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正交各向异性彈性平面应力的近似計算方法*

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提 要

本文在列赫尼茨基方法的基础之上提供一个正交各向异性彈性平面应力的近似計算方法．利用这个方法，我們处理了下面三种問題：(1)具有圓孔梁的純弯曲；(2)具有圓形空洞的板的拉伸；(3)具有橢圓空洞的無限平面．

1941年列赫尼茨基曾在苏联科学院报發表微弱各向异性体中应力的近似計算方法．用这个方法計算应力时必須計算大量的复变数的积分，而且应用范圍受到很大的限制．本文目的是在列赫尼茨基方法的基础上提供一个正交各向异性彈性平面应力的近似計算方法．这个方法比起列氏原有的方法在形式及运算上較为簡便，应用范圍也較为广泛．

本文由正交各向异性彈性平面問題的微分方程，利用小参变数法得到便于运算的微分方程組．再依列赫尼茨基所采用的步驟推出以兩种解析函数表示的应力計算表达式．

推导方法

应力函数所滿足的正交各向异性微分方程是：

$$\frac{1}{E_2}\frac{\partial^4 F}{\partial x^4}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial^2 x\partial y^2}+\frac{1}{E_1}\frac{\partial^4 F}{\partial y^4}=0, \tag{1}$$

式中 E_1，E_2 是沿主軸 x 和 y 的楊氏模数；$G=G_{12}$ 是剪切模数，$\nu_1=\nu_{12}$ 是泊松系数；F 是应力函数，它和应力分量的关系是

$$\sigma_x=\frac{\partial^2 F}{\partial y^2},\qquad \sigma_y=\frac{\partial^2 F}{\partial x^2},\qquad \tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y}. \tag{2}$$

將(1)式以新变数 x' 及 y' 表示之，則有

* 1957年3月13日收到．

$$\frac{\partial^4 F}{\partial x'^4}+2\mu\frac{\partial^4 F}{\partial x'^2\partial y'^2}+\frac{\partial^4 F}{\partial y'^4}=0, \tag{3}$$

其中

$$x'=x\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}},\quad y'=y\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}},\quad \mu=\frac{1}{2}\sqrt{E_1E_2}\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right).$$

一般工程上所用的正交各向异性材料，μ 的値的范圍大致如下：

$$2>\mu>0.$$

(一)当 μ 的数値大于 1 小于 2 时，我們可以將 μ 写成：

$$\mu=1+\lambda,$$

其中 λ 是小于 1 的实数.

將 λ 当做小参变数采用类似胡海昌先生所用过的方法，將(3)式写成

$$\frac{\partial^4 F}{\partial x'^4}+2(1+\lambda)\frac{\partial^4 F}{\partial x'^2 y'^2}+\frac{\partial^4 F}{\partial y'^4}=0. \tag{4}$$

(4)式的解答以小参变数表之如下：

$$F=F_0+\lambda F_1+\lambda^2F_2+\cdots+\lambda^nF_n. \tag{5}$$

將(5)式代入(4)式，并比較 λ 的同次方的系数，于是得到

$$\left.\begin{aligned}&\frac{\partial^4 F_0}{\partial x'^4}+2\frac{\partial^4 F_0}{\partial x'^2\partial y'^2}+\frac{\partial^4 F_0}{\partial y'^4}=0,\\&\frac{\partial^4 F_r}{\partial x'^4}+2\frac{\partial^4 F_r}{\partial x'^2\partial y'^2}+\frac{\partial^4 F_r}{\partial y'^4}=-2\frac{\partial F_{r-1}}{\partial x'^2\partial y'^2}.\end{aligned}\right\} \tag{6}$$

其中　$r=1,\ 2,\ 3,\cdots,n.$

將(6)式的解答以解析函数 $\varphi_i(z')$ 及 $\psi_i(z')$ 表示之，則得

$$\left.\begin{aligned}&F_0(z')=\mathrm{Re}\Big(\bar{z}'\varphi_0(z')+\chi_0(z')\Big),\\&F_r(z')=\mathrm{Re}\Big[\bar{z}'\varphi_r(z')+\chi_r(z')+\frac{1}{48}\Big(3\bar{z}'^2\,\psi'_{r-1}(z')+\bar{z}'^3\,\varphi''_{r-1}(z')\Big)\Big].\end{aligned}\right\} \tag{7}$$

其中 $r=1,\ 2,\ 3,\cdots,\ n$, Re 代表实数部分；$z'=x'+iy'$；$\bar{z}'=x'-iy'$；$\chi'_i(z')=\psi_i(z')$.

(7)式的一級导数的关系为

$$\left.\begin{aligned}&\frac{\partial F}{\partial x'}+i\frac{\partial F}{\partial y'}=z'\bar{\varphi}'_0(z')+\varphi_0(z')+\bar{\psi}_0(\bar{z}')+\\&+\sum_{r=1}^{r=n}\lambda^r\Big[z'\bar{\varphi}'_r(z')+\varphi_r(z')+\bar{\psi}_r(z')+\frac{1}{16}\Big(z'^2\bar{\psi}''_{r-1}(z')+2\bar{z}'\psi_{r-1}(z')\Big)+\\&+\frac{1}{48}\Big(z'^3\bar{\varphi}'''_{r-1}(z')+3\bar{z}'^2\varphi''_{r-1}(z')\Big)\Big].\end{aligned}\right\} \tag{8}$$

应力分量的一般表达式为

$$\left.\begin{aligned}&\sigma_{y'}+\sigma_{x'}=\mathrm{Re}\Big[4\varphi_0'(z')+\sum_{r=1}^{r=n}\lambda^r\Big(4\varphi_r'(z')+\frac{1}{2}\bar{z}'\psi_{r-1}''+\frac{1}{4}\bar{z}'^2\varphi_{r-1}'''(z')\Big)\Big],\\&\sigma_{y'}-\sigma_{x'}+2i\tau_{x'y'}=2\Big(\bar{z}'\varphi_0''(z')+\psi_0'(z')\Big)+\sum_{r=1}^{r=n}\lambda^r\Big[2\Big(\bar{z}'\varphi_r''(z')+\psi_r'(z')\Big)+\\&\quad+\frac{1}{4}\bar{\psi}_{r-1}'(z')+\frac{1}{8}\bar{z}'^2\psi_{r-1}'''(z')+\frac{1}{4}z'\bar{\varphi}_{r-1}''(z')+\frac{1}{24}\bar{z}'^3\varphi_{r-1}''''(z')\Big].\end{aligned}\right\}\qquad(9)$$

如果作用于边界上的外力，不依赖于材料的弹性常数，根据(8)式得知满足边界条件的解析函数，有以下的关系：

$$\frac{\partial F}{\partial x'}+i\frac{\partial F}{\partial y'}=\int_0^s(iX_n-Y_n)ds+C,\qquad(10)$$

$$\left.\begin{aligned}&z'\bar{\varphi}_0'(z')+\bar{\psi}_0(z')+\varphi_0(z')=\int_0^s(iX_n-Y_n)ds+C_0,\\&z'\bar{\varphi}_r'(z')+\bar{\psi}_r(z')+\varphi_r(z')=-\frac{1}{16}\Big(z'^2\bar{\psi}_{r-1}''(z')+2\bar{z}'\psi_{r-1}'(z')\Big)-\\&\qquad-\frac{1}{48}\Big(z'^3\bar{\varphi}_{r-1}'''(z')+3\bar{z}'^2\varphi_{r-1}''(z')\Big)+C_r,\end{aligned}\right\}\qquad(11)$$

其中 $r=1,2,\cdots,n$.

$$C_0+\lambda C_1+\lambda^2C_2+\cdots+\lambda^nC_n=C.\qquad(12)$$

(二)当 μ 的值小于1时，我們可以將 μ 写成

$$\mu=1-\lambda,$$

其中 λ 是小于1的实数.

(3)式可写为

$$\frac{\partial^4F}{\partial x'^4}+2(1-\lambda)\frac{\partial^4F}{\partial x'^2\partial y'^2}+\frac{\partial^4F}{\partial y'^4}=0.$$

其解答为

$$F=F_0-\lambda F_1-\lambda^2F_2-\cdots-\lambda^nF_n.\qquad(13)$$

$$\left.\begin{aligned}&F_0(z')=\mathrm{Re}\Big(\bar{z}'\varphi_0(z')+\chi_0(z')\Big),\\&F_1(z')=\mathrm{Re}\Big[\bar{z}'\varphi_1(z')+\chi_1(z')+\frac{1}{48}\Big(3\bar{z}'^2\psi_0'(z')+\bar{z}'^3\varphi_0''(z')\Big)\Big],\\&F_r(z')=\mathrm{Re}\Big[\bar{z}'\varphi_r(z')+\chi_r(z')-\frac{1}{48}\Big(3\bar{z}'^2\psi_{r-1}'(z')+\bar{z}'^3\varphi_{r-1}''(z')\Big)\Big].\end{aligned}\right\}\qquad(14)$$

其中 $r=2,3,\cdots,n$.

应力分量的表达式为

$$\begin{aligned}\sigma_{y'}+\sigma_{x'}=\mathrm{Re}\Big[&4\varphi_0'(z')-\lambda\Big(4\varphi_1'(z')+\frac{1}{2}\bar{z}\psi_0''(z')+\frac{1}{4}\bar{z}'^2\varphi_0'''(z')\Big)-\\&-\sum_{r=2}^{r=n}\lambda^r\Big(\varphi_r'(z')-\frac{1}{2}\bar{z}\psi_{r-1}''(z')-\frac{1}{4}\bar{z}^2\varphi_{r-1}'''(z')\Big)\Big];\end{aligned}\qquad(15)$$

$$
\left.\begin{aligned}
\sigma_{y'}-\sigma_{x'}+2i\tau_{x'y'}=&2\left(\bar{z}'\varphi_0''(z')+\overline{\psi}_0'(z')\right)-\lambda\Big[2\left(\bar{z}'\varphi_1''(z')+\overline{\psi}_1'(z')\right)+\\
&+\frac{1}{4}\overline{\psi}_0'(z')+\frac{1}{4}z'\overline{\varphi}_0''(z')+\frac{1}{8}\bar{z}'^2\psi_0'''(z')+\frac{1}{24}\bar{z}^3\varphi_0''''(z')\Big]-\\
&-\sum_{r=2}^{r=n}\lambda^r\Big[2\left(\bar{z}'\varphi_r''(z')+\overline{\psi}_r'(z')\right)-\frac{1}{4}\overline{\psi}_{r-1}'(z')-\frac{1}{8}\bar{z}'^2\psi_{r-1}'''(z')-\\
&-\frac{1}{4}z'\varphi_{r-1}''(z')-\frac{1}{24}\bar{z}^3\varphi_{r-1}''''(z')\Big].
\end{aligned}\right\}\quad(16)
$$

满足边界条件的解析函数的关系为

$$
\left.\begin{aligned}
&z'\overline{\varphi}_0'(z')+\overline{\psi}_0(z')+\varphi_0(z')=\int_0^s(iX_n-Y_n)ds+C_0,\\
&z'\overline{\varphi}_1'(z')+\overline{\psi}_1(z')+\varphi_1(z')=-\frac{1}{16}\left(z'^2\overline{\psi}_0''(z')+2\bar{z}'\psi_0'(z')\right)-\\
&\qquad-\frac{1}{48}\left(z'^3\overline{\varphi}_0'''(z')+3\bar{z}'^2\varphi_0''(z')\right)+C_2,\\
&z'\overline{\varphi}_r'(z')+\overline{\psi}_r(z')+\varphi_r(z')=\frac{1}{16}\left(z'^2\overline{\psi}_{r-1}''(z')+2\bar{z}'\psi_{r-1}'(z')\right)+\\
&\qquad+\frac{1}{48}\left(z'^3\overline{\varphi}_{r-1}'''(z')+3\bar{z}'^2\varphi_{r-1}''(z')\right)+C_r\\
&(r=2,\ 3,\cdots,\ n).
\end{aligned}\right\}\quad(17)
$$

上式中出現的函数 φ_0 及 ψ_0 是相当各向同性的解析函数，这些函数不难由莫斯海里什維里所采用的复变函数的积分求出；其他有关的解析函数可分别由(17)式相繼求出．以下我們討論这个方法的应用．

簡 單 应 用

现在我們用这个方法处理正交各向异性板或梁的弯曲問題，如梁或板具有空洞利用这个方法也很方便．在这里我們假定分布于空洞边上的外力的合力等于零，其次將所考虑的区域保角变換在單位圓以外的区域．設保角变換函数是

$$z'=\omega(\zeta).$$

經变換后(11)式可改写为

$$
\left.\begin{aligned}
&\frac{\omega(\sigma)}{\overline{\omega}('\sigma)}\overline{\Phi}_0'(\sigma)+\Phi_0(\sigma)+\overline{\Psi}_0(\sigma)=f_0(\sigma),\\
&\frac{\omega(\sigma)}{\overline{\omega}'(\sigma)}\overline{\Phi}_r'(\sigma)+\Phi_r(\sigma)+\overline{\Psi}_r(\sigma)=-\frac{1}{16}[\omega^2(\sigma)\overline{R}_{r-1}(\sigma)+2\overline{\omega}(\sigma)Q_{r-1}(\sigma)]-\\
&\qquad\qquad-\frac{1}{48}[\omega^3(\sigma)\overline{W}_{r-1}(\sigma)+3\overline{\omega}^2(\sigma)V_{r-1}(\sigma)]\\
&(r=1,\ 2,\ 3,\cdots,\ n).
\end{aligned}\right\}\quad(18)
$$

应力分量的表达式变为

$$
\left.\begin{aligned}
\sigma_\theta+\sigma_\rho=&\operatorname{Re}\Big[4U_0(\zeta)+\sum_{r=1}^{r=n}\lambda^r\Big(4U_r(\zeta)+\frac{1}{2}\overline{\omega}(\zeta)R_{r-1}(\zeta)+\\
&+\frac{1}{4}\overline{\omega}^2(\zeta)W_{r-1}(\zeta)\Big)\Big],\\
\sigma_\theta-\sigma_\rho+2i\tau_{\theta\rho}=&\frac{\zeta^2}{\rho^2}\frac{\omega'(\zeta)}{\overline{\omega}'(\zeta)}\Big\{2\Big(V_0(\zeta)\overline{\omega}(\zeta)+Q_0(\zeta)\Big)+\\
&+\sum_{r=1}^{r=n}\lambda^r\Big[2\Big(\overline{\omega}(\zeta)V_r(\zeta)+Q_r(\zeta)\Big)+\frac{1}{4}\bar{Q}_{r-1}(\zeta)+\\
&+\frac{1}{8}\overline{\omega}^2(\zeta)T_{r-1}(\zeta)+\frac{1}{4}\omega(\zeta)\bar{V}_{r-1}(\zeta)+\frac{1}{24}\overline{\omega}^3(\zeta)H_{r-1}(\zeta)\Big]\Big\},
\end{aligned}\right\}\tag{19}
$$

其中

$$\varphi_i(z')=\varphi_i(\omega(\zeta))=\Phi_i(\zeta),\quad \psi_i(z')=\psi_i(\omega(\zeta))=\Psi_i(\zeta),\quad \varphi_i'(z')=\frac{\Phi_i'(\zeta)}{\omega'(\zeta)}=U_i(\zeta);$$

$$\varphi_i''(z')=\left(\frac{\Phi_i'(\zeta)}{\omega'(\zeta)}\right)'\frac{1}{\omega'(\zeta)}=V_i(\zeta),\quad \varphi_i'''(z')=W_i(\zeta),\quad \varphi_i''''(z')=H_i(\zeta);$$

$$\psi_i'(z')=\frac{\Psi_i'(\zeta)}{\omega'(\zeta)}=Q_i(\zeta),\quad \psi_i''(z')=R_i(\zeta),\quad \psi_i'''(z')=T_i(\zeta);$$

$$\zeta=\rho e^{i\theta}\ 及\ \sigma=e^{i\theta}.$$

(一)具有圓孔梁的純弯曲

对于各向同性的梁来說,其解析函数 φ_0 及 ψ_0 是我們所熟知的,它等于

$$\Phi_0(\zeta)=\frac{iMR^2}{8J}\left(\zeta^2-\frac{1}{\zeta^2}\right),\quad \Psi_0(\zeta)=\frac{iMR^2}{8J}\left(\frac{1}{\zeta^2}-\frac{2}{\zeta^4}-\zeta^2\right),$$

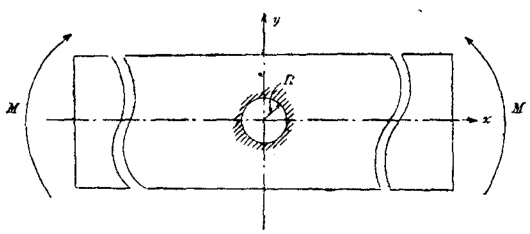

圖 1.

其中 M 是梁兩端所受之弯矩; J 是梁横截面的轉动慣量; R 是圓孔的半徑. 由(18)式我們得到

$$
\begin{aligned}
f_1=&-\frac{1}{16}[\overline{\omega}^2R_0+2\overline{\omega}Q_0]-\frac{1}{48}[\omega^3W_0+3\omega^2V_0]=\\
=&\frac{iMR^2}{8\times16J}\left(\frac{4}{\zeta^4}-\frac{16}{\zeta^6}+4-6\zeta^6+40\zeta^8+2\zeta^2\right)+\frac{iMR^2}{8\times48J}\left(\frac{18}{\zeta^6}-\frac{6}{\zeta^2}-24\zeta^8\right).
\end{aligned}
$$

由此得到

$$\Phi_1(\zeta)=-\frac{1}{2\pi i}\int_\gamma\frac{f_1d\sigma}{\sigma-\zeta}=\frac{iMR^2}{8\times16J}\left(\frac{4}{\zeta^4}-\frac{16}{\zeta^6}-\frac{2}{\zeta^2}\right),$$

$$\Psi_1(\zeta) = -\frac{1}{2\pi i}\int_r \frac{\overline{f_1}(\sigma)d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_r \frac{\overline{\omega}(\sigma)\Phi_1'(\sigma)}{\omega(\sigma)'}\frac{d\sigma}{\sigma-\zeta} =$$

$$= \frac{iMR^2}{8\times 16J}\left[\frac{2}{\zeta^2} - \frac{4}{\zeta^4} + \frac{10}{\zeta^6} - \frac{28}{\zeta^8}\right].$$

同法，我們得到

$$\Phi_2(\zeta) = \frac{iMR^2}{8\times 16\times 16}\left[\frac{8}{\zeta^4} - \frac{20}{\zeta^6} + \frac{40}{\zeta^8} - \frac{28}{\zeta^{10}}\right],$$

$$\Psi_2(\zeta) = \frac{iMR^2}{8\times 16\times 16}\left[\frac{20}{\zeta^6} - \frac{56}{\zeta^8} + \frac{60}{\zeta^{10}} + \frac{616}{\zeta^{12}}\right].$$

如 λ 的值甚小时，我們可以略去其二次以上的各項，由(19)式得到应力分量为

$$\sigma_\theta = \frac{MR}{8J}\left[\left(\frac{8}{\rho^5} + 2\rho - \frac{2}{\rho^3}\right)\sin 3\theta - \left(6\rho + \frac{2}{\rho^3}\right)\sin\theta\right] +$$

$$+\lambda\frac{MR}{16J}\left[\left(\frac{28}{\rho^9} - \frac{75}{2}\frac{1}{\rho^7} + \frac{8}{\rho^5} - \frac{5}{2}\frac{1}{\rho^3}\right)\sin 7\theta + \left(\frac{1}{2}\frac{1}{\rho^3} + \frac{6}{\rho^5} - \frac{15}{2}\frac{1}{\rho^7}\right)\sin 5\theta +\right.$$

$$\left.+\left(\frac{2}{\rho^5} - \frac{1}{2}\frac{1}{\rho^3} - \frac{\rho}{2}\right)\sin 3\theta + \left(\frac{3}{2}\rho - \frac{1}{2}\frac{1}{\rho}\right)\sin\theta\right] +$$

$$+\lambda^2\frac{MR}{256J}\left[\left(\frac{1091}{\rho^9} - \frac{924}{\rho^{13}} - \frac{315}{\rho^7} + \frac{70}{\rho^{11}}\right)\sin 11\theta + \left(\frac{165}{\rho^7} - \frac{80}{\rho^9} - \frac{75}{\rho^{11}}\right)\sin 9\theta +\right.$$

$$\left.+\left(\frac{5}{\rho^3} + \frac{16}{\rho^5} - \frac{75}{\rho^7} + \frac{56}{\rho^9}\right)\sin 7\theta + \left(\frac{1}{\rho^3} + \frac{12}{\rho^5} - \frac{15}{\rho^7}\right)\sin 5\theta\right],$$

$$\sigma_\rho = \frac{MR}{8J}\left[\left(\frac{2}{\rho^3} - 2\rho\right)\sin\theta + \left(\frac{10}{\rho^3} - 2\rho - \frac{8}{\rho^5}\right)\sin 3\theta\right] +$$

$$+\lambda\frac{MR}{16J}\left[\left(\frac{135}{2}\frac{1}{\rho^7} - \frac{48}{\rho^5} + \frac{9}{2}\frac{1}{\rho^3} - \frac{28}{\rho^9}\right)\sin 7\theta + \left(\frac{15}{2}\frac{1}{\rho^7} - \frac{14}{\rho^5} + \frac{11}{2}\frac{1}{\rho^3}\right)\sin 5\theta +\right.$$

$$\left.+\left(\frac{\rho}{2} + \frac{5}{2}\frac{1}{\rho^3} - \frac{2}{\rho^5}\right)\sin 3\theta + \left(\frac{\rho}{2} + \frac{1}{2}\frac{1}{\rho^3}\right)\sin\theta\right] +$$

$$+\lambda^2\frac{MR^2}{256}\left[\left(\frac{1995}{\rho^7} - \frac{3107}{\rho^9} + \frac{70}{\rho^{11}} + \frac{924}{\rho^{13}}\right)\sin 11\theta + \left(\frac{75}{\rho^{11}} - \frac{80}{\rho^9} + \frac{255}{\rho^7} - \frac{240}{\rho^5}\right)\sin 9\theta +\right.$$

$$\left.+\left(\frac{19}{\rho^3} - \frac{96}{\rho^5} + \frac{135}{\rho^7} - \frac{56}{\rho^9}\right)\sin 7\theta + \left(\frac{11}{\rho^3} - \frac{28}{\rho^5} + \frac{15}{\rho^7}\right)\sin 5\theta\right],$$

$$\tau_{\rho\theta} = \frac{MR}{8J}\left[\left(\frac{16}{\rho^5} - 4\rho - \frac{12}{\rho^3}\right)\cos 3\theta + \left(4\rho - \frac{2}{\rho^3}\right)\cos\theta\right] +$$

$$+\lambda\frac{MR}{8J}\left[\left(\frac{28}{\rho^9} - \frac{105}{2}\frac{1}{\rho^7} + \frac{32}{\rho^5} - \frac{13}{2}\frac{1}{\rho^3}\right)\cos 7\theta + \left(\frac{10}{\rho^5} - \frac{15}{2}\frac{1}{\rho^7} - \frac{7}{2}\frac{1}{\rho^3}\right)\cos 5\theta +\right.$$

$$\left.+\left(\frac{2}{\rho^5} - \frac{3}{2}\frac{1}{\rho^3} + \frac{\rho}{2}\right)\cos 3\theta - \left(\frac{1}{2}\frac{1}{\rho^3} + \frac{\rho}{2}\right)\cos\theta\right] +$$

$$+\lambda^2\frac{MR}{128}\left[\left(\frac{2211}{\rho^9} - \frac{924}{\rho^{13}} - \frac{1365}{\rho^7}\right)\cos 11\theta + \left(\frac{160}{\rho^5} - \frac{75}{\rho^7} - \frac{75}{\rho^{11}}\right)\cos 9\theta +\right.$$

$$\left.+\left(\frac{56}{\rho^9} - \frac{105}{\rho^7} + \frac{64}{\rho^5} - \frac{13}{\rho^3}\right)\cos 7\theta + \left(\frac{20}{\rho^5} - \frac{15}{\rho^7} - \frac{7}{\rho^3}\right)\cos 5\theta\right].$$

以上是在 ζ-平面上的应力分量，至于 z-平面的应力分量可以通过变数 ζ 与 z 的关

系来計算．ζ 与 z 的关系的表达式为

$$z=x+iy=\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}}x'+\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}}y'=\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}}+\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}}\right]\frac{z'}{2}+\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}}-\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}}\right]\frac{\bar{z}'}{2}=$$

$$=\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}}+\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}}\right]\frac{\omega(\zeta)}{2}+\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{8}}-\left(\frac{E_2}{E_1}\right)^{\frac{1}{8}}\right]\frac{\bar{\omega}(\zeta)}{2}.$$

(二)具有圓形空洞的板的拉伸

正交各向异性板的兩对外边上承受均匀分布的力 P 而被拉伸，令坐标的原点位于圓洞的中心，并使其兩軸与其主方向平行．首先設外边的分布力 P 与 x-軸成 α 角．

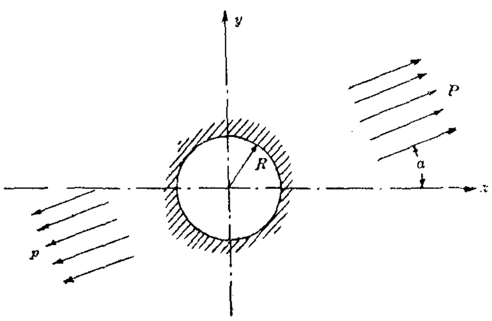

圖 2.

在此情形下，保角变换函数是

$$z=\omega(\zeta)=R\zeta,$$

其中 R 是圓孔的半徑；$\zeta=\rho e^{i\theta}$．

相当于各向同性板的解析函数是已知的，它等于

$$\Phi_0(\zeta)=\frac{PR}{4}\left(\zeta+\frac{2e^{2i\alpha}}{\zeta}\right),$$

$$\Psi_0(\zeta)=-\frac{PR}{2}\left(\zeta e^{-2i\alpha}-\frac{1}{\zeta}-\frac{e^{2i\alpha}}{\zeta^3}\right).$$

由(11)式求出 $\Phi_1(\zeta)$，$\Phi_2(\zeta)$，$\Psi_1(\zeta)$ 及 $\Psi_2(\zeta)$ 如下：

$$\Phi_1(\zeta)=\frac{PR}{32}\left(\frac{2e^{-2i\alpha}}{\zeta}-\frac{2}{\zeta^3}+\frac{4e^{2i\alpha}}{\zeta^5}\right),$$

$$\Psi_1(\zeta)=\frac{PR}{32}\left[\frac{10e^{2i\alpha}}{\zeta^7}-\frac{4}{\zeta^5}+\frac{2e^{-2i\alpha}}{\zeta^3}\right],$$

$$\Phi_2(\zeta)=\frac{PR}{32\times16}\left[\frac{20e^{2i\alpha}}{\zeta^9}+\frac{8e^{-2i\alpha}}{\zeta^5}-\frac{16}{\zeta^7}\right],$$

$$\Psi_2(\zeta)=\frac{PR}{32\times16}\left[\frac{20e^{-2i\alpha}}{\zeta^7}-\frac{32}{\zeta^9}-\frac{100e^{2i\alpha}}{\zeta^{11}}\right].$$

应力分量为

$$\sigma_\theta=P\left[\left(\frac{1}{2}+\frac{1}{2}\frac{1}{\rho^2}\right)-\left(\frac{1}{2}+\frac{3}{2}\frac{1}{\rho^4}\right)\cos(2\alpha-2\theta)\right]+$$

$$+\lambda\frac{P}{8}\left[\left(\frac{5}{\rho^6}-\frac{3}{\rho^4}\right)\cos4\theta+\left(\frac{20}{\rho^6}-\frac{35}{2}\frac{1}{\rho^8}-\frac{3}{\rho^4}-\frac{1}{\rho^2}\right)\cos(2\alpha-6\theta)-\right.$$

$$\left.-\frac{3}{2}\frac{1}{\rho^4}\cos(2\theta+2\alpha)+\left(\frac{1}{\rho^2}-\frac{3}{2}\frac{1}{\rho^4}\right)\cos(2\alpha+6\theta)-\frac{1}{2}\cos(2\theta-2\alpha)\right]+$$

$$+\frac{\lambda^2P}{128}\left[\left(\frac{360}{\rho^{10}}-\frac{675}{\rho^8}+\frac{140}{\rho^6}+\frac{275}{\rho^{12}}\right)\cos(2\alpha-10\theta)+\right.$$

$$+\left(\frac{2}{\rho^2}-\frac{3}{\rho^4}\right)\cos(6\theta-2\alpha)+\left(\frac{72}{\rho^{10}}-\frac{12}{\rho^4}-\frac{168}{\rho^8}+\frac{100}{\rho^6}\right)\cos8\theta+$$

$$\left.+\left(\frac{5}{\rho^6}-\frac{10}{\rho^4}-\frac{2}{\rho^2}\right)\cos(2\alpha+6\theta)-\frac{35}{\rho^8}\cos(2\alpha+10\theta)+\frac{60}{\rho^6}\cos10\theta\right],$$

$$\sigma_\rho = P\left[\left(\frac{1}{2}-\frac{1}{2}\frac{1}{\rho^2}\right)+\left(\frac{1}{2}-\frac{2}{\rho^2}+\frac{3}{2}\frac{1}{\rho^4}\right)\cos(2\alpha-2\theta)\right]+$$

$$+\lambda\frac{P}{8}\left[\left(\frac{9}{\rho^4}-\frac{4}{\rho^2}-\frac{5}{\rho^6}\right)\cos 4\theta+\left(\frac{35}{2}\frac{1}{\rho^8}-\frac{40}{\rho^6}+\frac{27}{\rho^4}-\frac{5}{\rho^2}\right)\cos(2\alpha-6\theta)-\right.$$

$$\left.-\left(\frac{2}{\rho^2}+\frac{3}{2}\frac{1}{\rho^4}\right)\cos(2\theta+2\alpha)+\frac{1}{2}\cos(2\theta-2\alpha)+\left(\frac{3}{2}\frac{1}{\rho^4}-\frac{1}{\rho^2}\right)\cos(2\alpha+6\theta)\right]+$$

$$+\lambda^2\frac{P}{128}\left[\left(\frac{1759}{\rho^8}-\frac{540}{\rho^{10}}-\frac{980}{\rho^6}-\frac{275}{\rho^{12}}\right)\cos(2\alpha-10\theta)+\right.$$

$$+\left(\frac{132}{\rho^4}+\frac{280}{\rho^8}-\frac{340}{\rho^6}-\frac{72}{\rho^{10}}\right)\cos 8\theta+\frac{35}{\rho^8}\cos(2\alpha+10\theta)+$$

$$\left.+\left(\frac{56}{\rho^4}-\frac{45}{\rho^6}-\frac{10}{\rho^2}\right)\cos(2\alpha+6\theta)-\frac{60}{\rho^6}\cos 10\theta+\left(\frac{3}{\rho^4}-\frac{2}{\rho^2}\right)\cos(6\theta-2\alpha)\right],$$

$$\tau_{\rho\theta} = P\left[\frac{2}{\rho^2}\sin(2\alpha-2\theta)-\sin(2\theta-2\alpha)-\frac{3}{\rho^4}\sin(2\alpha-2\theta)\right]+$$

$$+\lambda\frac{P}{8}\left[\left(\frac{12}{\rho^4}-\frac{2}{\rho^2}-\frac{10}{\rho^6}\right)\sin 4\theta+\left(\frac{3}{\rho^4}-\frac{2}{\rho^2}-1\right)\sin(2\alpha+2\theta)+\right.$$

$$\left.+\left(\frac{60}{\rho^6}-\frac{35}{\rho^8}-\frac{30}{\rho^4}+\frac{4}{\rho^2}\right)\sin(2\alpha-6\theta)+\left(\frac{2}{\rho^2}-\frac{3}{\rho^4}\right)\sin(2\alpha+6\theta)\right]+$$

$$+\lambda^2\frac{P}{128}\left[\left(\frac{450}{\rho^{10}}+\frac{275}{\rho^{12}}+\frac{560}{\rho^6}-\frac{1235}{\rho^8}\right)\sin(2\alpha-10\theta)+\right.$$

$$+\left(\frac{30}{\rho^4}-\frac{25}{\rho^6}-\frac{4}{\rho^2}\right)\sin(2\alpha+6\theta)+\left(\frac{48}{\rho^4}-\frac{220}{\rho^6}+\frac{224}{\rho^8}-\frac{72}{\rho^{10}}\right)\sin 8\theta+$$

$$\left.+\left(\frac{2}{\rho^2}-\frac{13}{4}\frac{1}{\rho^4}\right)\sin(6\theta-2\alpha)-\frac{35}{\rho^8}\sin(2\alpha+10\theta)+\frac{60}{\rho^6}\sin 10\theta\right].$$

当 $\alpha=0$ 时，应力分量为

$$\sigma_\theta = P\left[\left(\frac{1}{2}+\frac{1}{2}\frac{1}{\rho^2}\right)-\left(\frac{1}{2}+\frac{3}{2}\frac{1}{\rho^4}\right)\cos 2\theta\right]+$$

$$+\lambda\frac{P}{8}\left[\left(\frac{5}{\rho^6}-\frac{3}{\rho^4}\right)\cos 4\theta+\left(\frac{20}{\rho^6}-\frac{35}{2}\frac{1}{\rho^8}-\frac{3}{\rho^4}-\frac{1}{\rho^2}\right)\cos 6\theta-\right.$$

$$\left.-\frac{3}{2}\frac{1}{\rho^4}\cos 2\theta+\left(\frac{1}{\rho^2}-\frac{3}{2}\frac{1}{\rho^4}\right)\cos 6\theta-\frac{1}{2}\cos 2\theta\right]+$$

$$+\lambda^2\frac{P}{128}\left[\left(\frac{360}{\rho^{10}}-\frac{675}{\rho^8}+\frac{140}{\rho^6}+\frac{275}{\rho^{12}}\right)\cos 10\theta+\left(\frac{2}{\rho^2}-\frac{3}{\rho^4}\right)\cos 6\theta+\right.$$

$$+\left(\frac{5}{\rho^6}-\frac{10}{\rho^4}-\frac{2}{\rho^2}\right)\cos 6\theta+\left(\frac{72}{\rho^{10}}-\frac{12}{\rho^4}-\frac{168}{\rho^8}+\frac{100}{\rho^6}\right)\cos 8\theta-$$

$$\left.-\frac{35}{\rho^8}\cos 10\theta+\frac{60}{\rho^6}\cos 10\theta\right],$$

$$\sigma_\rho=P\Big[\Big(\frac{1}{2}-\frac{1}{2}\,\frac{1}{\rho^2}\Big)+\Big(\frac{1}{2}-\frac{2}{\rho^2}+\frac{3}{2}\,\frac{1}{\rho^4}\Big)\cos 2\theta\Big]+\lambda\,\frac{P}{8}\Big[\Big(\frac{9}{\rho^4}-\frac{4}{\rho^2}-\frac{5}{\rho^6}\Big)\cos 4\theta+$$

$$+\Big(\frac{1}{2}-\frac{2}{\rho^2}-\frac{3}{2}\,\frac{1}{\rho^4}\Big)\cos 2\theta+\Big(\frac{35}{2}\,\frac{1}{\rho^8}-\frac{40}{\rho^6}+\frac{57}{2}\,\frac{1}{\rho^4}-\frac{6}{\rho^2}\Big)\cos 6\theta+$$

$$+\lambda^2\frac{P}{128}\Big[\Big(\frac{1794}{\rho^8}-\frac{540}{\rho^{10}}-\frac{1040}{\rho^6}-\frac{275}{\rho^{12}}\Big)\cos 10\theta+\Big(\frac{132}{\rho^4}+\frac{280}{\rho^8}-\frac{340}{\rho^6}-\frac{72}{\rho^{10}}\Big)\cos 8\theta+$$

$$+\Big(\frac{59}{\rho^4}-\frac{45}{\rho^6}-\frac{12}{\rho^2}\Big)\cos 6\theta\Big],$$

$$\tau_{r\theta}=P\Big[\Big(\frac{-2}{\rho^2}-1+\frac{3}{\rho^4}\Big)\sin 2\theta\Big]+\lambda\,\frac{P}{8}\Big[\Big(\frac{12}{\rho^4}-\frac{2}{\rho^2}-\frac{10}{\rho^6}\Big)\sin 4\theta+$$

$$+\Big(\frac{3}{\rho^4}-\frac{2}{\rho^2}-1\Big)\sin 2\theta+\Big(\frac{-60}{\rho^6}+\frac{35}{\rho^8}+\frac{27}{\rho^4}-\frac{2}{\rho^2}\Big)\sin 6\theta\Big]+$$

$$+\lambda^2\frac{P}{128}\Big[\Big(\frac{1200}{\rho^8}-\frac{450}{\rho^{10}}-\frac{275}{\rho^{12}}-\frac{500}{\rho^6}\Big)\sin 10\theta+$$

$$+\Big(\frac{48}{\rho^4}-\frac{220}{\rho^6}+\frac{224}{\rho^8}-\frac{72}{\rho^{10}}\Big)\sin 8\theta+\Big(\frac{107}{4}\,\frac{1}{\rho^4}-\frac{25}{\rho^6}-\frac{2}{\rho^2}\Big)\sin 6\theta\Big].$$

(三)具有椭圓空洞的無限平面

設作用椭圓空洞边界的力是均分布的法向力 P. 取椭圓中心为坐标原点. 令保角变換函数是

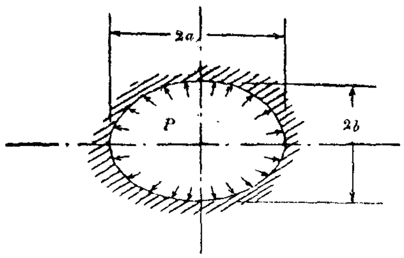

圖 3.

$$z'=\omega(\zeta)=R\Big(\zeta+\frac{m}{\zeta}\Big),$$

其中

$$R=\frac{a+b}{2},$$

$$m=\frac{a-b}{a+b}.$$

相当各向同性的解析函数为

$$\Phi_0(\zeta)=-\frac{PRm}{\zeta},$$

$$\Psi_0(\zeta)=-\frac{PR}{\zeta}-\frac{PRm(1+m\zeta^2)}{\zeta(\zeta^2-m)}.$$

由(8)式我們求出解析函数 $\Phi_1(\zeta)$, $\Phi_2(\zeta)$, $\Psi_1(\zeta)$ 及 $\Psi_2(\zeta)$ 为

$$\Phi_1(\zeta)=-\frac{PR}{8}\Big[\frac{(1+m\zeta^2)(\zeta^2+m^3)\zeta}{(\zeta^2-m)^3}\Big],$$

$$\Psi_1(\zeta)=-\frac{PR}{8}(\zeta^2m+1)\Big[\frac{(5+2m^2)\zeta^5+(2m^5+2m^3-2m)\zeta^3}{(\zeta^2-m)^5}\Big],$$

$$\Phi_2(\zeta)=\frac{-PR(m\zeta^2+1)}{64}\Big[\frac{S_1\zeta^9+S_2\zeta^7+S_3\zeta^5+S_4\zeta^3}{(\zeta^2-m)^7}\Big]+$$

$$+\frac{PR(m\zeta^2+1)^2}{64}\Big[\frac{M_1\zeta^9+M_2\zeta^7+M_3\zeta^5+M_4\zeta^3+M_5\zeta}{(\zeta^2-m)^7}\Big],$$

$$\Psi_2(\zeta)=\frac{PR}{128}\left[\frac{(H_1\zeta^{11}+H_2\zeta^9+H_3\zeta^7+H_4\zeta^5+H_5\zeta^3)(m\zeta^2+1)^2}{(\zeta^2-m)^9}\right]-$$
$$-\frac{PR}{128}\left[\frac{(N_1\zeta^{11}+N_2\zeta^9+N_3\zeta^7+N_4\zeta^5+N_5\zeta^3+N_6\zeta)(m\zeta^3+1)^3}{(\zeta^2-m)^9}\right]-$$
$$-\frac{\zeta(\zeta^2m+1)}{\zeta^2-m}\Phi_2'(\zeta);$$

其中

$$M_1=2m,$$
$$M_2=12m^4+28m^2+24,$$
$$M_3=36m^5+60m^3+42m,$$
$$M_4=12m^6+28m^4+4m^2,$$
$$M_5=2m^5;$$
$$N_1=6m,$$
$$N_2=60m^4+152m^2+120,$$
$$N_3=360m^5+672m^3+510m,$$
$$N_4=360m^6+682m^4+330m^2,$$
$$N_5=60m^7+162m^5+20m^3,$$
$$N_6=6m^6;$$
$$S_1=15m+6m^3,$$
$$S_2=10m^6+24m^4+35m^2+25,$$
$$S_3=10m^7+24m^5+14m^3+11m,$$
$$S_4=6m^6+6m^4-6m^2;$$
$$H_1=60m+24m^3,$$
$$H_2=60m^6+204m^4+360m^2+150,$$
$$H_3=160m^7+384m^5+392m^3+288m,$$
$$H_4=60m^8+204m^6+144m^4+6m^2,$$
$$H_5=24m^7+24m^5-24m^3.$$

应力分量可由(9)式决定之.

結　論

这个方法的应用范圍比較广泛，可以处理一些实际問題．不但可以解决边緣受力板的弯曲問題，而且也可以結合 A. И. 盧尔也所采用的方法处理薄板負担任意載荷的弯曲問題．此外，如結合广义的施娃尔茨交替法，也可以解决复联区域彈性平面問題．应用时，如 $\mu=1+\lambda$ 接近于1时，可以略去 λ 二次以上的各項，这样在运算方面比較簡

單. 如 M 的值接近 2 时, 需要取很多項才能达到所要的准确度. 在此情形下, 用这个方法处理問題时, 必須計算大量的复变数的积分, 运算步驟就特别繁难了.

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AN APPROXIMATE METHOD FOR DETERMINING THE PLANE STRESS OF ORTHOTROPIC MATERIAL

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ABSTRACT

In this paper, an approximate expression for solving plane stress problem of orthotropic material is worked out based upon the procedures of Lecknitzky's work. For the purpose of applications, three simple examples are given, namely: (1) The bending of beam with circular hole, (2) circular hole in a plate under simple tension, (3) elliptic hole in an infinite plane with uniformly distributed pressure acting on the edge of hole.