[w_1''(\alpha)\,W_m + w_3''(\alpha)\,W_m^3] = \nu\sum\nolimits_r''(\alpha),\\
&\sum\nolimits_r(\alpha) = \sum\nolimits_r'(\alpha) + \sum\nolimits_r''(\alpha),\\
&\sum\nolimits_t(\alpha) = \sum\nolimits_t'(\alpha) + \sum\nolimits_t''(\alpha) = \nu\sum\nolimits_r(\alpha).
\end{aligned}
\right\}\qquad (41)
$$

表 2. 不同的比值 a/b 所對應的各個函數 $w_1'(\rho)$, $w_1''(\rho)$, $s_2(\rho)$, $t_2(\rho)$, $w_3'(\rho)$, $w_3''(\rho)$, $s_4(\rho)$, $t_4(\rho)$ 在 $\rho=\alpha$ 及 $\rho=1$ 處之值 ($\nu=0.3$)

函數 \ a/b		5	4	3	2	1.5	1.25
α	情形 A	25	16	9	4	2.25	1.5625
	情形 B	1/25	1/16	1/9	1/4	1/2.25	1/1.5625
$w_1''(\alpha)$	情形 A	-7.083×10^{-4}	-2.129×10^{-3}	-9.436×10^{-3}	-9.822×10^{-2}	-0.7735	-4.747
	情形 B	-30.08	-20.94	-13.96	-9.915	-11.53	-21.01
$w_3''(\alpha)$	情形 A	6.864×10^{-5}	1.733×10^{-4}	5.550×10^{-4}	2.620×10^{-3}	7.213×10^{-3}	1.751×10^{-2}
	情形 B	1.185	0.7425	0.3995	0.1587	0.07810	0.04030
$s_2(\alpha)$	情形 A	0.01102	0.01772	0.03182	0.06668	0.1035	0.1310
	情形 B	0.3415	0.3285	0.3072	0.2664	0.2290	0.2020
$s_4(\alpha)$	情形 A	-2.075×10^{-4}	-3.092×10^{-4}	-4.531×10^{-4}	-5.057×10^{-4}	-2.411×10^{-4}	$+0.01046$
	情形 B	-6.736×10^{-3}	-5.658×10^{-3}	-4.079×10^{-3}	-1.828×10^{-3}	-6.648×10^{-4}	-0.008643
$w_1'(1)$	情形 A	0.4250	0.5109	0.6794	1.179	2.175	4.172
	情形 B	-1.155	-1.227	-1.379	-1.859	-2.847	-4.840
$w_1''(1)$	情形 A	-0.4427	-0.5449	-0.7643	-1.572	-3.916	-11.59
	情形 B	-0.04812	-0.08178	-0.1723	-0.6197	-2.278	-8.605
$w_3'(1)$	情形 A	0.03335	0.03425	0.03352	0.02679	0.01728	0.007331
	情形 B	-0.03427	-0.03327	-0.03086	-0.02407	-0.01597	-0.01091
$w_3''(1)$	情形 A	-0.05210	-0.05906	-0.06866	-0.08091	-0.08598	-0.08088
	情形 B	-0.05161	-0.05485	-0.06018	-0.06974	-0.07718	-0.08478
$t_2(1)$	情形 A	0.1595	0.2159	0.3331	0.6902	1.386	2.743
	情形 B	-0.6283	-0.6850	-0.8019	-1.155	-1.846	-3.200
$t_4(1)$	情形 A	0.01110	0.01143	0.01105	0.008463	0.005220	-0.002942
	情形 B	-0.01262	-0.01193	-0.01063	-0.007764	-0.004940	-0.003827

附註: $w_1'(\alpha)=0$, $w_3'(\alpha)=0$, $s_2(1)=0$, $s_4(1)=0$ 對於任何比值 a/b 均成立,故不列入表內; $t_2(\alpha)$ 和 $t_4(\alpha)$ 可自關係式 $t_2(\alpha)=\nu s_2(\alpha)$, $t_4(\alpha)=\nu s_4(\alpha)$ 算出,故亦不列入表內.

在這些計算應力的公式中, $w_1'(\rho)$, $s_2(\rho)$, $t_2(\rho)$, $w_3'(\rho)$, $s_4(\rho)$ 和 $t_4(\rho)$ 已分別在 (12), (17), (18), (23), (30) 和 (31) 各式中給出; 而 $w_1''(\rho)$ 及 $w_3''(\rho)$ 則可自 (12) 及 (23) 二式分別求得如次:

$$w_1''(\rho) = \frac{C_1\alpha}{\rho^2}\,. \tag{42}$$

$$\begin{aligned} w_3''(\rho) = -\frac{C_1^3}{24}\Big\{ & 2\rho - 12\alpha\ln\rho - 3(C_2+1-\alpha) + \\ & + \frac{6}{\rho}\left[C_2 + \alpha(C_2+1) - 2\alpha^2(1-\ln\rho)\right] + \\ & + \frac{6}{\rho^2}\left[\alpha(C_2-2\alpha^2) - C_3 - \alpha C_2\ln\rho + \alpha^3(\ln\rho)^2\right]\Big\}\,. \end{aligned} \tag{43}$$

我們取泊松比 $\nu = 0.3$, 則在情形 A 及情形 B 中, 對於各種不同的外半徑與內半徑之比 a/b 所對應的各個函數 $w_1'(\rho)$, $w_1''(\rho)$, $s_2(\rho)$, $t_2(\rho)$, $w_3'(\rho)$, $w_3''(\rho)$, $s_4(\rho)$, $t_4(\rho)$ 在 $\rho=1$ 及 $\rho=\alpha$ 處之值如表 2 所示. 將這些數值代入 (40), (41) 諸式, 我們就得到當 $\nu = 0.3$ 時板的內邊及外邊的應力分量了.

彎矩作用邊（$\rho = 1$）**的應力分量**

甲．在情形 A——板的內邊：

i) 當 $a/b = 5$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r'(1) = 0,\\
&\sum\nolimits_t'(1) = 0.1595\,W_m^2 + 0.01110\,W_m^4,\\
&\sum\nolimits_r''(1) = \pm(0.6045\,W_m + 0.1105\,W_m^3),\\
&\sum\nolimits_t''(1) = \mp(0.5209\,W_m + 0.02197\,W_m^3);
\end{aligned}\right\}\quad (44\text{-}A_1)$$

ii) 當 $a/b = 4$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r'(1) = 0,\\
&\sum\nolimits_t'(1) = 0.2159\,W_m^2 + 0.01143\,W_m^4,\\
&\sum\nolimits_r''(1) = \pm(0.7730\,W_m + 0.1336\,W_m^3),\\
&\sum\nolimits_t''(1) = \mp(0.6122\,W_m + 0.01650\,W_m^3);
\end{aligned}\right\}\quad (44\text{-}A_2)$$

iii) 當 $a/b = 3$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r'(1) = 0,\\
&\sum\nolimits_t'(1) = 0.3331\,W_m^2 + 0.01105\,W_m^4,\\
&\sum\nolimits_r''(1) = \pm(1.172\,W_m + 0.1702\,W_m^3),\\
&\sum\nolimits_t''(1) = \mp(0.7710\,W_m + 0.004325\,W_m^3);
\end{aligned}\right\}\quad (44\text{-}A_3)$$

iv) 當 $a/b = 2$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r'(1) = 0,\\
&\sum\nolimits_t'(1) = 0.6902\,W_m^2 + 0.008463\,W_m^4,\\
&\sum\nolimits_r''(1) = \pm(2.925\,W_m + 0.2306\,W_m^3),\\
&\sum\nolimits_t''(1) = \mp(1.070\,W_m - 0.02491\,W_m^3);
\end{aligned}\right\}\quad (44\text{-}A_4)$$

v) 當 $a/b = 1.5$ 時,

$$\left.\begin{aligned} &\sum\nolimits_r'(1) = 0\,, \\ &\sum\nolimits_t'(1) = 1.386\, W_m^2 + 0.005220\, W_m^4\,, \\ &\sum\nolimits_r''(1) = \pm(9.085\, W_m + 0.2714\, W_m^3)\,, \\ &\sum\nolimits_t''(1) = \mp(0.8690\, W_m - 0.05287\, W_m^3)\,; \end{aligned}\right\} \qquad (44\text{-}A_5)$$

vi) 當 $a/b = 1.25$ 時,

$$\left.\begin{aligned} &\sum\nolimits_r'(1) = 0\,, \\ &\sum\nolimits_t'(1) = 2.743\, W_m^2 - 0.002942\, W_m^4\,, \\ &\sum\nolimits_r''(1) = \pm(32.24\, W_m + 0.2764\, W_m^3)\,, \\ &\sum\nolimits_t''(1) = \pm(2.778\, W_m + 0.07081\, W_m^3)\,. \end{aligned}\right\} \qquad (44\text{-}A_6)$$

乙. 在情形 B——板的外邊:

i) 當 $a/b = 5$ 時,

$$\left.\begin{aligned} &\sum\nolimits_r'(1) = 0\,, \\ &\sum\nolimits_t'(1) = -\,0.6283\, W_m^2 - 0.01262\, W_m^4\,, \\ &\sum\nolimits_r''(1) = \pm(2.901\, W_m + 0.2683\, W_m^3)\,, \\ &\sum\nolimits_t''(1) = \pm(2.778\, W_m + 0.1371\, W_m^3)\,; \end{aligned}\right\} \qquad (44\text{-}B_1)$$

ii) 當 $a/b = 4$ 時,

$$\left.\begin{aligned} &\sum\nolimits_r'(1) = 0\,, \\ &\sum\nolimits_t'(1) = -\,0.6850\, W_m^2 - 0.01193\, W_m^4\,, \\ &\sum\nolimits_r''(1) = \pm(3.193\, W_m + 0.2777\, W_m^3)\,, \\ &\sum\nolimits_t''(1) = \pm(2.985\, W_m + 0.1383\, W_m^3)\,; \end{aligned}\right\} \qquad (44\text{-}B_2)$$

iii）當 $a/b=3$ 時，

$$\left.\begin{aligned}&\sum\nolimits_r'(1)=0,\\&\sum\nolimits_t'(1)=-0.8019\,W_m^2-0.01063\,W_m^4,\\&\sum\nolimits_r''(1)=\pm(3.880\,W_m+0.2914\,W_m^3),\\&\sum\nolimits_t''(1)=\pm(3.442\,W_m+0.1384\,W_m^3);\end{aligned}\right\}\qquad(44\text{-}B_3)$$

iv）當 $a/b=2$ 時，

$$\left.\begin{aligned}&\sum\nolimits_r'(1)=0,\\&\sum\nolimits_t'(1)=-1.155\,W_m^2-0.007764\,W_m^4,\\&\sum\nolimits_r''(1)=\pm(6.638\,W_m+0.3101\,W_m^3),\\&\sum\nolimits_t''(1)=\pm(5.063\,W_m+0.1328\,W_m^3);\end{aligned}\right\}\qquad(44\text{-}B_4)$$

v）當 $a/b=1.5$ 時，

$$\left.\begin{aligned}&\sum\nolimits_r'(1)=0,\\&\sum\nolimits_t'(1)=-1.846\,W_m^2-0.004940\,W_m^4,\\&\sum\nolimits_r''(1)=\pm(14.99\,W_m+0.3180\,W_m^3),\\&\sum\nolimits_t''(1)=\pm(9.201\,W_m+0.1218\,W_m^3);\end{aligned}\right\}\qquad(44\text{-}B_5)$$

vi）當 $a/b=1.25$ 時，

$$\left.\begin{aligned}&\sum\nolimits_r'(1)=0,\\&\sum\nolimits_t'(1)=-3.200\,W_m^2-0.003827\,W_m^4,\\&\sum\nolimits_r''(1)=\pm(42.67\,W_m+0.3336\,W_m^3),\\&\sum\nolimits_t''(1)=\pm(20.80\,W_m+0.1181\,W_m^3).\end{aligned}\right\}\qquad(44\text{-}B_6)$$

平夾邊（$\rho = \alpha$）的應力分量

甲. 在情形 A——板的外邊:

i) 當 $a/b = 5$（$\alpha = 25$）時,

$$\left.\begin{aligned}
\sum_r{}'(\alpha) &= 0.01102\, W_m^2 - 0.0002075\, W_m^4\,, \\
\sum_t{}'(\alpha) &= 0.003306\, W_m^2 - 0.00006224\, W_m^4\,, \\
\sum_r{}''(\alpha) &= \pm(0.06430\, W_m - 0.006231\, W_m^3)\,, \\
\sum_t{}''(\alpha) &= \pm(0.01929\, W_m - 0.001869\, W_m^3)\,;
\end{aligned}\right\} \qquad (45\text{-}A_1)$$

ii) 當 $a/b = 4$（$\alpha = 16$）時,

$$\left.\begin{aligned}
\sum_r{}'(\alpha) &= 0.01772\, W_m^2 - 0.0003092\, W_m^4\,, \\
\sum_t{}'(\alpha) &= 0.005317\, W_m^2 - 0.00009277\, W_m^4\,, \\
\sum_r{}''(\alpha) &= \pm(0.1237\, W_m - 0.01007\, W_m^3)\,, \\
\sum_t{}''(\alpha) &= \pm(0.03710\, W_m - 0.003020\, W_m^3)\,;
\end{aligned}\right\} \qquad (45\text{-}A_2)$$

iii) 當 $a/b = 3$（$\alpha = 9$）時,

$$\left.\begin{aligned}
\sum_r{}'(\alpha) &= 0.03182\, W_m^2 - 0.0004531\, W_m^4\,, \\
\sum_t{}'(\alpha) &= 0.009545\, W_m^2 - 0.0001359\, W_m^4\,, \\
\sum_r{}''(\alpha) &= \pm(0.3084\, W_m - 0.01814\, W_m^3)\,, \\
\sum_t{}''(\alpha) &= \pm(0.09252\, W_m - 0.005442\, W_m^3)\,;
\end{aligned}\right\} \qquad (45\text{-}A_3)$$

iv) 當 $a/b = 2$（$\alpha = 4$）時,

$$\left.\begin{aligned}
\sum_r{}'(\alpha) &= 0.06668\, W_m^2 - 0.0005057\, W_m^4\,, \\
\sum_t{}'(\alpha) &= 0.02000\, W_m^2 - 0.0001517\, W_m^4\,, \\
\sum_r{}''(\alpha) &= \pm(1.427\, W_m - 0.03805\, W_m^3)\,, \\
\sum_t{}''(\alpha) &= \pm(0.4280\, W_m - 0.01142\, W_m^3)\,;
\end{aligned}\right\} \qquad (45\text{-}A_4)$$

v) 當 $a/b = 1.5$ ($\alpha = 2.25$) 時,

$$\left.\begin{aligned}
&\sum\nolimits_r'(\alpha) = 0.1035\ W_m^2 - 0.0002411\ W_m^4\,,\\
&\sum\nolimits_t'(\alpha) = 0.03106\ W_m^2 - 0.00007232\ W_m^4\,,\\
&\sum\nolimits_r''(\alpha) = \pm(6.320\ W_m - 0.05894\ W_m^3)\,,\\
&\sum\nolimits_t''(\alpha) = \pm(1.896\ W_m - 0.01768\ W_m^3)\,;
\end{aligned}\right\} \tag{45-A$_5$}$$

vi) 當 $a/b = 1.25$ ($\alpha = 1.5625$) 時,

$$\left.\begin{aligned}
&\sum\nolimits_r'(\alpha) = 0.1310\ W_m^2 + 0.01046\ W_m^4\,,\\
&\sum\nolimits_t'(\alpha) = 0.03929\ W_m^2 + 0.003137\ W_m^4\,,\\
&\sum\nolimits_r''(\alpha) = \pm(26.93\ W_m - 0.09935\ W_m^3)\,,\\
&\sum\nolimits_t''(\alpha) = \pm(8.080\ W_m - 0.02980\ W_m^3)\,.
\end{aligned}\right\} \tag{45-A$_6$}$$

乙. 在情形 B——板的內邊:

i) 當 $a/b = 5$ $\left(\alpha = \dfrac{1}{25}\right)$ 時,

$$\left.\begin{aligned}
&\sum\nolimits_r'(\alpha) = 0.3415\ W_m^2 - 0.006736\ W_m^4\,,\\
&\sum\nolimits_t'(\alpha) = 0.1024\ W_m^2 - 0.002021\ W_m^4\,,\\
&\sum\nolimits_r''(\alpha) = \pm(4.369\ W_m - 0.1722\ W_m^3)\,,\\
&\sum\nolimits_t''(\alpha) = \pm(1.311\ W_m - 0.05165\ W_m^3)\,;
\end{aligned}\right\} \tag{45-B$_1$}$$

ii) 當 $a/b = 4$ $\left(\alpha = \dfrac{1}{16}\right)$ 時,

$$\left.\begin{aligned}
&\sum\nolimits_r'(\alpha) = 0.3285\ W_m^2 - 0.005658\ W_m^4\,,\\
&\sum\nolimits_t'(\alpha) = 0.09856\ W_m^2 - 0.001697\ W_m^4\,,\\
&\sum\nolimits_r''(\alpha) = \pm(4.752\ W_m - 0.1685\ W_m^3)\,,\\
&\sum\nolimits_t''(\alpha) = \pm(1.426\ W_m - 0.05055\ W_m^3)\,;
\end{aligned}\right\} \tag{45-B$_2$}$$

iii) 當 $a/b = 3\left(\alpha = \frac{1}{9}\right)$ 時,

$$\left.\begin{aligned}
\sum_r'(\alpha) &= 0.3072\ W_m^2 - 0.004079\ W_m^4\ ,\\
\sum_t'(\alpha) &= 0.09217\ W_m^2 - 0.001224\ W_m^4\ ,\\
\sum_r''(\alpha) &= \pm(5.632\ W_m - 0.1612\ W_m^3)\ ,\\
\sum_t''(\alpha) &= \pm(1.690\ W_m - 0.04836\ W_m^3)\ ;
\end{aligned}\right\} \tag{45-B$_3$}$$

iv) 當 $a/b = 2\left(\alpha = \frac{1}{4}\right)$ 時,

$$\left.\begin{aligned}
\sum_r'(\alpha) &= 0.2664\ W_m^2 - 0.001828\ W_m^4\ ,\\
\sum_t'(\alpha) &= 0.07993\ W_m^2 - 0.0005485\ W_m^4\ ,\\
\sum_r''(\alpha) &= \pm(9.001\ W_m - 0.1441\ W_m^3)\ ,\\
\sum_t''(\alpha) &= \pm(2.700\ W_m - 0.04323\ W_m^3)\ ;
\end{aligned}\right\} \tag{45-B$_4$}$$

v) 當 $a/b = 1.5\left(\alpha = \frac{1}{2.25}\right)$ 時,

$$\left.\begin{aligned}
\sum_r'(\alpha) &= 0.2290\ W_m^2 - 0.0006648\ W_m^4\ ,\\
\sum_t'(\alpha) &= 0.06871\ W_m^2 - 0.0001994\ W_m^4\ ,\\
\sum_r''(\alpha) &= \pm(18.61\ W_m - 0.1261\ W_m^3)\ ,\\
\sum_t''(\alpha) &= \pm(5.583\ W_m - 0.03782\ W_m^3)\ ;
\end{aligned}\right\} \tag{45-B$_5$}$$

vi) 當 $a/b = 1.25\left(\alpha = \frac{1}{1.5625}\right)$ 時,

$$\left.\begin{aligned}
\sum_r'(\alpha) &= 0.2020\ W_m^2 - 0.008643\ W_m^4\ ,\\
\sum_t'(\alpha) &= 0.06060\ W_m^2 - 0.002593\ W_m^4\ ,\\
\sum_r''(\alpha) &= \pm(48.82\ W_m - 0.09367\ W_m^3)\ ,\\
\sum_t''(\alpha) &= \pm(14.65\ W_m - 0.02810\ W_m^3)\ .
\end{aligned}\right\} \tag{45-B$_6$}$$

於是，我們有情形 A 及情形 B 的當 $\nu=0.3$ 時的應力設計公式如下：

甲．在情形 A

i）當 $a/b=5$ $(\alpha=25)$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1)=0.6045\,W_m+0.1105\,W_m^3,\\
&\sum\nolimits_t(1)=0.5209\,W_m+0.1595\,W_m^2+0.02197\,W_m^3+0.01110\,W_m^4,\\
&\sum\nolimits_r(\alpha)=0.06430\,W_m+0.01102\,W_m^2-0.006231\,W_m^3-0.0002075\,W_m^4,\\
&\sum\nolimits_t(\alpha)=0.01929\,W_m+0.003306\,W_m^2-0.001869\,W_m^3-0.00006224\,W_m^4;
\end{aligned}\right\}\quad(46\text{-}A_1)$$

ii）當 $a/b=4$ $(\alpha=16)$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1)=0.7730\,W_m+0.1336\,W_m^3,\\
&\sum\nolimits_t(1)=0.6122\,W_m+0.2159\,W_m^2+0.01650\,W_m^3+0.01143\,W_m^4,\\
&\sum\nolimits_r(\alpha)=0.1237\,W_m+0.01772\,W_m^2-0.01007\,W_m^3-0.0003092\,W_m^4,\\
&\sum\nolimits_t(\alpha)=0.03710\,W_m+0.005317\,W_m^2-0.003020\,W_m^3-0.00009277\,W_m^4;
\end{aligned}\right\}\quad(46\text{-}A_2)$$

iii）當 $a/b=3$ $(\alpha=9)$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1)=1.172\,W_m+0.1702\,W_m^3,\\
&\sum\nolimits_t(1)=0.7710\,W_m+0.3331\,W_m^2+0.004325\,W_m^3+0.01105\,W_m^4,\\
&\sum\nolimits_r(\alpha)=0.3084\,W_m+0.03182\,W_m^2-0.01814\,W_m^3-0.0004531\,W_m^4,\\
&\sum\nolimits_t(\alpha)=0.09252\,W_m+0.009545\,W_m^2-0.005442\,W_m^3-0.0001359\,W_m^4;
\end{aligned}\right\}\quad(46\text{-}A_3)$$

iv）當 $a/b=2$ $(\alpha=4)$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1)=2.925\,W_m+0.2306\,W_m^3,\\
&\sum\nolimits_t(1)=1.070\,W_m+0.6902\,W_m^2-0.02491\,W_m^3+0.008463\,W_m^4,\\
&\sum\nolimits_r(\alpha)=1.427\,W_m+0.06668\,W_m^2-0.03805\,W_m^3-0.0005057\,W_m^4,\\
&\sum\nolimits_t(\alpha)=0.4280\,W_m+0.02000\,W_m^2-0.01142\,W_m^3-0.0001517\,W_m^4;
\end{aligned}\right\}\quad(46\text{-}A_4)$$

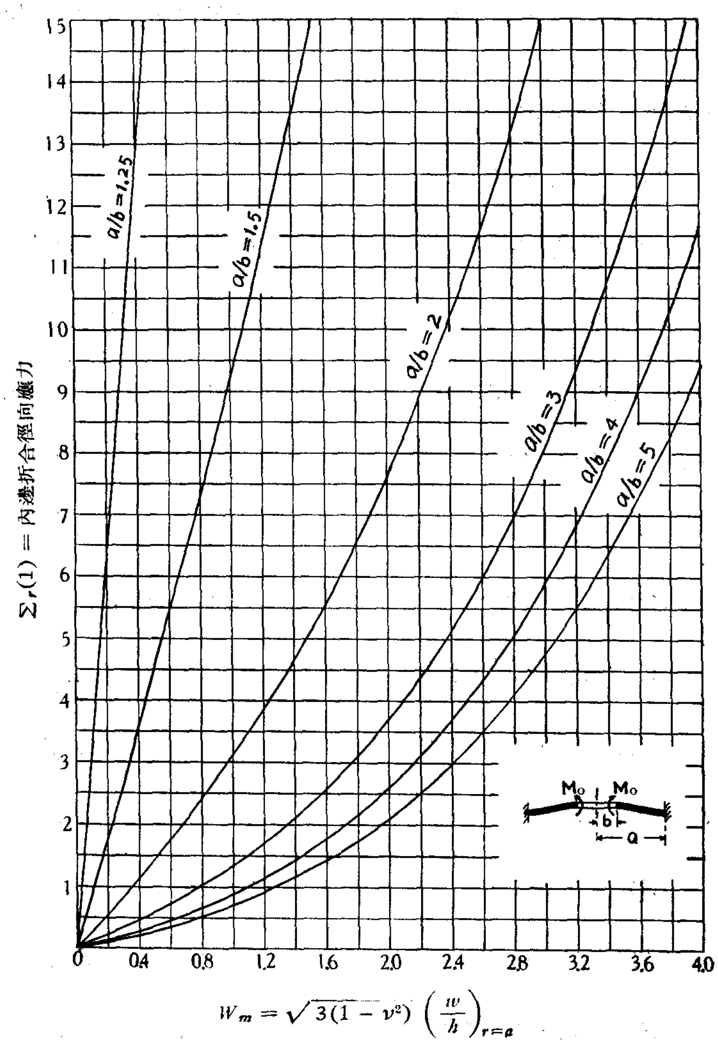

圖 3-*A*. 在情形 *A* 中，在不同的比值 a/b 下，最大撓度對彎矩作用邊（內邊）折合徑向應力的曲線（$\nu=0.3$）

v）當 $a/b = 1.5$ （$\alpha = 2.25$）時，

$$\left.\begin{aligned}
\textstyle\sum_r(1) &= 9.085\, W_m + 0.2714\, W_m^3, \\
\textstyle\sum_t(1) &= 0.8690\, W_m + 1.386\, W_m^2 - 0.05287\, W_m^3 + 0.005220\, W_m^4, \\
\textstyle\sum_r(\alpha) &= 6.320\, W_m + 0.1035\, W_m^2 - 0.05894\, W_m^3 - 0.0002411\, W_m^4, \\
\textstyle\sum_t(\alpha) &= 1.896\, W_m + 0.03106\, W_m^2 - 0.01768\, W_m^3 - 0.00007232\, W_m^4;
\end{aligned}\right\}\quad (46\text{-}A_5)$$

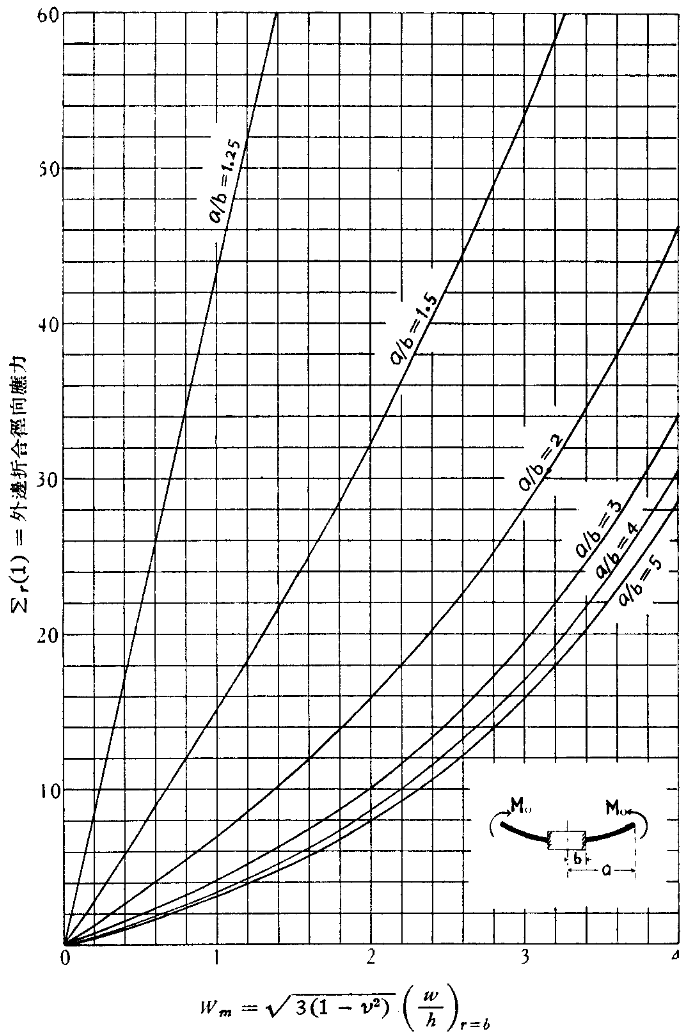

圖 3-B. 在情形 B 中，在不同的比值 a/b 下，最大撓度對彎矩作用邊（外邊）折合徑向應力的曲線（$\nu=0.3$）

vi) 當 $a/b = 1.25$ ($\alpha = 1.5625$) 時，

$$\left.\begin{aligned}
&\sum_r(1) = 32.24\, W_m + 0.2764\, W_m^3, \\
&\sum_t(1) = 2.778\, W_m + 2.743\, W_m^2 + 0.07081\, W_m^3 - 0.002942\, W_m^4, \\
&\sum_r(\alpha) = 26.93\, W_m + 0.1310\, W_m^2 - 0.09935\, W_m^3 + 0.01046\, W_m^4, \\
&\sum_t(\alpha) = 8.080\, W_m + 0.03929\, W_m^2 - 0.02980\, W_m^3 + 0.003137\, W_m^4.
\end{aligned}\right\} \quad (46\text{-}A_6)$$

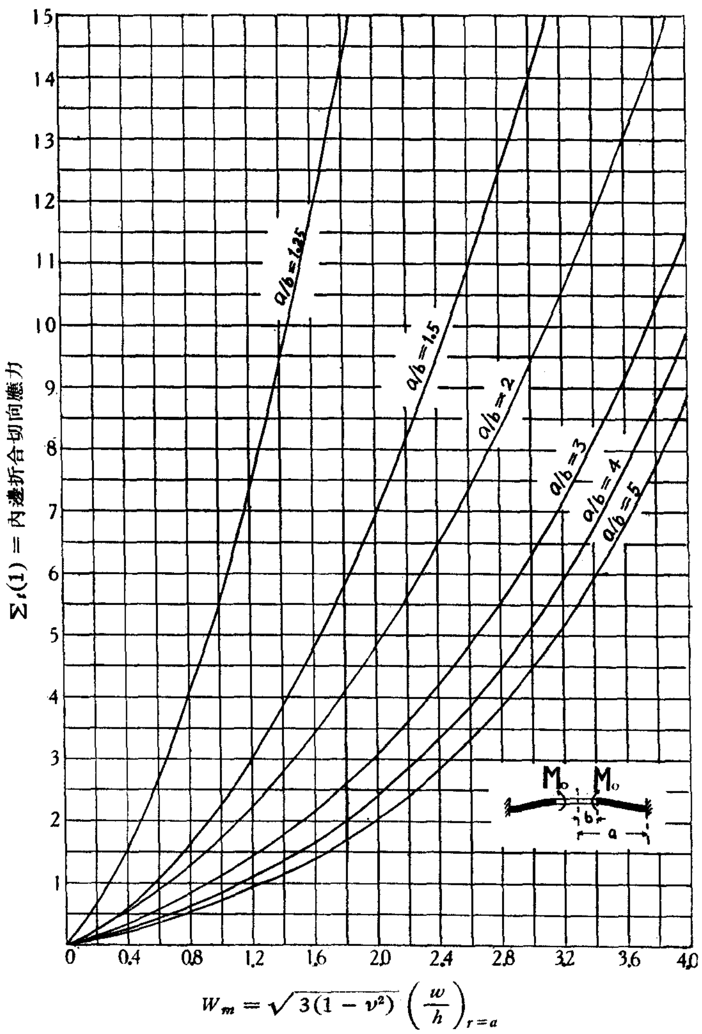

圖 4-A. 在情形 A 中，在不同的比值 a/b 下，最大撓度對彎矩作用邊（內邊）折合切向應力的曲綫（$\nu=0.3$）

乙．在情形 B

i）當 $a/b=5$（$\alpha=1/25$）時，

$$
\left.
\begin{aligned}
&\sum\nolimits_r(1) = 2.901\,W_m + 0.2683\,W_m^3,\\
&\sum\nolimits_t(1) = 2.778\,W_m + 0.6283\,W_m^2 + 0.1371\,W_m^3 + 0.01262\,W_m^4,\\
&\sum\nolimits_r(\alpha) = 4.369\,W_m + 0.3415\,W_m^2 - 0.1722\,W_m^3 - 0.006736\,W_m^4,\\
&\sum\nolimits_t(\alpha) = 1.311\,W_m + 0.1024\,W_m^2 - 0.05165\,W_m^3 - 0.002021\,W_m^4;
\end{aligned}
\right\}\qquad (46\text{-}B_1)
$$

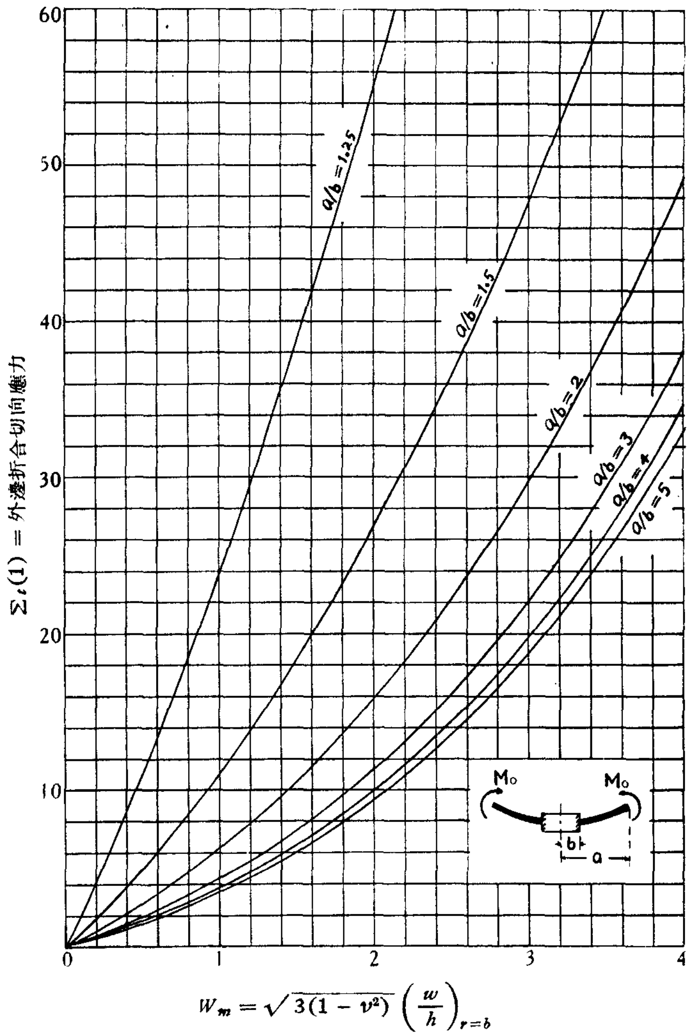

圖 4-B. 在情形 B 中，在不同的比值 a/b 下，最大撓度對彎矩作用邊（外邊）折合切向應力的曲線 ($\nu=0.3$)

ii) 當 $a/b=4$ ($\alpha=1/16$) 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1)=3.193\,W_m+0.2777\,W_m^3,\\
&\sum\nolimits_t(1)=2.985\,W_m+0.6850\,W_m^2+0.1383\,W_m^3+0.01193\,W_m^4,\\
&\sum\nolimits_r(\alpha)=4.752\,W_m+0.3285\,W_m^2-0.1685\,W_m^3-0.005658\,W_m^4,\\
&\sum\nolimits_t(\alpha)=1.426\,W_m+0.09856\,W_m^2-0.05055\,W_m^3-0.001697\,W_m^4;
\end{aligned}\right\}\qquad(46\text{–}B_2)$$

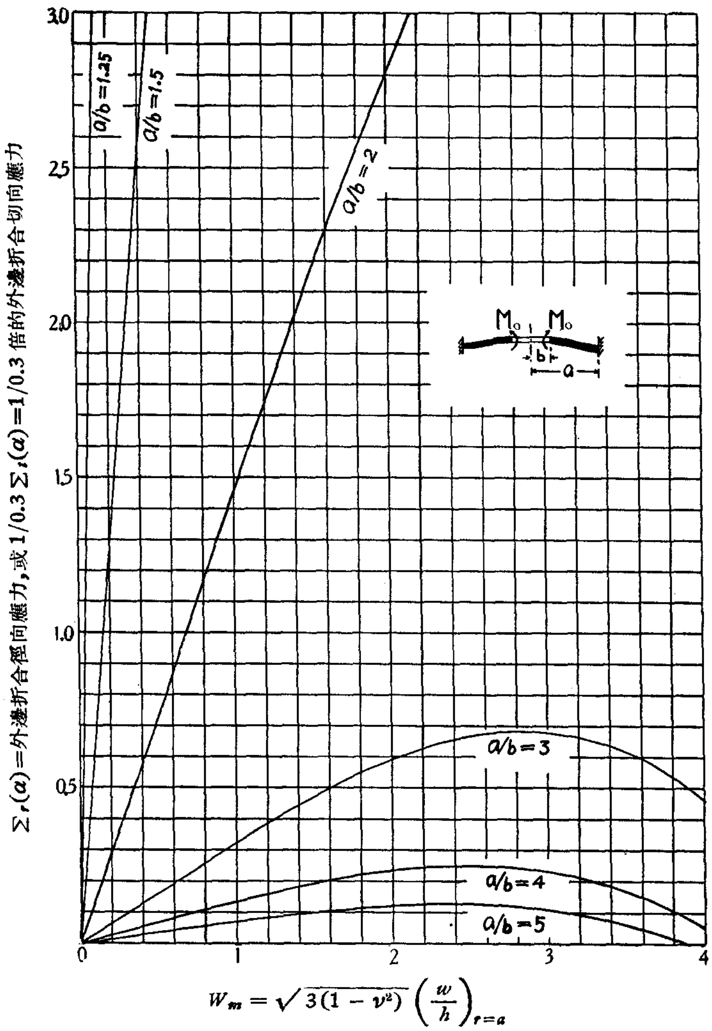

圖 5-A. 在情形 A 中，在不同的比值 a/b 下，最大撓度對平夾邊（外邊）折合徑向應力——或 1/0.3 倍的折合切向應力——的曲線 ($\nu=0.3$)

iii) 當 $a/b = 3$ $(\alpha = 1/9)$ 時,

$$\left.\begin{aligned}
&\sum\nolimits_r(1) = 3.880\ W_m + 0.2914\ W_m^3,\\
&\sum\nolimits_t(1) = 3.442\ W_m + 0.8019\ W_m^2 + 0.1384\ W_m^3 + 0.01063\ W_m^4,\\
&\sum\nolimits_r(\alpha) = 5.632\ W_m + 0.3072\ W_m^2 - 0.1612\ W_m^3 - 0.004079\ W_m^4,\\
&\sum\nolimits_t(\alpha) = 1.690\ W_m + 0.09217\ W_m^2 - 0.04836\ W_m^3 - 0.001224\ W_m^4;
\end{aligned}\right\} \quad (46\text{-}B_3)$$

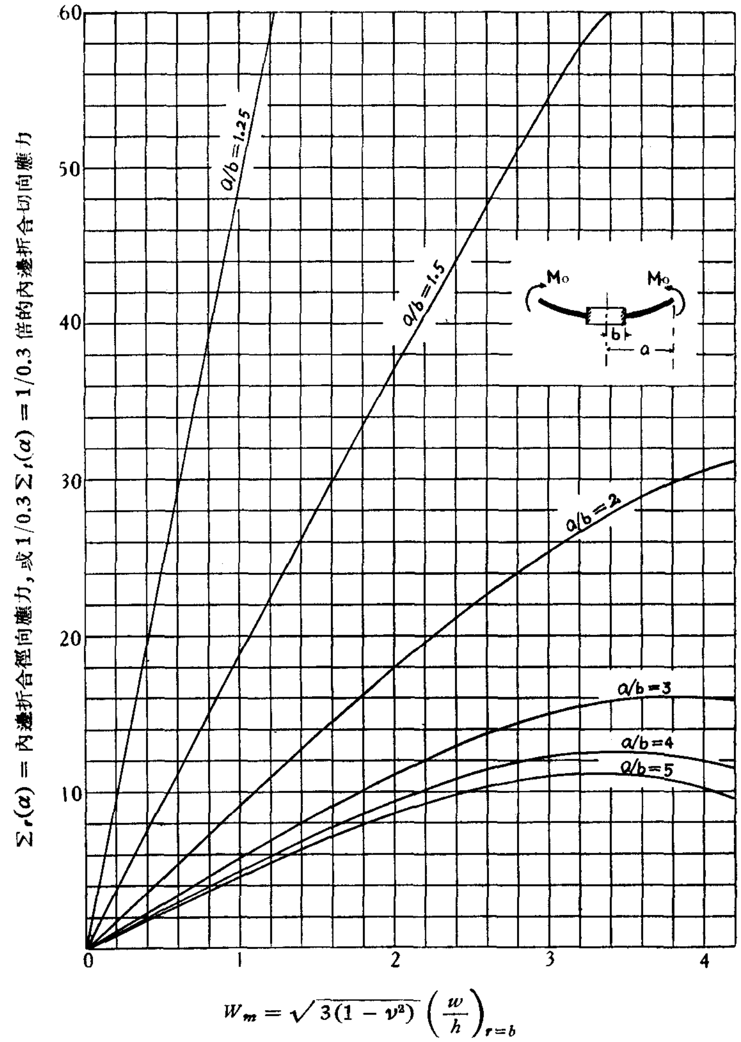

圖 5-*B*. 在情形 *B* 中，在不同的比值 a/b 下，最大撓度對平夾邊(內邊)折合徑向應力——或 1/0.3 倍的折合切向應力——的曲線 ($\nu=0.3$)

iv) 當 $a/b=2$ $(\alpha=1/4)$ 時，

$$\left.\begin{aligned}
&\sum\nolimits_r(1) = 6.638\, W_m + 0.3101\, W_m^3,\\
&\sum\nolimits_t(1)=5.063\, W_m+1.155\, W_m^2+0.1328\, W_m^3+0.007764\, W_m^4,\\
&\sum\nolimits_r(\alpha)=9.001\, W_m+0.2664\, W_m^2-0.1441\, W_m^3-0.001828\, W_m^4,\\
&\sum\nolimits_t(\alpha)=2.700\, W_m+0.07993\, W_m^2-0.04323\, W_m^3-0.0005485\, W_m^4;
\end{aligned}\right\}\quad (46\text{-}B_4)$$

v) 當 $a/b = 1.5$ ($\alpha = 1/2.25$) 時,

$$\left.\begin{aligned}
&\sum\nolimits_r(1) = 14.99\, W_m + 0.3180\, W_m^3, \\
&\sum\nolimits_t(1) = 9.201\, W_m + 1.846\, W_m^2 + 0.1218\, W_m^3 + 0.004940\, W_m^4, \\
&\sum\nolimits_r(\alpha) = 18.61\, W_m + 0.2290\, W_m^2 - 0.1261\, W_m^3 - 0.0006648\, W_m^4, \\
&\sum\nolimits_t(\alpha) = 5.583\, W_m + 0.06871\, W_m^2 - 0.03782\, W_m^3 - 0.0001994\, W_m^4;
\end{aligned}\right\} (46\text{-}B_5)$$

vi) 當 $a/b = 1.25$ ($\alpha = 1/1.5625$) 時,

$$\left.\begin{aligned}
&\sum\nolimits_r(1) = 42.67\, W_m + 0.3336\, W_m^3, \\
&\sum\nolimits_t(1) = 20.80\, W_m + 3.200\, W_m^2 + 0.1181\, W_m^3 + 0.003827\, W_m^4, \\
&\sum\nolimits_r(\alpha) = 48.82\, W_m + 0.2020\, W_m^2 - 0.09367\, W_m^3 - 0.008643\, W_m^4, \\
&\sum\nolimits_t(\alpha) = 14.65\, W_m + 0.06060\, W_m^2 - 0.02810\, W_m^3 - 0.002593\, W_m^4.
\end{aligned}\right\} (46\text{-}B_6)$$

在情形 A 和情形 B 中,在不同的比值 a/b 下,最大撓度對彎矩作用邊及平夾邊的徑向、切向應力的曲線如圖 3—5 所示.

參　考　文　獻

[1] 黃擇言;彈性圓薄板在均佈邊緣力矩作用下的彎曲,物理學報, **12** (1956), 597—606.

[2] Динник, А. Н. 院士主編, Справочник по технической механике, стр. 558, Гостехиздат, 1949. 或見 Roark, R. J., Formulas for Stress and Strain, 3rd ed., p. 201, McGraw-Hill, 1954.

CIRCULAR RING PLATE BENT WITH LARGE DEFLECTION BY UNIFORM MOMENT ALONG ONE EDGE AND CLAMPED AT ANOTHER EDGE

Huang Tse-yen

(Peking Institute of Geological Exploration)

Abstract

In a previous paper[1], it has been shown that the perturbation method is much superior to the method of numerical integration in treating the problem of the bending with large deflection of a circular plate by uniform edge moment. Now, we extend the perturbation method successfully to the case of a circular ring plate with one edge clamped and uniform moment along another edge. As in the previous paper, all our results are presented in such a form that the direct application to design problem is possible.