第 23 卷 第 2 期　　物 理 学 报　　Vol. 23, No. 2
1974 年 3 月　　ACTA PHYSICA SINICA　　March, 1974

标量-张量引力波*

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提　　要

本文的结果大部分已在文献[1]中发表，这里给以完整的证明. 此外，还证明了 D 型的标量-张量波不存在.

一、旋量分析简介

本文使用的四维时空流形的度规

$$ds^2 = g_{jk}dx^j dx^k \tag{1.1}$$

的号差为 $(+, -, -, -)$；这里指标 $j, k, \cdots$ 由 0—3. 考虑如下的 Einstein 方程:

$$R_{jk} - \frac{1}{2} g_{jk}R = b\left[\frac{a}{\varphi^2}\left(\varphi_{;j}\varphi_{;k} - \frac{1}{2} g_{jk}\varphi_{;p}\varphi^{;p}\right) + \frac{1}{\varphi}\varphi_{;jk}\right], \tag{1.2}$$

$$g^{jk}\varphi_{;jk} = 0. \tag{1.3}$$

其中 b 与 a 为任意正实常数，φ 是一标量场. 当取 $b = 1$ 时，就是 Brans-Dicke[2] 所引进的方程. 由于方程(1.3)是双曲型二阶线性偏微分方程，寻求其波动解，自然要求存在波面，在数学上即要求 φ 还要适合

$$g^{jk}\varphi_{;j}\varphi_{;k} = 0. \tag{1.4}$$

这里是用旋量分析的方法研究方程(1.2—1.4)的解.

首先采用拟正交就范四维标架 (Quasi-Orthonormal tetrad) $\{e^j_{(a)}\}$，使得度规张量的逆变支量 g^{jk} 可写为

$$g^{jk} = e^j_{(0)}e^k_{(1)} + e^j_{(1)}e^k_{(0)} - e^j_{(2)}e^k_{(3)} - e^j_{(3)}e^k_{(2)}, \tag{1.5}$$

其中 $e^j_{(3)} = \overline{e^j_{(2)}}$.

命 $\sigma^a_{A\bar{B}}$ 为 Pauli 矩阵元，即

$$\sigma^0_{A\bar{B}} = \delta^1_A\delta^1_{\bar{B}},\ \sigma^1_{A\bar{B}} = \delta^2_A\delta^2_{\bar{B}},\ \sigma^2_{A\bar{B}} = \delta^1_A\delta^2_{\bar{B}},\ \sigma^3_{A\bar{B}} = \delta^2_A\delta^1_{\bar{B}}. \tag{1.6}$$

这里指标 $A, B, \cdots$ 表由 1—2. 于是每一张量对应有一旋量. 例如能量-动量张量 T_{jk} 对应的旋量为

$$T_{A\bar{C}B\bar{D}} = T_{jk}e^j_{(a)}e^k_{(b)}\sigma^a_{A\bar{C}}\sigma^b_{B\bar{D}}. \tag{1.7}$$

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* 1972 年 11 月 15 日收到.

关于旋量的基本运算可参阅文献 [3] 或 [4], 在那里还证明了外尔 (Weyl) 张量所对应的旋量可写为

$$C_{A\bar{E}B\bar{F}C\bar{G}D\bar{H}} = \epsilon_{\bar{E}\bar{F}}\epsilon_{\bar{G}\bar{H}}\Psi_{ABCD} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi_{EFGH}}, \tag{1.8}$$

其中

$$\epsilon_{AB} = \epsilon_{\bar{A}\bar{B}} = \delta^1_A\delta^2_B - \delta^2_A\delta^1_B.$$

注意,这里用的旋量指标 $\bar{A}, \bar{B}, \cdots$ 等通常用 $\dot{A}, \dot{B}, \cdots$. 此外,命

$$T = T_{A\bar{C}}{}^{A\bar{C}}, \quad \varphi_{AB\bar{C}\bar{D}} = \frac{1}{4}(T_{A\bar{C}B\bar{D}} + T_{B\bar{C}A\bar{D}}). \tag{1.9}$$

Newman-Penrose[5] 把黎曼联络所对应的旋量联络 $\Gamma_{ABC\bar{D}}$ 用下面的符号表示

$\Gamma_{ABC\bar{D}} =$ (1.10)

$C\bar{D}$ \ AB	11	12 或 21	22
$1\bar{1}$	κ	ϵ	π
$2\bar{1}$	ρ	α	λ
$1\bar{2}$	σ	β	μ
$2\bar{2}$	τ	γ	ν

并引进微分算子

$$X_{1\bar{1}} = D = e^j_{(0)}\frac{\partial}{\partial x^j}, \quad X_{2\bar{2}} = \Delta = e^j_{(1)}\frac{\partial}{\partial x^j},$$
$$X_{1\bar{2}} = \delta = e^j_{(2)}\frac{\partial}{\partial x^j}, \quad X_{2\bar{1}} = \bar{\delta} = e^j_{(3)}\frac{\partial}{\partial x^j}. \tag{1.11}$$

他们实质上是通过 É. Cartan 的结构方程[6] 以及黎曼曲率张量的 Géhéniau-Debever[7] 的分解旋量形式,把 Einstein 方程化为下面的一系列一阶方程. 首先是 N-P 方程:

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \varphi_{11\bar{1}\bar{1}}, \tag{1.12$_1$}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + 3\beta + \bar{\alpha})\kappa + \Psi_{1111}, \tag{1.12$_2$}$$

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma - (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_{1112} + \varphi_{11\bar{1}\bar{2}}, \tag{1.12$_3$}$$

$$D\alpha - \bar{\delta}\epsilon = (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \bar{\kappa}\gamma - \kappa\lambda + (\epsilon + \rho)\pi + \varphi_{12\bar{1}\bar{1}}, \tag{1.12$_4$}$$

$$D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_{1112}, \tag{1.12$_5$}$$

$$D\gamma - \Delta\epsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \kappa\nu + \Psi_{1122} + \varphi_{12\bar{1}\bar{2}} - \frac{1}{24}T, \tag{1.12$_6$}$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\epsilon - \bar{\epsilon})\lambda + \varphi_{22\bar{1}\bar{1}}, \tag{1.12$_7$}$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_{1122} + \frac{1}{12}T, \tag{1.12$_8$}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_{1222} + \varphi_{22\bar{1}\bar{2}}, \tag{1.12$_9$}$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_{2222}, \tag{1.12$_{10}$}$$

$$\delta\rho-\bar{\delta}\sigma=\rho(\bar{\alpha}+\beta)-\sigma(3\alpha-\bar{\beta})+(\rho-\bar{\rho})\tau+(\mu-\bar{\mu})\kappa-\Psi_{1112}+\varphi_{11\bar{1}\bar{2}}, \quad (1.12)_{11}$$

$$\delta\alpha-\bar{\delta}\beta=\mu\rho-\lambda\sigma+\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta+\gamma(\rho-\bar{\rho})+\epsilon(\mu-\bar{\mu})$$
$$-\Psi_{1122}+\varphi_{12\bar{1}\bar{2}}+\frac{1}{24}T, \quad (1.12)_{12}$$

$$\delta\lambda-\bar{\delta}\mu=(\rho-\bar{\rho})\nu+(\mu-\bar{\mu})\pi+\mu(\alpha+\bar{\beta})+\lambda(\bar{\alpha}-3\beta)-\Psi_{1222}+\varphi_{22\bar{1}\bar{2}}, \quad (1.12)_{13}$$

$$\delta\nu-\Delta\mu=\mu^2+\lambda\bar{\lambda}+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu+\varphi_{22\bar{2}\bar{2}}, \quad (1.12)_{14}$$

$$\delta\gamma-\Delta\beta=(\tau-\bar{\alpha}-\beta)\gamma+\mu\tau-\sigma\nu-\epsilon\bar{\nu}-\beta(\gamma-\bar{\gamma}-\mu)+\alpha\bar{\lambda}+\varphi_{12\bar{2}\bar{2}}, \quad (1.12)_{15}$$

$$\delta\tau-\Delta\sigma=\mu\sigma+\lambda\rho+(\tau+\beta-\bar{\alpha})\tau-(3\gamma-\bar{\gamma})\sigma-\kappa\bar{\nu}+\varphi_{11\bar{2}\bar{2}}, \quad (1.12)_{16}$$

$$\Delta\rho-\bar{\delta}\tau=-(\rho\bar{\mu}+\lambda\sigma)+(\bar{\beta}-\alpha-\bar{\tau})\tau+(\gamma+\bar{\gamma})\rho+\nu\kappa-\Psi_{1122}-\frac{1}{12}T, \quad (1.12)_{17}$$

$$\Delta\alpha-\bar{\delta}\gamma=(\rho+\epsilon)\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+(\bar{\beta}-\bar{\tau})\gamma-\Psi_{1222}. \quad (1.12)_{18}$$

其次是 Bianchi 恒等式[1)]:

$$D\Psi_{1112}-\bar{\delta}\Psi_{1111}+(-\pi+4\alpha)\Psi_{1111}-(2\epsilon+4\rho)\Psi_{1112}+3\kappa\Psi_{1122}$$
$$=D\varphi_{11\bar{1}\bar{2}}-\delta\varphi_{11\bar{1}\bar{1}}+(-\bar{\pi}+2\bar{\alpha}+2\beta)\varphi_{11\bar{1}\bar{1}}-2(\epsilon+\bar{\rho})\varphi_{11\bar{1}\bar{2}}$$
$$+\bar{\kappa}\varphi_{11\bar{2}\bar{2}}-2\sigma\varphi_{12\bar{1}\bar{1}}+2\kappa\varphi_{12\bar{1}\bar{2}}, \quad (1.13)_1$$

$$D\Psi_{1122}-\bar{\delta}\Psi_{1112}+\frac{1}{12}DT+\lambda\Psi_{1111}+(-2\pi+2\alpha)\Psi_{1112}-3\rho\Psi_{1122}+2\kappa\Psi_{1222}$$
$$=\bar{\delta}\varphi_{11\bar{1}\bar{2}}-\Delta\varphi_{11\bar{1}\bar{1}}+(2\gamma+2\bar{\gamma}-\bar{\mu})\varphi_{11\bar{1}\bar{1}}-2(\alpha+\bar{\tau})\varphi_{11\bar{1}\bar{2}}$$
$$+\bar{\sigma}\varphi_{11\bar{2}\bar{2}}-2\tau\varphi_{12\bar{1}\bar{1}}+2\rho\varphi_{12\bar{1}\bar{2}}, \quad (1.13)_2$$

$$D\Psi_{1222}-\bar{\delta}\Psi_{1122}+\frac{1}{24}\bar{\delta}T+2\lambda\Psi_{1112}-3\pi\Psi_{1122}+2(\epsilon-\rho)\Psi_{1222}+\kappa\Psi_{2222}$$
$$=\bar{\delta}\varphi_{12\bar{1}\bar{2}}-\Delta\varphi_{12\bar{1}\bar{1}}+\nu\varphi_{11\bar{1}\bar{1}}-\lambda\varphi_{11\bar{1}\bar{2}}+(2\bar{\gamma}-\bar{\mu})\varphi_{12\bar{1}\bar{1}}-2\bar{\tau}\varphi_{12\bar{1}\bar{2}}$$
$$+\bar{\sigma}\varphi_{12\bar{2}\bar{2}}-\tau\varphi_{22\bar{1}\bar{1}}+\rho\varphi_{22\bar{1}\bar{2}}, \quad (1.13)_3$$

$$D\Psi_{2222}-\bar{\delta}\Psi_{1222}+3\lambda\Psi_{1122}-2(\alpha+2\pi)\Psi_{1222}+(4\epsilon-\rho)\Psi_{2222}$$
$$=\bar{\delta}\varphi_{22\bar{1}\bar{2}}-\Delta\varphi_{22\bar{1}\bar{1}}+2\nu\varphi_{12\bar{1}\bar{1}}-2\lambda\varphi_{12\bar{1}\bar{2}}+(2\bar{\gamma}-2\gamma-\bar{\mu})\varphi_{22\bar{1}\bar{1}}$$
$$+2(\alpha-\bar{\tau})\varphi_{22\bar{1}\bar{2}}+\bar{\sigma}\varphi_{22\bar{2}\bar{2}}, \quad (1.13)_4$$

$$\delta\Psi_{1112}-\Delta\Psi_{1111}+(4\gamma-\mu)\Psi_{1111}-2(\beta+2\tau)\Psi_{1112}+3\sigma\Psi_{1122}$$
$$=D\varphi_{11\bar{2}\bar{2}}-\delta\varphi_{11\bar{1}\bar{2}}+\bar{\lambda}\varphi_{11\bar{1}\bar{1}}+2(\beta-\bar{\pi})\varphi_{11\bar{1}\bar{2}}+(2\bar{\epsilon}-2\epsilon-\bar{\rho})\varphi_{11\bar{2}\bar{2}}$$
$$-2\sigma\varphi_{12\bar{1}\bar{2}}+2\kappa\varphi_{12\bar{2}\bar{2}}, \quad (1.13)_5$$

$$\delta\Psi_{1122}-\Delta\Psi_{1112}+\frac{1}{12}\delta T+\nu\Psi_{1111}+2(\gamma-\mu)\Psi_{1112}-3\tau\Psi_{1122}+2\sigma\Psi_{1222}$$
$$=\bar{\delta}\varphi_{11\bar{2}\bar{2}}-\Delta\varphi_{11\bar{1}\bar{2}}+\bar{\nu}\varphi_{11\bar{1}\bar{1}}+2(\gamma-\bar{\mu})\varphi_{11\bar{1}\bar{2}}-(2\alpha-2\bar{\beta}+\bar{\tau})\varphi_{11\bar{2}\bar{2}}$$
$$-2\tau\varphi_{12\bar{1}\bar{2}}+2\rho\varphi_{12\bar{2}\bar{2}}, \quad (1.13)_6$$

1) Newman-Penrose[5] 只给出真空场与电磁场的 Bianchi 恒等式．而 Pirani[4] 的讲义中，所引用的 Bianchi 恒等式与这里有所不同．原因是那里所引用的恒等式共有 11 个之多，并非彼此独立的，而独立的 Bianchi 恒等式只有 8 个．

$$\delta\Psi_{1222}-\Delta\Psi_{1122}+\frac{1}{24}\Delta T+2\nu\Psi_{1112}-3\mu\Psi_{1122}+(2\beta-2\tau)\Psi_{1222}+\sigma\Psi_{2222}$$
$$=\bar\delta\varphi_{12\bar2\bar2}-\Delta\varphi_{12\bar1\bar2}+\nu\varphi_{11\bar1\bar2}-\lambda\varphi_{11\bar2\bar2}+\bar\nu\varphi_{12\bar1\bar1}-2\bar\mu\varphi_{12\bar1\bar2}+(2\bar\beta-\bar\tau)\varphi_{12\bar2\bar2}$$
$$-\tau\varphi_{22\bar1\bar2}+\rho\varphi_{22\bar2\bar2}, \tag{1.13$_7$}$$

$$\delta\Psi_{2222}-\Delta\Psi_{1222}+3\nu\Psi_{1122}-2(\gamma+2\mu)\Psi_{1222}+(4\beta-\tau)\Psi_{2222}$$
$$=\bar\delta\varphi_{22\bar2\bar2}-\Delta\varphi_{22\bar1\bar2}+2\nu\varphi_{12\bar1\bar2}-2\lambda\varphi_{12\bar2\bar2}+\bar\nu\varphi_{22\bar1\bar1}-2(\gamma+\bar\mu)\varphi_{22\bar1\bar2}$$
$$+(2\alpha+2\bar\beta-\bar\tau)\varphi_{22\bar2\bar2}. \tag{1.13$_8$}$$

此外还要利用微分算子 D, Δ, δ, $\bar\delta$ 的对易关系:

$$\Delta D-D\Delta=(\gamma+\bar\gamma)D+(\varepsilon+\bar\varepsilon)\Delta-(\bar\tau+\pi)\delta-(\tau+\bar\pi)\bar\delta, \tag{1.14$_1$}$$
$$\delta D-D\delta=(\bar\alpha+\beta-\bar\pi)D+\kappa\Delta-(\bar\rho+\varepsilon-\bar\varepsilon)\delta-\sigma\bar\delta, \tag{1.14$_2$}$$
$$\delta\Delta-\Delta\delta=-\bar\nu D+(\tau-\bar\alpha-\beta)\Delta+(\mu-\gamma+\bar\gamma)\delta+\bar\lambda\bar\delta, \tag{1.14$_3$}$$
$$\bar\delta\delta-\delta\bar\delta=-(\mu-\bar\mu)D-(\rho-\bar\rho)\Delta+(\alpha-\bar\beta)\delta-(\bar\alpha-\beta)\bar\delta. \tag{1.14$_4$}$$

在证明过程中，经常要选取合适的拟正交标架，这相应于对旋量作么模群 $SL(2, C)$ 的变换. 熟知 $SL(2, C)$ 可以由下面的四种变换生成:

$$\begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \quad A>0; \qquad \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+\frac{i}{2}\theta} \end{pmatrix} \quad \theta \text{ 为实};$$
$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad B \text{ 为复}; \qquad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

相应于这四种变换的拟正交标架的变换关系，以及旋量系数 $\Gamma_{ABC\bar D}$ 的变换关系如下:

(1) 称为类时旋转

$$\begin{aligned}
&\tilde e^i_{(0)}=Ae^i_{(0)}, && \tilde e^i_{(1)}=A^{-1}e^i_{(1)}, && \tilde e^i_{(2)}=e^i_{(2)},\\
&\tilde\kappa=A^2\kappa, && \tilde\varepsilon=A\varepsilon+\frac{1}{2}DA, && \tilde\pi=\pi,\\
&\tilde\rho=A\rho, && \tilde\alpha=\alpha+\frac{1}{2}A^{-1}\bar\delta A, && \tilde\lambda=A^{-1}\lambda,\\
&\tilde\sigma=A\sigma, && \tilde\beta=\beta+\frac{1}{2}A^{-1}\delta A, && \tilde\mu=A^{-1}\mu,\\
&\tilde\tau=\tau, && \tilde\gamma=A^{-1}\gamma+\frac{1}{2}A^{-2}\Delta A, && \tilde\nu=A^{-2}\nu.
\end{aligned} \tag{1.15}$$

(2) 称为类空旋转

$$\begin{aligned}
&\tilde e^i_{(0)}=e^i_{(0)}, && \tilde e^i_{(1)}=e^i_{(1)}, && \tilde e^i_{(2)}=e^{i\theta}e^i_{(2)},\\
&\tilde\kappa=e^{i\theta}\kappa, && \tilde\varepsilon=\varepsilon+\frac{i}{2}D\theta, && \tilde\pi=e^{-i\theta}\pi,\\
&\tilde\rho=\rho, && \tilde\alpha=e^{-i\theta}\left(\alpha+\frac{i}{2}\bar\delta\theta\right), && \tilde\lambda=e^{-2i\theta}\lambda,\\
&\tilde\sigma=e^{2i\theta}\sigma, && \tilde\beta=e^{i\theta}\left(\beta+\frac{i}{2}\delta\theta\right), && \tilde\mu=\mu,\\
&\tilde\tau=e^{i\theta}\tau, && \tilde\gamma=\gamma+\frac{i}{2}\Delta\theta, && \tilde\nu=e^{-i\theta}\nu.
\end{aligned} \tag{1.16}$$

（3）称为类光旋转或零旋转

$$
\begin{aligned}
&\tilde{e}^j_{(0)} = e^j_{(0)},\\
&\tilde{e}^j_{(1)} = e^j_{(1)} + B\bar{B}e^j_{(0)} - Be^j_{(2)} - \bar{B}\bar{e}^j_{(2)},\\
&\tilde{e}^j_{(2)} = e^j_{(2)} - \bar{B}e^j_{(0)},\\
&\tilde{\kappa} = \kappa,\\
&\tilde{\rho} = \rho - B\kappa,\\
&\tilde{\sigma} = \sigma - \bar{B}\kappa,\\
&\tilde{\tau} = \tau - B\sigma - \bar{B}\rho + B\bar{B}\kappa,\\
&\tilde{\epsilon} = \epsilon - B\kappa,\\
&\tilde{\alpha} = \alpha - B(\rho + \epsilon) + B^2\kappa,\\
&\tilde{\beta} = \beta - B\sigma - \bar{B}\epsilon + B\bar{B}\kappa,\\
&\tilde{\gamma} = \gamma - B(\tau + \beta) - \bar{B}\alpha + B^2\sigma + B\bar{B}(\rho + \epsilon) - B^2\bar{B}\kappa,\\
&\tilde{\pi} = \pi - 2B\epsilon + B^2\kappa - DB,\\
&\tilde{\lambda} = \lambda - (2\alpha + \pi)B + B^2(2\epsilon + \rho) - B^3\kappa - (\bar{\delta} - BD)B,\\
&\tilde{\mu} = \mu - 2B\beta - \bar{B}\pi + B^2\sigma + 2B\bar{B}\epsilon - B^2\bar{B}\kappa - (\delta - \bar{B}D)B,\\
&\tilde{\nu} = \nu - B(2\gamma + \mu) - \bar{B}\lambda + B^2(\tau + 2\beta) + B\bar{B}(2\alpha + \pi) - B^3\sigma\\
&\qquad - B^2\bar{B}(\rho + 2\epsilon) + B^3\bar{B}\kappa - (\Delta - B\delta - \bar{B}\bar{\delta} + B\bar{B}D)B.
\end{aligned}
\tag{1.17}
$$

（4）称为倒换

$$
\begin{aligned}
&\tilde{e}^j_{(0)} = e^j_{(1)}, &&\tilde{e}^j_{(1)} = e^j_{(0)}, &&\tilde{e}^j_{(2)} = -\bar{e}^j_{(2)},\\
&\tilde{\kappa} = \nu, &&\tilde{\epsilon} = \gamma, &&\tilde{\pi} = \tau,\\
&\tilde{\rho} = \mu, &&\tilde{\alpha} = \beta, &&\tilde{\lambda} = \sigma,\\
&\tilde{\sigma} = \lambda, &&\tilde{\beta} = \alpha, &&\tilde{\mu} = \rho,\\
&\tilde{\tau} = \pi, &&\tilde{\gamma} = \epsilon, &&\tilde{\nu} = \kappa.
\end{aligned}
\tag{1.18}
$$

二、标量-张量引力波的一般性质

我们取标架向量

$$e^j_{(0)} = g^{jk}\frac{\partial\varphi}{\partial x^k}. \tag{2.1}$$

易见由方程

$$\frac{dx^j}{dr} = e^j_{(0)} \tag{2.2}$$

所确定的曲线簇，据条件 (1.4)，是一零 (Null) 短程线簇．这里选取坐标

$$x^0 = \varphi, \qquad x^1 = r, \tag{2.3}$$

于是有

$$e^j_{(0)} = \delta^j_1, \qquad e^0_{(a)} = \delta^1_a. \tag{2.4}$$

末一式是由 (2.1) 得出的 $g^{j0} = \delta^j_1$ 及 (1.5) 推得的．

命 $e^j_{(0)}$ 对应的旋量为

$$\epsilon^A\epsilon^{\bar{B}} = \sigma_0^{A\bar{B}}.$$

据 (1.6), 不妨假定 $\epsilon^A = \delta^A_1$. 由此可知, $\frac{\partial\varphi}{\partial x^k}$ 对应的旋量为

$$\nabla_{A\bar{C}}\varphi = \epsilon_A \bar{\epsilon}_C, \qquad \epsilon_A = \delta^2_A. \tag{2.5}$$

其中 $\nabla_{A\bar{C}}$ 表旋量的协变微分. 于是有

$$\begin{aligned}\nabla_{B\bar{D}}\nabla_{A\bar{C}}\varphi &= \bar{\epsilon}_C(X_{B\bar{D}}\epsilon_A - \epsilon^{PQ}\Gamma_{QAB\bar{D}}\epsilon_P) + \epsilon_A(X_{B\bar{D}}\bar{\epsilon}_C - \epsilon^{\bar{P}\bar{Q}}\overline{\Gamma_{QCD\bar{B}}}\bar{\epsilon}_P) \\ &= \Gamma_{1AB\bar{D}}\delta^2_C + \overline{\Gamma_{1CD\bar{B}}}\delta^2_A.\end{aligned} \tag{2.6}$$

另一方面, 由 $\varphi_{;jk} = \varphi_{;kj}$ 可知

$$\nabla_{B\bar{D}}\nabla_{A\bar{C}}\varphi = \nabla_{A\bar{C}}\nabla_{B\bar{D}}\varphi. \tag{2.7}$$

利用 (1.10) 的符号, 从上两式知, 必须

$$\kappa = \epsilon + \bar{\epsilon} = \bar{\rho} - \rho = \bar{\alpha} + \beta - \tau = 0. \tag{2.8}$$

此外, 方程 (1.3) 的旋量形式为

$$\epsilon^{AB}\epsilon^{\bar{C}\bar{D}}\nabla_{B\bar{D}}\nabla_{A\bar{C}}\varphi = 0.$$

以 (2.6) 与 (2.8) 代入上式得出

$$\rho + \bar{\rho} = 0,$$

故有

$$\rho = 0. \tag{2.9}$$

由于本文中能量-动量张量 T_{jk} 取 (1.2) 式的右边, 据 (1.7) 及 (1.9) 式,

$$T = 0, \quad \varphi_{AB\bar{C}\bar{D}} = \frac{b}{4}\left[\frac{a}{\varphi^2}(\nabla_{A\bar{C}}\varphi\nabla_{B\bar{D}}\varphi + \nabla_{B\bar{C}}\varphi\nabla_{A\bar{D}}\varphi) + \frac{1}{\varphi}(\nabla_{A\bar{C}}\nabla_{B\bar{D}}\varphi + \nabla_{B\bar{C}}\nabla_{A\bar{D}}\varphi)\right]. \tag{2.10}$$

特别是 $\varphi_{11\bar{1}\bar{1}} = 0$. 由 $(1.12)_1$ 知, 必须

$$\sigma = 0. \tag{2.11}$$

由于 $-\frac{1}{2}(\rho + \bar{\rho})$, $\frac{1}{2}|\rho - \bar{\rho}|$ 以及 $|\sigma|$ 分别代表引力波射线的扩度 (expansion)、挠度 (twist) 及切度 (shear) [例如见文献 [4] 中 (4.19) 与 (4.16) 式], 故得出

结论 1 标量-张量引力波的射线簇必定是无扩、无挠又无切的.

据 (2.5), (2.6) 和 (2.11) 知,

$$\begin{aligned}&\varphi_{12\bar{2}\bar{2}} = \varphi_{21\bar{2}\bar{2}} = \overline{\varphi_{22\bar{1}\bar{2}}} = \overline{\varphi_{12\bar{2}\bar{2}}} = \frac{b\tau}{2\varphi}, \\ &\varphi_{22\bar{2}\bar{2}} = \frac{b}{2}\left[\frac{a}{\varphi^2} + \frac{1}{\varphi}(\gamma + \bar{\gamma})\right], \quad \text{其它 } \varphi_{AB\bar{C}\bar{D}} = 0.\end{aligned} \tag{2.12}$$

作类空旋转. 由 (1.16) 知, 可选取 θ 使对于新的标架有 $\tilde{\epsilon} - \bar{\tilde{\epsilon}} = \epsilon - \bar{\epsilon} + iD\theta = 0$, 但关系式 (2.8—2.11) 对于新的标架仍然成立, 因此, 不妨假定对于原来的标架就有

$$\kappa = \epsilon = \rho = \sigma = \tau - \bar{\alpha} - \beta = 0. \tag{2.13}$$

而由 $(1.12)_2$ 和 $(1.12)_{11}$ 知, 必须

$$\Psi_{1111} = \Psi_{1112} = 0. \tag{2.14}$$

故下面的齐次四次式的因子分解

$$\Psi_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = (\alpha_1\zeta^1 + \alpha_2\zeta^2)(\beta_1\zeta^1 + \beta_2\zeta^2)(\gamma_1\zeta^1 + \gamma_2\zeta^2)(\delta_1\zeta^1 + \delta_2\zeta^2) \tag{2.15}$$

中，α_1, β_1, γ_1, δ_1 最少有两个为零. 据 Петров 分类的旋量形式[3,4]，Ψ_{ABCD} 必非 I 型，设 $\alpha_1=\beta_1=0$，则旋量 α_A 与 β_A 与 ϵ_A 都成比例. 而由 α_A 定义的向量

$$l^i=\alpha^A\overline{\alpha^B}\sigma^a_{A\dot{B}}e^i_{(a)}$$

（称为外尔张量的主特征方向）必与 $e^i_{(0)}$ 成比例. 换言之，标量-张量引力波的传播方向必与外尔张量的一个主特征方向重合.

结论 2　不存在 I 型的标量-张量引力波（即必定是代数上特殊的），并且标量-张量引力波的传播方向必定与外尔张量的一个多重的主特征方向重合.

为方便起见，引进新的符号 U, $X^n(n=0,2,3)$ 及 ω, $\xi^n(n=0,2,3)$，使得

$$\begin{aligned}e^i_{(1)}&=X^0\delta^i_0+U\delta^i_1+X^2\delta^i_2+X^3\delta^i_3,\\ e^i_{(2)}&=\xi^0\delta^i_0+\omega\delta^i_1+\xi^2\delta^i_2+\xi^3\delta^i_3.\end{aligned}\tag{2.16}$$

由于已选定 $e^i_{(0)}=\delta^i_1$，当把 (1.14) 作用于坐标 x^0 与 $x^n(n=0,2,3)$，得出标架方程

$$DU=-(\gamma+\bar{\gamma})-(\epsilon+\bar{\epsilon})U+(\bar{\tau}+\pi)\omega+(\tau+\bar{\pi})\bar{\omega},\tag{2.17$_1$}$$

$$DX^n=-(\epsilon+\bar{\epsilon})X^n+(\bar{\tau}+\pi)\xi^n+(\tau+\bar{\pi})\bar{\xi}^n,\tag{2.17$_2$}$$

$$D\omega=-(\bar{\alpha}+\beta-\bar{\pi})-\kappa U+(\bar{\rho}+\epsilon-\bar{\epsilon})\omega+\sigma\bar{\omega},\tag{2.17$_3$}$$

$$D\xi^n=-\kappa X^n+(\bar{\rho}+\epsilon-\bar{\epsilon})\xi^n+\sigma\bar{\xi}^n,\tag{2.17$_4$}$$

$$\delta U-\Delta\omega=-\bar{\nu}+(\tau-\bar{\alpha}-\beta)U+(\mu-\gamma+\bar{\gamma})\omega+\bar{\lambda}\bar{\omega},\tag{2.17$_5$}$$

$$\delta X^n-\Delta\xi^n=(\tau-\bar{\alpha}-\beta)X^n+(\mu-\gamma+\bar{\gamma})\xi^n+\bar{\lambda}\bar{\xi}^n,\tag{2.17$_6$}$$

$$\bar{\delta}\omega-\delta\bar{\omega}=-(\mu-\bar{\mu})-(\rho-\bar{\rho})U+(\alpha-\bar{\beta})\omega-(\bar{\alpha}-\beta)\bar{\omega},\tag{2.17$_7$}$$

$$\bar{\delta}\xi^n-\delta\bar{\xi}^n=-(\rho-\bar{\rho})X^n+(\alpha-\bar{\beta})\xi^n-(\bar{\alpha}-\beta)\bar{\xi}^n.\tag{2.17$_8$}$$

在标量-张量引力波的情形，据 (2.4) 第二式知，必须

$$X^0=1,\qquad \xi^0=0.\tag{2.18}$$

此外，由 (2.12—2.14) 知，N-P 方程 (1.12) 化为

$$D\tau=D\alpha=D\beta=0,\qquad \text{即 } \tau,\ \beta,\ \alpha \text{ 不含 } r,\tag{2.19$_1$}$$

$$D\gamma=(\tau+\bar{\pi})\alpha+(\bar{\tau}+\pi)\beta+\tau\pi+\Psi_2,\tag{2.19$_2$}$$

$$D\lambda-\bar{\delta}\pi=\pi^2+(\alpha-\bar{\beta})\pi,\tag{2.19$_3$}$$

$$D\mu-\delta\pi=\pi\bar{\pi}-\pi(\bar{\alpha}-\beta)+\Psi_2,\tag{2.19$_4$}$$

$$D\nu-\Delta\pi=(\pi+\bar{\tau})\mu+(\tau+\bar{\pi})\lambda+(\gamma-\bar{\gamma})\pi+\Psi_3+\bar{\varphi}_1,\tag{2.19$_5$}$$

$$\Delta\lambda-\bar{\delta}\nu=-(\mu+\bar{\mu})\lambda-(3\gamma-\bar{\gamma})\lambda+(3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu-\Psi_4,\tag{2.19$_6$}$$

$$\delta\alpha-\bar{\delta}\beta=\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta-\Psi_2,\tag{2.19$_7$}$$

$$\delta\lambda-\bar{\delta}\mu=(\mu-\bar{\mu})\pi+\mu(\alpha+\bar{\beta})+\lambda(\bar{\alpha}-3\beta)-\Psi_3+\bar{\varphi}_1,\tag{2.19$_8$}$$

$$\delta\nu-\Delta\mu=\mu^2+\lambda\bar{\lambda}+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu+\varphi_2,\tag{2.19$_9$}$$

$$\delta\gamma-\Delta\beta=(\tau-\bar{\alpha}-\beta)\gamma+\mu\tau-\beta(\gamma-\bar{\gamma}-\mu)+\alpha\bar{\lambda}+\varphi_1,\tag{2.19$_{10}$}$$

$$\delta\tau=\tau(\tau+\beta-\bar{\alpha}),\tag{2.19$_{11}$}$$

$$-\bar{\delta}\tau=\tau(\bar{\beta}-\alpha-\bar{\tau})-\Psi_2,\tag{2.19$_{12}$}$$

$$\Delta\alpha-\bar{\delta}\gamma=-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+(\bar{\beta}-\bar{\tau})\gamma-\Psi_3;\tag{2.19$_{13}$}$$

其中

$$\begin{aligned}&\Psi_2=\Psi_{1122},\qquad \Psi_3=\Psi_{1222},\qquad \Psi_4=\Psi_{2222},\\ &\varphi_1=\frac{b\tau}{2\varphi},\qquad \varphi_2=\frac{b}{2}\left[\frac{a}{\varphi^2}+\frac{1}{\varphi}(\gamma+\bar{\gamma})\right].\end{aligned}\tag{2.20}$$

而 Bianchi 恒等式可利用 N-P 方程化为

$$D\Psi_2 = 0, \tag{2.21$_1$}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 - 3\pi\Psi_2 = 0, \tag{2.21$_2$}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 + 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 = 0, \tag{2.21$_3$}$$

$$\delta\Psi_2 - 3\tau\Psi_2 = 0, \tag{2.21$_4$}$$

$$\delta\Psi_3 - \Delta\Psi_2 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 = \frac{b}{2\varphi}\Psi_2, \tag{2.21$_5$}$$

$$\delta\Psi_4 - \Delta\Psi_3 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (\Delta\beta - \tau)\Psi_4 = \frac{b}{2\varphi^2}\left(a + \frac{b}{2} + 1\right)\bar{\tau}. \tag{2.21$_6$}$$

标架方程化为

$$DU = -(\gamma + \bar{\gamma}) + (\bar{\tau} + \pi)\omega + (\tau + \bar{\pi})\bar{\omega}, \tag{2.22$_1$}$$

$$DX^n = (\bar{\tau} + \pi)\xi^n + (\tau + \bar{\pi})\bar{\xi}^n, \tag{2.22$_2$}$$

$$D\omega = -(\bar{\alpha} + \beta - \bar{\pi}), \tag{2.22$_3$}$$

$$D\xi^n = 0, \tag{2.22$_4$}$$

$$\delta U - \Delta\omega = -\bar{\nu} + (\mu - \gamma + \bar{\gamma})\omega + \bar{\lambda}\bar{\omega}, \tag{2.22$_5$}$$

$$\delta X^n - \Delta\xi^n = (\mu - \gamma + \bar{\gamma})\xi^n + \bar{\lambda}\bar{\xi}^n, \tag{2.22$_6$}$$

$$\bar{\delta}\omega - \delta\bar{\omega} = -(\mu - \bar{\mu}) + (\alpha - \bar{\beta})\omega - (\bar{\alpha} - \beta)\bar{\omega}, \tag{2.22$_7$}$$

$$\bar{\delta}\xi^n - \delta\bar{\xi}^n = (\alpha - \bar{\beta})\xi^n - (\bar{\alpha} - \beta)\bar{\xi}^n. \tag{2.22$_8$}$$

命 P 为不含 r 的函数,使得

$$\alpha - \bar{\beta} = \bar{\delta}\log P, \tag{2.23}$$

而命

$$\xi^n = P\eta^n, \tag{2.24}$$

其中 η^n 据 $(2.22)_4$ 是不含 r, 于是 $(2.22)_8$ 化为

$$\sum_{m=0,2,3}\left(\bar{\eta}^m \frac{\partial\eta^n}{\partial x^m} - \eta^m \frac{\partial\bar{\eta}^n}{\partial x^m}\right) = 0.$$

熟知(例如见文献 [8]) 可作适当的坐标变换 $x^n \to f^n(x^0, x^2, x^3)(n = 0, 2, 3)$, 使得对新的坐标有

$$\eta^0 = 0, \qquad \eta^2 = \frac{1}{\sqrt{2}}, \qquad \eta^3 = \frac{i}{\sqrt{2}}. \tag{2.25}$$

据 (1.5) 式,

$$(g^{jk}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2(U - \omega\bar{\omega}) & X^2 - \dfrac{\omega\bar{P} + \bar{\omega}P}{\sqrt{2}} & X^3 - \dfrac{\omega\bar{P} - \bar{\omega}P}{\sqrt{2}\,i} \\ 0 & X^2 - \dfrac{\omega\bar{P} + \bar{\omega}P}{\sqrt{2}} & -P\bar{P} & 0 \\ 0 & X^3 - \dfrac{\omega\bar{P} - \bar{\omega}P}{\sqrt{2}\,i} & 0 & -P\bar{P} \end{pmatrix}. \tag{2.26}$$

因此,最一般的标量-张量引力波的度规如果存在的话,必须是下面的形式:

$$ds^2 = Hd\varphi^2 + 2d\varphi dr - \frac{1}{P\bar{P}}|dz - Kd\varphi|^2, \tag{2.27}$$

其中

$$z = x^2 + ix^3, \qquad K = X^2 + iX^3 - \sqrt{2}\,\omega\bar{P},$$
$$H = -2U + 2\omega\bar{\omega} + \frac{K\bar{K}}{P\bar{P}}. \tag{2.28}$$

总可作适当的类光旋转 (1.17) 使 π 不含 r. 由 $(2.19)_2$ 知, γ 最多是 r 的一次函数, 由 (2.22) 知, ω 与 X^n 亦然, 而 U 最多是 r 的二次函数. 因此, K 最多是 r 的一次函数, H 最多是 r 的二次函数. 当外尔旋量比 II 型更为特殊时, H 与 K 可以更明显的全部确定之. 这将分别在下面讨论.

三、D 型的情形

此时在 (2.15) 的因子分解中, α_A, β_A, γ_A, δ_A 必定可分为两个一组, 每组的两个旋量成比例. 由于假定了 $\alpha_A \sim \beta_A$, 并且 $\alpha_1 = \beta_1 = 0$; 必定有 $\gamma_A \sim \delta_A$. 但 $\gamma_1 \neq 0$, 可作类光旋转使

$$\tilde{\gamma}_1 = \gamma_1, \qquad \tilde{\gamma}_2 = -B\gamma_1 + \gamma_2 = 0.$$

据 (1.17) 可知, 对新的标架, (2.4), (2.13) 和 (2.14) 仍然成立. 因此, 不妨假定原来的标架就有 $\gamma_2 = \delta_2 = 0$. 于是由 (2.15) 知, 必须

$$\Psi_3 = \Psi_{1222} = 0, \qquad \Psi_4 = \Psi_{2222} = 0. \tag{3.1}$$

为方便起见, 可写为

$$\Psi = \Psi_2 = \Psi_{1122} (\neq 0). \tag{3.2}$$

由 $(2.21)_3$ 知, 必须

$$\lambda = 0. \tag{3.3}$$

而 $(2.21)_6$ 化为

$$3\nu\Psi = \frac{b}{2\varphi^2}\left(a + \frac{b}{2} + 1\right)\bar{\tau}. \tag{3.4}$$

由此知 ν 不含 r.

把 $(1.14)_1$ 两边的算符作用于 Ψ, 利用 (2.21) 可得

$$D\mu = \pi\bar{\pi} - \tau\bar{\tau}. \tag{3.5}$$

同理, 以 $(1.14)_4$ 作用于 Ψ, 利用 $(2.19)_{12}$ 得

$$\delta\pi = \pi(\bar{\alpha} - \beta) - \tau\bar{\tau} - \Psi. \tag{3.6}$$

积分 $(2.22)_2$, $(2.19)_2$ 及 (3.5) 有

$$X^n = [(\bar{\tau} + \pi)\xi^n + (\tau + \bar{\pi})\bar{\xi}^n]r + X^{0n},$$
$$\gamma = [(\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta + \tau\pi + \Psi]r + \gamma^0,$$
$$\mu = (\pi\bar{\pi} - \tau\bar{\tau})r + \mu^0,$$

其中 X^{0n}, γ^0, μ^0 不含 r. 由此知

$$\Delta\pi = \sum_{n=0,2,3} X^n \frac{\partial\pi}{\partial x^n} = [(\bar{\tau} + \pi)\delta\pi + (\tau + \bar{\pi})\bar{\delta}\pi]r + \sum_{n=0,2,3} X^{0n}\frac{\partial\pi}{\partial x^n}. \tag{3.7}$$

以此代入 $(2.19)_5$，得

$$0 = \Delta\pi + (\pi + \bar{\tau})\mu + (\gamma - \bar{\gamma})\pi + \frac{b\bar{\tau}}{2\varphi}$$
$$= \{[(\bar{\tau} + \pi)\delta\pi + (\tau + \bar{\pi})\bar{\delta}\pi] + (\bar{\tau} + \pi)(\pi\bar{\pi} - \tau\bar{\tau}) + \pi[(\tau + \bar{\pi})(\alpha - \bar{\beta})$$
$$- (\bar{\tau} + \pi)(\bar{\alpha} - \beta) + \tau\pi - \bar{\tau}\bar{\pi} + \Psi - \bar{\Psi}]\}r + \sum_{n=0,2,3} X^{0n}\frac{\partial\pi}{\partial x^n}$$
$$+ (\bar{\tau} + \pi)\mu^0 + \pi(\gamma^0 - \bar{\gamma}^0) + \frac{b\bar{\tau}}{2\varphi}.$$

把上式中的 $\bar{\delta}\pi$ 与 $\delta\pi$ 值用 $(2.19)_3$ 及 (3.6) 代入，并比较含 r 项的系数得

$$2(\bar{\tau} + \pi)\tau\bar{\tau} + \bar{\tau}\Psi + \pi\bar{\Psi} = 0.$$

以 $(\tau + \bar{\pi})$ 乘上式而比较两边的虚部得出

$$(\tau\bar{\tau} - \pi\bar{\pi})(\Psi - \bar{\Psi}) = 0. \tag{3.8}$$

故必须 $\Psi = \bar{\Psi}$ 或 $\pi\bar{\pi} = \tau\bar{\tau}$. 若前者成立，则据 $(2.21)_2$ 与 $(2.21)_4$ 知，$\tau = -\bar{\pi}$，后者亦必成立. 因此恒有

$$\pi\bar{\pi} = \tau\bar{\tau}. \tag{3.9}$$

由 (3.5) 知 μ 不含 r.

以 $\bar{\delta}$ 作用 (3.4) 的两边得

$$3(\Psi\bar{\delta}\nu + \nu\bar{\delta}\Psi) = \frac{b\left(a + \frac{1}{2}b + 1\right)}{2\varphi^2}\bar{\delta}\bar{\tau}.$$

以 $(2.19)_6$，$(2.19)_{11}$ 及 $(2.21)_2$ 代入上式，得

$$\frac{b\left(a + \frac{1}{2}b + 1\right)}{2\varphi^2}\bar{\tau}(-2\alpha - 4\pi) = \frac{b\left(a + \frac{1}{2}b + 1\right)}{2\varphi^2}2\bar{\tau}\bar{\beta}.$$

由于 a 与 b 皆假定为正数，由上式知，必须

$$\bar{\tau}(\bar{\tau} + 2\pi) = 0.$$

故或者 $\tau = 0$，或者 $\bar{\tau} = -2\pi$. 如果后者成立，则根据 (3.9) 必须 $\tau = 0$，即恒有 $\tau = 0$. 据 $(2.19)_{12}$ 知，$\Psi = 0$，这就不是 D 型. 因此，不存在 D 型的标量-张量引力波.

结论 3 标量-张量引力波存在的必要条件，是外尔张量的任一个多重的主特征方向与标量-张量引力波的传播方向重合.

四、III 型、N 型、O 型的标量-张量引力波 ($\tau = 0$)

当外尔张量为 III 型或更特殊的情形(即 N 型、O 型)时，(2.15) 的分解式中 α_A，β_A，γ_A，δ_A 最少有 3 个成比例，设 $\alpha_A \sim \beta_A \sim \gamma_A$，于是有 $\alpha_1 = \beta_1 = \gamma_1 = 0$. 由此知，必须

$$\Psi_2 = \Psi_{1122} = 0. \tag{4.1}$$

作不含 r 的类空旋转，使

$$\tau = \bar{\tau}. \tag{4.2}$$

取 $(2.19)_{11}$ 的复共轭与 $(2.19)_{12}$ 相加得

$$2\tau(\bar{\beta} - \alpha) = 0. \tag{4.3}$$

由此知，或者 $\tau = 0$，或者 $\bar{\beta} = \alpha$.

作类光旋转 (1.17)，使得关系式 (2.4)，(2.12—2.14)，(4.1—4.2) 仍然保持. 可选取 B 使 $\tilde{\pi} = \pi - DB = 0$. 因此，不妨假定原来就有

$$\pi = 0. \tag{4.4}$$

现在讨论 $\tau = 0$ 的情况

此时

$$\alpha + \bar{\beta} = 0. \tag{4.5}$$

由 $(2.19)_{2-4}$ 与 $(2.22)_{1-3}$ 知，此时

$$\gamma,\ \lambda,\ \mu,\ X^n \text{ 和 } \omega \text{ 不含 } r,$$
$$U = U_0 - (\gamma + \bar{\gamma})r, \qquad U_0 \text{ 不含 } r. \tag{4.6}$$

作类光旋转 (1.17)，取 B 不含 r，并使 $\tilde{\lambda} = \lambda - 2\alpha B - \bar{\delta} B = 0$，即不妨假定

$$\lambda = 0. \tag{4.7}$$

作坐标变换 $x^0 \to x^0,\ x^n \to f^n(x^0,\ x^2,\ x^3)(n = 2,\ 3)$，使得

$$X^n = \delta_0^n. \tag{4.8}$$

取 P 为不含 r 的函数适合

$$\Delta P = -(\mu - \gamma + \bar{\gamma})P. \tag{4.9}$$

而命

$$\xi^n = P\eta^n. \tag{4.10}$$

其中 η^n 不含 r. 以此代入 $(2.22)_6$ 得出

$$\Delta \eta^n = 0.$$

此示 η^n 不含 x^0，因之可作坐标变换 $x^n \to g^n(x^2,\ x^3)$ $(n = 2,\ 3)$，使得 $\eta^2 = \frac{1}{\sqrt{2}} P_1$，$\eta^3 = iP_1/\sqrt{2}$，其中 P_1 仅包含 x^2，x^3. 但把 P 换为 PP_1 时，方程 (4.9) 仍成立. 因此不妨假定 $P_1 = 1$，即

$$\eta^2 = \frac{1}{\sqrt{2}}, \qquad \eta^3 = \frac{i}{\sqrt{2}}. \tag{4.11}$$

以 (4.10) 代入 $(2.22)_8$ 得

$$\sqrt{2}\left(\eta^n \bar{P}\ \frac{\partial P}{\partial z} - \bar{\eta}^n P\ \frac{\partial \bar{P}}{\partial \bar{z}}\right) = 2\alpha P\eta^n - 2\bar{\alpha}\bar{P}\bar{\eta}^n. \tag{4.12}$$

其中

$$z = x^2 + ix^3,$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x^2} - \frac{i}{2}\frac{\partial}{\partial x^3}, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x^2} + \frac{i}{2}\frac{\partial}{\partial x^3}.$$

在 (4.12) 中取 $n = 2$，并与 $n = 3$ 时的式除以 i 相加得

$$\alpha = \frac{1}{\sqrt{2}}\bar{P}\ \frac{\partial \log P}{\partial z}. \tag{4.13}$$

以此代入 $(2.19)_7$ 知，$P\bar{P}$ 适合

$$\frac{\partial^2 \log P\bar{P}}{\partial z\, \partial \bar{z}} = 0. \tag{4.14}$$

由此知，存在对 $\bar{z}$ 的解析函数 Q，使

$$\log P\bar{P} = \log Q + \log \bar{Q} = \log Q\bar{Q}.$$

故有

$$P = Qe^{i\theta_0}. \tag{4.15}$$

其中 θ_0 为 x^2, x^3 的实函数．由此知，(4.13) 可写为

$$\alpha = \frac{i}{\sqrt{2}}\bar{P}\frac{\partial\theta_0}{\partial z}. \tag{4.16}$$

作类空旋转 (1.16) 有

$$\tilde{\xi}^n = e^{i\theta}\xi^n = e^{i\theta}Qe^{i\theta_0}\eta^n,$$
$$\tilde{\bar{\alpha}} + \tilde{\beta} = (\bar{\alpha} + \beta)e^{i\theta},$$
$$\tilde{\alpha} = e^{-i\theta}\left(\alpha + \frac{i}{2}\bar{\delta}\theta\right) = \frac{i\bar{P}}{\sqrt{2}}e^{-i\theta}\frac{\partial}{\partial z}(\theta_0 + \theta).$$

取 $\theta = -\theta_0$，这是不包含 x^0 与 r，使得原来已成立的关系式不变．因此，不妨假定有

$$\alpha = \beta = 0, \qquad P = Q \qquad \text{对 } \bar{z} \text{ 解析}. \tag{4.17}$$

由 $(2.19)_{10}$ 与 $(2.19)_{13}$ 得

$$\bar{\Psi}_3 = \delta(\gamma + \bar{\gamma}).$$

而 $(2.19)_8$ 化为

$$\Psi_3 = \bar{\delta}\mu, \tag{4.18}$$

因此有

$$\bar{\delta}(\mu - \gamma - \bar{\gamma}) = 0.$$

此示

$$\mu = \gamma + \bar{\gamma} + \phi.$$

其中 $\phi = \phi(x^0, \bar{z})$ 对 $\bar{z}$ 是复解析的．

作类光旋转 (1.17)，取 B 是不含 r 而对 $\bar{z}$ 复解析的函数，使适合 $\delta B = \phi$，则由 (1.17) 知

$$\tilde{\gamma} = \gamma, \qquad \tilde{\lambda} = \lambda - \bar{\delta}B = \lambda = 0,$$
$$\tilde{\mu} = \mu - \delta B = \gamma + \bar{\gamma} + \phi - \delta B = \tilde{\gamma} + \tilde{\bar{\gamma}}.$$

故不妨假定原来就有

$$\mu = \gamma + \bar{\gamma}. \tag{4.19}$$

但此时 (4.8) 的 X^n 变为

$$X^n = \delta^n_0 - BP\eta^n - \bar{B}\bar{\eta}^n\bar{P},$$

而 ξ^n 不变，即 (4.10) 仍然成立．由此知，$X^2 + iX^3 = -\sqrt{2}\,\bar{B}\bar{P}$ 对 z 是解析的，故存在坐标变换 $Z \to F(x^0, z)$，对 z 是解析的，使得 $X^n \to \delta^n_0$, $\xi^n \to \frac{\partial\bar{F}}{\partial\bar{z}}\xi^n = \frac{\partial\bar{F}}{\partial\bar{z}}P\eta^n$, $P\frac{\partial\bar{F}}{\partial\bar{z}}$ 仍然对 $\bar{z}$ 解析．因此，可假定原来的坐标与标架就使 (4.19) 成立．

若命 $\omega = P\omega_1$，则 $(2.22)_7$ 化为

$$\frac{\partial\omega_1}{\partial z} - \frac{\partial\bar{\omega}_1}{\partial\bar{z}} = 0. \tag{4.20}$$

因此，存在不含 r 的实函数 h，使

$$\omega_1 = \frac{\partial h}{\partial \bar{z}}, \qquad 即 \quad \omega = P\frac{\partial h}{\partial \bar{z}}.$$

作坐标变换 $r \to r - \frac{1}{\sqrt{2}} h(x^0, x^2, x^3)$，使得对新的坐标有

$$\omega = 0. \tag{4.21}$$

由 (4.9) 与 (4.19) 知

$$\Delta \log P\bar{P} = -(\mu + \bar{\mu}) = -2\mu. \tag{4.22}$$

由 $(2.22)_5$ 知

$$\delta U = -\bar{\nu}.$$

以 (4.6) 代入上式，有

$$\delta U_0 = \delta\mu - \bar{\nu}.$$

由此知

$$\bar{\delta}\delta U_0 = \bar{\delta}\delta\mu - \overline{\delta\nu}.$$

由 (4.22) 与 (4.14) 知上式右边第一项为零，第二项以 $(2.19)_{10}$ 代入，并利用 (4.22) 得

$$\frac{\partial^2 U_0}{(\partial x^2)^2} + \frac{\partial^2 U_0}{(\partial x^3)^2} = -\frac{\partial^2}{\partial\varphi^2}\frac{1}{P\bar{P}} - \frac{b}{2\varphi}\frac{\partial}{\partial\varphi}\frac{1}{P\bar{P}} - \frac{ab}{\varphi^2 P\bar{P}}. \tag{4.23}$$

据 (2.31) 与 (2.32)，此时度规为

$$ds^2 = -2Ud\varphi^2 + 2d\varphi dr - |f|^2|dz|^2, \tag{4.24}$$

其中

$$\begin{aligned} &f = \frac{1}{P} \text{ 是不含 } r \text{ 而对 } z \text{ 解析的任意函数,} \\ &U = U_0 - r\frac{\partial}{\partial\varphi}\log|f|, \text{ 其中 } U_0 \text{ 是不含 } r \text{ 的任一函数适合} \\ &\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = -\frac{\partial^2|f|^2}{\partial\varphi^2} - \frac{b}{2\varphi}\frac{\partial|f|^2}{\partial\varphi} - \frac{ab|f|^2}{\varphi^2}. \end{aligned} \tag{4.25}$$

这里 $x = x^2$, $y = x^3$.

现考虑比 III 型更特殊的情形：

(1) 设外尔张量为 N 型

此时 (2.15) 右边中 $\alpha_A \sim \beta_A \sim \gamma_A \sim \delta_A$，故 $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$. 因此

$$\Psi_3 = \Psi_{1222} = 0. \tag{4.26}$$

由 (4.18) 知 $\bar{\delta}\mu = 0$，即 μ 是 $\bar{z}$ 的解析函数. 可作类光旋转 (1.17)，取 B 是 $\bar{z}$ 的解析函数，使 $\tilde{\mu} = \mu - \delta B = 0$. 并作坐标变换 $Z \to F(x^0, z)$，对 z 解析，使 (4.8) 仍成立，即不妨假定

$$\mu = 0. \tag{4.27}$$

据 $(2.19)_{10,13}$ 有 $\delta\gamma = \bar{\delta}\gamma = 0$，故 γ 仅包含 x^0. 作类时旋转 (1.15)，取 A 仅包含 x^0，使 $\tilde{\gamma} + \bar{\tilde{\gamma}} = 0$；又作类空旋转 (1.16)，取 θ 仅包含 x^0，使 $\tilde{\gamma} - \bar{\tilde{\gamma}} = 0$. 注意此时 (4.17) 仍成立，故不妨假定

$$\gamma = 0. \tag{4.28}$$

由(4.9)知，P不含x^0，因之$f=\bar{P}^{-1}$亦然．可作坐标变换$z\to\int f dz$，使得度规化为

$$ds^2=-2Ud\varphi^2+2d\varphi dr-|dz|^2, \tag{4.29}$$

其中U为不含r的函数，适合方程

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=-\frac{ab}{\varphi^2}. \tag{4.30}$$

这证明标量-张量引力波中，pp波亦存在；特别当$b\to 0$时，这就是熟知的广义相对论真空场的pp波(例如见 Ehlers 及 Kundt 的文章[9])．

(2) 当外尔张量为O型

此时

$$\Psi_4=\Psi_{2222}=0. \tag{4.31}$$

由$(2.19)_6$知$\bar{\delta}\nu=0$，即ν是$\bar{z}$的解析函数．由$(2.19)_9$知

$$\delta\nu=\varphi_2=\frac{ab}{2\varphi^2}.$$

积分之有

$$\nu=\frac{ab}{2\sqrt{2}\,\varphi^2}\bar{z}+\sqrt{2}A(x^0).$$

其中A仅包含x^0，以此代入$(2.22)_5$得

$$\sqrt{2}\frac{\partial U}{\partial\bar{z}}=\frac{-ab}{2\sqrt{2}\,\varphi^2}z-\sqrt{2}\bar{A}.$$

注意U为实，积分之有

$$U=-\frac{ab}{4\varphi^2}z\bar{z}-Az-\bar{A}\bar{z}-B, \tag{4.32}$$

其中A与B为仅含x^0的任意函数．故O型的度规为

$$ds^2=2\left[\frac{ab}{4\varphi^2}z\bar{z}+Az+\bar{A}\bar{z}+B\right]d\varphi^2+2d\varphi dr-|dz|^2. \tag{4.33}$$

五、III型、N型、O型的标量-张量引力波($\tau\neq 0$)

讨论$\tau\neq 0$的情况

此时由(4.3)知必须

$$\alpha=\bar{\beta}, \tag{5.1}$$

因此$(2.22)_8$化为

$$\bar{\delta}\xi^n-\delta\bar{\xi}^n=0.$$

故存在坐标变换$x^n\to f^n(x^0,x^2,x^3)(n=2,3)$，使得

$$\xi^2=\frac{1}{\sqrt{2}},\quad \xi^3=\frac{i}{\sqrt{2}}. \tag{5.2}$$

由$(2.19)_{11}$知$\delta\tau=\tau^2$，积分之有(注意τ为实)

$$\tau=-\frac{1}{\sqrt{2}(x^2+c)}.$$

其中 c 仅包含 x^0，不妨假定 $c=0$，否则作坐标变换 $x^2\to x^2+c$ 使之如此，即

$$\tau=-\frac{1}{\sqrt{2}\,x^2}. \tag{5.3}$$

据 (5.1) 及 (2.13) 知

$$\alpha=\beta=\frac{1}{2}\tau=-\frac{1}{2}\delta\log\tau^{-1}=-\frac{1}{2}\bar{\delta}\log\tau^{-1}.$$

作类时旋转 (1.15)，选取 A 不含 r，并同时作坐标变换 $r\to rA^{-1}$，使得 (2.4) 变为

$$e^i_{(0)}\to e^i_{(0)}=\delta^i_1,\qquad e^0_{(a)}\to A^{-1}\delta^0_a.$$

据 (1.15) 知

$$\tilde{\alpha}=\alpha+\frac{1}{2}A^{-1}\bar{\delta}A=-\frac{1}{2}\bar{\delta}(\log\tau^{-1}-\log A),$$

$$\tilde{\beta}=\beta+\frac{1}{2}A^{-1}\delta A=-\frac{1}{2}\delta(\log\tau^{-1}-\log A).$$

故不妨假定

$$\alpha=\beta=0. \tag{5.4}$$

注意此时 $\tau-\bar{\alpha}-\beta\neq 0$，但仍有

$$\kappa=\epsilon=\rho=\sigma=\pi=0. \tag{5.5}$$

而经此旋转

$$\nabla_{A\bar{B}}\varphi=\tau\delta^2_A\delta^2_B,\quad \varphi_{12\bar{2}\bar{2}}=\varphi_{21\bar{2}\bar{2}}=\overline{\varphi_{22\bar{1}\bar{2}}}=\overline{\varphi_{12\bar{2}\bar{2}}}=\frac{b\tau^2}{2\varphi},$$

$$\varphi_{22\bar{2}\bar{2}}=\text{实},\quad \text{其它 } \varphi_{AB\bar{C}\bar{D}}=0. \tag{5.6}$$

而 (2.4) 变为

$$e^i_{(0)}=\delta^i_1,\qquad e^0_{(a)}=\tau\delta^0_a. \tag{5.7}$$

作不含 r 的类光旋转 (1.17)，取 B 使 $\tilde{\gamma}=\gamma-B\tau=0$，即不妨假定

$$\gamma=0. \tag{5.8}$$

由 $(2.19)_{10}$ 可得

$$\mu=-\frac{b\tau}{2\varphi}. \tag{5.9}$$

又 $(2.19)_{13}$ 即

$$\Psi_3=-\tau\lambda, \tag{5.10}$$

因此 $(2.19)_5$ 化为

$$D\nu=0,\quad \text{即 } \nu \text{ 不含 } r. \tag{5.11}$$

据 $(2.17)_3$，ω 不含 r，而据 $(2.17)_7$ 有

$$\bar{\delta}\omega-\delta\bar{\omega}=0.$$

故存在不含 r 的实函数 h，使 $\omega=\delta h$，因而可作坐标变换 $r\to r-h$，使

$$\omega=0. \tag{5.12}$$

积分 $(2.17)_2$ 有

$$X^n=\tau(\xi^n+\bar{\xi}^n)r+X^{0n},\quad X^{0n}\text{ 不含 } r. \tag{5.13}$$

以此代入 $(2.17)_6$ 比较 r 的系数有

$$\delta\left(\frac{X^{0n}}{\tau}\right)=\frac{\mu}{\tau}\xi^n+\frac{\bar{\lambda}}{\tau}\bar{\xi}^n. \tag{5.14}$$

由此知

$$\delta\left(\frac{X^{02}-iX^{03}}{\tau}\right)=-\frac{b}{\sqrt{2}\,\varphi},$$

积分之有

$$\frac{X^{02}-iX^{03}}{\tau}=-\frac{b}{2\varphi}\bar{z}+f_1, \tag{5.15}$$

其中 $f_1=f_1(x^0,z)$ 为对 z 解析的任意函数，故

$$\frac{X^2-iX^3}{\tau}=\sqrt{2}\,r-\frac{b}{2\varphi}\bar{z}+f_1. \tag{5.16}$$

现由 $(2.17)_1$ 知 U 不含 r，由 $(2.17)_5$ 知

$$\delta U=-\bar{\nu}+\tau U,$$

应用 $(2.19)_9$，(5.9) 有

$$\begin{aligned}\bar{\delta}\delta U&=-\overline{\delta\nu}+\tau^2U+\tau\bar{\delta}U\\&=-\Delta\mu-\mu^2-\lambda\bar{\lambda}-\tau\bar{\nu}-\varphi_2+\tau^2U+\tau\bar{\delta}U\\&=\tau(\delta+\bar{\delta})U+X^2\frac{\sqrt{2}\,b\tau^2}{2\varphi}-\frac{b\tau^2}{2\varphi^2}-\frac{b^2\tau^2}{4\varphi^2}-\lambda\bar{\lambda}-\varphi_2.\end{aligned} \tag{5.17}$$

另一方面，由 (5.14) 知

$$\delta\left(\frac{X^{02}+iX^{03}}{\tau}\right)=\sqrt{2}\,\frac{\bar{\lambda}}{\tau},$$

但以 δ 作用于 (5.15) 的复共轭得

$$\delta\left(\frac{X^{02}+iX^{03}}{\tau}\right)=\delta\bar{f}_1=\sqrt{2}\,\frac{\partial\bar{f}_1}{\partial\bar{z}}.$$

两式比较得

$$\lambda=\tau\frac{\partial f_1}{\partial z}. \tag{5.18}$$

又由 (5.6) 的第一式可知

$$\nabla_{B\bar{D}}\nabla_{A\bar{C}}\varphi=\delta_A^2\delta_C^2X_{B\bar{D}}\tau+\tau\delta_C^2\Gamma_{1AB\bar{D}}+\tau\delta_A^2\overline{\Gamma_{1CD\bar{B}}}.$$

由此及由 (2.11) 得

$$\varphi_2=\varphi_{22\bar{2}\bar{2}}=\frac{b}{2}\left(\frac{a}{\varphi^2}\nabla_{2\bar{2}}\varphi\nabla_{2\bar{2}}\varphi+\frac{1}{\varphi}\nabla_{2\bar{2}}\nabla_{2\bar{2}}\varphi\right)=\frac{ab}{2\varphi^2}\tau^2+X^2\frac{\sqrt{2}\,b\tau^2}{2\varphi}. \tag{5.19}$$

以 (5.18)，(5.19) 代入 (5.17) 得出 U 适合方程

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+\frac{2}{x}\frac{\partial U}{\partial x}=-\frac{1}{x^2}\left|\frac{\partial f_1}{\partial z}\right|^2-\frac{b\left(a+\frac{1}{2}b+1\right)}{2x^2\varphi^2}, \tag{5.20}$$

其中 $x=x^2$，$y=x^3$，而 $f_1=f_1(x^0,z)$ 为对 z 解析的任意函数.

由计算可知度规为

$$ds^2=-4Ux^2d\varphi^2-2\sqrt{2}\,xd\varphi dr-\left|\left(\sqrt{2}\,r-\frac{b}{2\varphi}z+\bar{f}_1\right)d\varphi-dz\right|^2. \tag{5.21}$$

为了便于把此度规写为能与 (4.24) 统一的形式，把 U 换为 $\frac{1}{2}U$；z 换为 $-(z+c_0)$；x 换为 $-(x+c_0)$，c_0 为实数；φ 换为 $\frac{\varphi}{c_0}$；r 换为 $\frac{r}{\sqrt{2}}$；f_1 换为 $c_0\left(f_1-\frac{bc_0}{2\varphi}\right)$，则 (5.21) 为

$$ds^2=-2U\left(\frac{x+c_0}{c_0}\right)^2 d\varphi^2+2\left(\frac{x+c_0}{c_0}\right)d\varphi dr$$
$$-\left|\left(\frac{r}{c_0}+\frac{b}{2\varphi}z+\bar{f}_1\right)d\varphi+dz\right|^2. \tag{5.22}$$

而在 (4.24) 中作变换 $Z\to\int fdz$，则为

$$ds^2=-2Ud\varphi^2+2d\varphi dr-|f_2d\varphi+dz|^2, \tag{5.23}$$

其中 $f_2=f_2(x^0,z)$ 是对 z 解析的任意函数．因此上面两度规能统一地写为如下形式：

$$ds^2=-2U\left(\frac{x+c_0}{c_0}\right)d\varphi^2+2\left(\frac{x+c_0}{c_0}\right)d\varphi dr$$
$$-\left|\left(\bar{f}_1+f_2+\frac{r}{c_0}\right)d\varphi+dz\right|^2. \tag{5.24}$$

在 $\tau\doteqdot 0$ 即有转 (rotating) 情形时，

$$f_1=f_1(x^0,z)\text{ 是对 }z\text{ 解析的任意函数},\ f_2=\frac{b}{2\varphi}z;$$

在 $\tau=0$ 即无转 (rotating free) 情形时，

$$f_1=0,\quad c_0=\infty,\quad f_2=f_2(x^0,z)\text{ 是对 }z\text{ 解析的任意函数}.$$

现在回到讨论度规 (5.21) 在外尔张量比 III 型更特殊的情形：

(1) 当外尔张量为 N 型

此时

$$\Psi_3=0. \tag{5.25}$$

由 (5.10) 知必须

$$\lambda=0. \tag{5.26}$$

由 (5.18) 知 f_1 不包含 z，作坐标变换 $z\to z+h(x^0)$，适当选取 h 使得度规 (5.21) 化为

$$ds^2=-4Ux^2d\varphi^2-2\sqrt{2}\,xd\varphi dr-\left|\left(\sqrt{2}\,r-\frac{b}{2\varphi}z\right)d\varphi-dz\right|^2, \tag{5.27}$$

其中 U 适合

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+\frac{2}{x}\frac{\partial U}{\partial x}=-\frac{b\left(a+\frac{1}{2}b+1\right)}{2x^2\varphi^2}. \tag{5.28}$$

(2) 当外尔张量为 O 型

除 (5.25) 外还有

$$\Psi_4=0. \tag{5.29}$$

由 $(2.19)_6$ 知

$$\bar{\delta}\left(\frac{\nu}{\tau}\right)=0,\quad\text{即}\quad\frac{\nu}{\tau}=-\sqrt{2}\,\bar{f}_3,$$

其中 $f_3=f_3(x^0,z)$ 为对 z 解析的函数．由 $(2.17)_5$ 有

$$\delta\left(\frac{U}{\tau}\right)=-\frac{\bar{\nu}}{\tau}=\sqrt{2}f_3,$$

积分之有

$$\frac{U}{\tau}=f_3\bar{z}+f_4,\quad f_4=f_4(x^0,z)\text{ 对 } z \text{ 解析.}$$

注意上式左边为实，必须 f_3 与 f_4 为 z 的线性函数，上式可写成

$$xU=Az\bar{z}+Bz+\bar{B}\bar{z}+C, \tag{5.30}$$

其中 A, B, C 仅含 x^0，即 xU 是 z 与 $\bar{z}$ 的二次函数. 但 U 不适合方程 (5.28)，因为 (5.28) 可写为

$$\frac{1}{x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(xU)=-\frac{b\left(a+\frac{1}{2}b+1\right)}{2x^2\varphi^2}. \tag{5.31}$$

以 (5.30) 代入上式左边，得

$$\frac{4A}{x}=-\frac{b\left(a+\frac{1}{2}b+1\right)}{2x^2\varphi^2}.$$

由于 A 仅含 $x^0=\varphi$；而上式说明 A 包含 x^{-1}，这是矛盾的. 故 $\tau\neq 0$ 时没有 O 型的标量-张量引力波.

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THE SCALAR-TENSOR GRAVITATIONAL WAVES

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Abstract

We present the complete proof of the results of our previous paper[1]. Furthermore, we also show that the D-type scalar-tensor waves does not exist.