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电、磁的聚焦与偏转复合系统的相对论象差理论

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提　　要

复数域为实数域上的可除结合代数[1]，本文在其中定义了内积，并以这个新的代数系统为工具，研究了电、磁的聚焦场与偏转场重迭时的复合系统，导出了该系统的相对论三级几何象差公式.

一、引　言

Glaser 在文献 [2] 中提出了研究聚焦场与偏转场重迭时的复合系统的三级几何象差理论的要求，但并未解决. 近来西门纪业[3]在一定条件下研究了这一问题. 本文在文献[3]的基础上，研究了广泛情况下，电、磁的聚焦场与偏转场重迭时的复合系统，导出了这系统的相对论三级几何象差公式.

复数域是实数域上的二维可除结合代数，本文在其中定义了内积，从而形成了一个新的代数系统（见附录一），它把二维欧氏空间理论和复数理论有机地结合在一起，因而能以简明的形式推导和表达一系列电子光学结论.

二、电磁场的傅里叶展开

令 Ω 为一个以 z 轴直线为旋转轴的圆柱体真空区域，今考察 Ω 中的无旋无源稳定电磁场. 由附录一及文献[2]可知，Ω 中的电势 φ 及磁标势 ψ 在圆柱坐标系 (ρ, θ, z) 下可按变量 θ 展开成傅里叶级数如下：

$$\varphi(\rho, \theta, z) = \sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{m!}{4^{\mu}\mu!(m+\mu)!}\rho^{m+2\mu}(\Phi_m^{(2\mu)}(z), e^{im\theta}),$$

$$\psi(\rho, \theta, z) = \sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{m!}{4^{\mu}\mu!(m+\mu)!}\rho^{m+2\mu}(\Psi_m^{(2\mu)}(z), e^{im\theta}), \tag{1}$$

其中 Φ_m, Ψ_m 为变量 z 的复值函数. 在本文中任一复数 p 的实部及虚部分别以 p_r 及 p_i 表示. 于是

$$\Phi_m = \Phi_{mr} + i\Phi_{mi}, \quad \Psi_m = \Psi_{mr} + i\Psi_{mi}. \tag{2}$$

我们规定

$$\Phi_{0i}=\Psi_{0i}=0, \tag{3}$$

故

$$\Phi_0=\Phi_{0r},\quad \Psi_0=\Psi_{0r}. \tag{4}$$

以下令

$$\Phi=\Phi_0. \tag{5}$$

由(1)式知Ω内的磁感强度场为

$$B_\theta=\sum_{m=1}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!m}{4^\mu\mu!(m+\mu)!}\rho^{m+2\mu-1}(\Psi_m^{(2\mu)},ie^{im\theta}),$$

$$B_z=\sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!}{4^\mu\mu!(m+\mu)!}\rho^{m+2\mu}(\Psi_m^{(2\mu+1)},e^{im\theta}),$$

$$B_\rho=\sum_{\substack{m=0\\m+2\mu\geq 1}}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!(m+2\mu)}{4^\mu\mu!(m+\mu)!}\rho^{m+2\mu-1}(\Psi_m^{(2\mu)},e^{im\theta}). \tag{6}$$

由附录二知，可以选择其磁矢势场(A_ρ, A_θ, A_z)如下:

$$\begin{aligned}\rho A_\theta&=\int_0^\rho \rho B_z d\rho,\\ A_z&=-\int_0^\rho B_\theta d\rho,\\ A_\rho&=0.\end{aligned} \tag{7}$$

故得

$$\begin{aligned}\rho A_\theta&=\sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!}{4^\mu\mu!(m+\mu)!(m+2\mu+2)}\rho^{m+2\mu+2}(\Psi_m^{(2\mu+1)},e^{im\theta}),\\ A_z&=\sum_{m=1}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!m}{4^\mu\mu!(m+\mu)!(m+2\mu)}\rho^{m+2\mu}(i\Psi_m^{(2\mu)},e^{im\theta}),\\ A_\rho&=0.\end{aligned} \tag{8}$$

今后以$B_0(z)$表示$B_z(\rho,\theta,z)|_{\rho=0}$，故

$$B_0=-\Psi_0' \tag{9}$$

上撇符号一律表示对z求导.

三、拉格朗日函数

上述区域Ω内的相对论电子轨迹方程可通过下述拉格朗日函数F由费马原理求得[4]

$$F=\sqrt{\varphi(1+\varepsilon\varphi)(1+\rho'^2+\rho^2\theta'^2)}-\eta(\rho' A_\rho+\rho\theta' A_\theta+A_z), \tag{10}$$

其中φ为规范化电势，

$$\varepsilon=\frac{e}{2m_0c_0^2},\quad \eta=\sqrt{\frac{e}{2m_0}}, \tag{11}$$

式中e, m_0分别为电子的电荷绝对值及静止质量，c_0为真空中的光速.

电子轨迹上的一点$P(\rho,\theta,z)$可用(u,z)表示，其中

$$u = \rho e^{i\theta}. \tag{12}$$

故由附录一得

$$(u, u) = \rho^2, \quad (u', u') = \rho'^2 + \rho^2\theta'^2, \quad (iu, u') = \rho^2\theta'. \tag{13}$$

所以

$$\varphi = \sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{m!}{4^{\mu}\mu!(m+\mu)!}(u, u)^{\mu}(\Phi_m^{(2\mu)}, u^m), \tag{14}$$

$$\begin{aligned}\rho' A_\rho + \rho\theta' A_\theta + A_z &= \sum_{m=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\frac{m!}{4^{\mu}\mu!(m+\mu)!}\left[\frac{1}{m+2\mu+2}(\Psi_m^{(2\mu+1)}, u^m)(u, u)^{\mu}(iu, u')\right.\\ &\quad\left.+\frac{m}{m+2\mu}(i\Psi_m^{(2\mu)}, u^m)(u, u)^{\mu}\right].\end{aligned} \tag{15}$$

因我们只考察聚焦、偏转复合系统，故

$$\Phi_{2n} = \Psi_{2n} = 0 \qquad (n = 1, 2, 3, \cdots). \tag{16}$$

今把 u, u', $\Phi_1^{(\mu)}$, $\Psi_1^{(\mu)}(\mu = 0, 1, 2)$, Φ_3, Ψ_3 都看作一级小量，则

$$F = F_0 + F_2 + F_4 + \cdots, \tag{17}$$

其中 F_n 为 n 级小量部份. 如令

$$V = \Phi(1 + \epsilon\Phi), \quad \sigma = 1 + 2\epsilon\Phi, \tag{18}$$

则有

$$F_0 = \sqrt{V}, \tag{19}$$

$$\begin{aligned}F_2 = \frac{\sqrt{V}}{2}(u', u') - \frac{\sigma\Phi''}{8\sqrt{V}}(u, u) - \frac{\eta B_0}{2}(iu, u') + \frac{\sigma}{2\sqrt{V}}(\Phi_1, u)\\ + \eta(i\Psi_1, u),\end{aligned} \tag{20}$$

$$F_4 = F_{4C} + F_{4D}, \tag{21}$$

$$\begin{aligned}F_{4C} = -\frac{\sqrt{V}}{8}(u', u')^2 + \frac{1}{128}\left(\frac{\sigma\Phi^{(4)}}{\sqrt{V}} - \frac{\Phi''^2}{V^{3/2}}\right)(u, u)^2\\ - \frac{\sigma\Phi''}{16\sqrt{V}}(u, u)(u', u') + \frac{\eta B_0''}{16}(u, u)(iu, u').\end{aligned} \tag{22}$$

F_{4D} 在费马原理中经部份积分法处理后可取以下形式:

$$\begin{aligned}F_{4D} = \left(\frac{\sigma\Phi_3}{2\sqrt{V}} + i\eta\Psi_3,\ u^3\right) - \left(\frac{\Phi_1}{(4V)^{3/4}}, u\right)^2 + \left(\frac{\Phi''\Phi_1}{16V^{3/2}} - \frac{\Phi'\Phi_1'}{32V^{3/2}}, u\right)(u, u)\\ + \left(\frac{\sigma\Phi_1}{4\sqrt{V}}, u\right)(u', u') + \left(\frac{\eta\Psi_1'}{3}, u\right)(iu, u')\\ + \left(\frac{\sigma\Phi_1'}{16\sqrt{V}} + i\frac{\eta\Psi_1'}{24}, u'\right)(u, u) + 2\left(\frac{\sigma\Phi_1'}{16\sqrt{V}} + i\frac{\eta\Psi_1'}{24}, u\right)(u, u').\end{aligned} \tag{23}$$

我们按电子光学中通常的方法引入旋转（圆柱）坐标系 (ρ, x, z)，它相对于固定(圆柱)坐标系 (ρ, θ, z) 的转角 $\Theta(z)$ 为

$$\Theta = \frac{\eta}{2}\int_{z_a}^{z}\frac{B_0}{\sqrt{V}}dz, \tag{24}$$

所以

$$\Theta' = \frac{\eta B_0}{2\sqrt{V}}. \tag{25}$$

于是空间同一点 P，在固定坐标系中以 (ρ, θ, z) 表示，在旋转坐标系中以 (ρ, x, z) 表示，并且有

$$x = \theta - \Theta. \tag{26}$$

这点 P 在旋转坐标系中可用 (v, z) 表示，其中

$$v = \rho e^{ix}, \tag{27}$$

故

$$u = ve^{i\Theta}. \tag{28}$$

今后令

$$\omega = e^{-i\Theta}. \tag{29}$$

故由附录一知

$$F_2 = \frac{\sqrt{V}}{2}(v', v') - \frac{\sigma\Phi'' + \eta^2 B_0^2}{8\sqrt{V}}(v, v) + \frac{\sigma}{2\sqrt{V}}(\Phi_1\omega, v) + \eta(i\Psi_1\omega, v), \tag{30}$$

$$\begin{aligned}-F_{4C} = &\frac{L}{4}(v, v)^2 + \frac{M}{2}(v,v)(v', v') + \frac{N}{4}(v', v')^2 \\ &+ \sqrt{V}[P(v, v) + Q(v', v')](iv, v') + VK(iv, v')^2,\end{aligned} \tag{31}$$

其中

$$\begin{aligned}
L &= \frac{1}{32}\left[\frac{\Phi''^2 + 2\sigma\Phi''\eta^2B_0^2 + \eta^4B_0^4}{V^{3/2}} - \frac{\sigma\Phi^{(4)} + 4\eta^2B_0B_0''}{\sqrt{V}}\right], \\
M &= \frac{\sigma\Phi'' + \eta^2B_0^2}{8\sqrt{V}}, \\
N &= \frac{\sqrt{V}}{2}, \\
P &= \frac{1}{16}\left[\frac{(\sigma\Phi'' + \eta^2B_0^2)\eta B_0}{V^{3/2}} - \frac{\eta B_0''}{\sqrt{V}}\right], \\
Q &= \frac{\eta B_0}{4\sqrt{V}}, \quad K = \frac{\eta^2 B_0^2}{8V^{3/2}}.
\end{aligned} \tag{32}$$

同理可得

$$\begin{aligned}-F_{4D} = I\Big\{&-(\alpha\omega^3, v^3) + (\lambda\omega, v)(v', v') + (\mu\omega, v)(v, v) \\ &+ (\nu\omega, v')(v, v) + 2(\nu\omega, v)(v, v') \\ &+ 2\sqrt{\frac{V}{V_a}}(\kappa\omega, v)(iv, v')\Big\} + I^2(\tau\omega, v)^2.\end{aligned} \tag{33}$$

上式中采用以下符号，即 $\mathscr{D}$, z_b 由下面 (44),(45) 式所定义，并令

$$I = |\mathscr{D}(z_b)|, \tag{34}$$

$$\phi_m = \Phi_m/I, \ \phi_m = \Psi_m/I \qquad (m = 1, 3), \tag{35}$$

则

$$\alpha=\frac{\sigma\phi_3}{2\sqrt{V}}+i\eta\phi_3,\ \kappa=-\sqrt{V_a}\left(\frac{\sigma\eta B_0\phi_1}{8V^{3/2}}+\frac{\eta\phi_1'}{6\sqrt{V}}\right),$$
$$\lambda=-\frac{\sigma\phi_1}{4\sqrt{V}},$$
$$\mu=-\frac{(\Phi''+\sigma\eta^2B_0^2)\phi_1}{16V^{3/2}}+\frac{1}{32}\left(\frac{\Phi'}{V^{3/2}}+i\frac{\sigma\eta B_0}{V}\right)\phi_1'-\frac{3\eta^2B_0\phi_1'}{16\sqrt{V}},$$
$$\nu=-\left(\frac{\sigma\phi_1'}{16\sqrt{V}}+i\frac{\eta\phi_1'}{24}\right),\quad \tau=-\frac{\phi_1}{(4V)^{3/4}}. \tag{36}$$

四、高 斯 轨 迹

由费马原理知在 Ω 中的电子轨迹方程为

$$(\nabla_{v'}F)'-\nabla_v F=0. \tag{37}$$

故由附录一及(30)式得相对论高斯轨迹方程为

$$(\sqrt{V}\,v')'+\frac{\sigma\Phi''+\eta^2B_0^2}{4\sqrt{V}}v=\left(\frac{\sigma\Phi_1}{2\sqrt{V}}+i\eta\Psi_1\right)\omega. \tag{38}$$

令实值函数 $s(z)$, $t(z)$ 为齐次方程

$$(\sqrt{V}v')'+\frac{\sigma\Phi''+\eta^2B_0^2}{4\sqrt{V}}v=0, \tag{39}$$

满足下述初条件的线性无关解

$$s(z_a)=1,\quad s'(z_a)=0;$$
$$t(z_a)=0,\quad t'(z_a)=1. \tag{40}$$

则方程(38)满足初条件

$$v(z_a)=v_a,\quad v'(z_a)=v_a' \tag{41}$$

的解为以下的高斯轨迹：

$$v_g=\mathscr{V}s+\mathscr{W}t, \tag{42}$$

其中

$$\mathscr{V}=v_a+\mathscr{D},\quad \mathscr{W}=v_a'+\mathscr{E}, \tag{43}$$
$$\mathscr{D}=-\frac{1}{\sqrt{V_a}}\int_{z_a}^{z}\left(\frac{\sigma\Phi_1}{2\sqrt{V}}+i\eta\Psi_1\right)\omega t\,dz,$$
$$\mathscr{E}=\frac{1}{\sqrt{V_a}}\int_{z_a}^{z}\left(\frac{\sigma\Phi_1}{2\sqrt{V}}+i\eta\Psi_1\right)\omega s\,dz. \tag{44}$$

由(42),(43)式知共轭于物平面 $z=z_a$ 的象平面 $z=z_b$ 满足

$$t(z_b)=0, \tag{45}$$

横向放大率为

$$\beta=s(z_b). \tag{46}$$

由(44)式知 $\mathscr{D}'s+\mathscr{E}'t=0$，因而

$$v_g'=\mathscr{V}s'+\mathscr{W}t'. \tag{47}$$

五、三级几何象差

由(37)式解得直到三级的相对论轨迹方程为

$$(\sqrt{V}v')' + \frac{\sigma\Phi'' + \eta^2 B_0^2}{4\sqrt{V}}v - \left(\frac{\sigma\Phi_1}{2\sqrt{V}} + i\eta\Psi_1\right)\omega = -[(\nabla_{v'}F_4)' - \nabla_v F_4]. \tag{48}$$

令上式满足高斯轨迹 v_g 同一初条件(41)式的解为

$$v = v_g + \Delta v, \tag{49}$$

则 Δv 满足初条件

$$(\Delta v)|_{z=z_a} = (\Delta v)'|_{z=z_a} = 0. \tag{50}$$

把(49)式代入(48)式，考虑到 v_g 满足(38)式，并略去高级小量，得

$$(\sqrt{V}\ (\Delta v)')' + \frac{\sigma\Phi'' + \eta^2 B_0^2}{4\sqrt{V}}\Delta v = -[(\nabla_{v_g'}F_{4_g})' - \nabla_{v_g}F_{4_g}], \tag{51}$$

其中 F_{4_g} 的下标 g 表示以高斯轨迹 v_g 代替轨迹 v. 由

$$\sqrt{V}\ (st' - s't) = \text{const} = \sqrt{V_a} \tag{52}$$

及 (40) 式，知 (51) 式满足 (50) 式的解为

$$\Delta v = \frac{1}{\sqrt{V_a}}\left\{-s\int_{z_a}^{z}(t'\nabla_{v_g'}F_{4_g} + t\nabla_{v_g}F_{4_g})dz + t\int_{z_a}^{z}(s'\nabla_{v_g'}F_{4_g} + s\nabla_{v_g}F_{4_g})dz + t(\nabla_{v_g'}F_{4_g})_{z_a}\right\}. \tag{53}$$

由 (42)，(47) 式及附录一易证

$$\nabla_{\mathscr{V}}F_{4_g} = s\nabla_{v_g}F_{4_g} + s'\nabla_{v_g'}F_{4_g},$$
$$\nabla_{\mathscr{W}}F_{4_g} = t\nabla_{v_g}F_{4_g} + t'\nabla_{v_g'}F_{4_g}. \tag{54}$$

把(54)式代入 (53) 式，得任意平面 $z = z$ 处的三级几何象差为

$$\Delta v = \frac{1}{\sqrt{V_a}}\left\{-s\int_{z_a}^{z}\nabla_{\mathscr{W}}F_{4_g}dz + t\int_{z_a}^{z}\nabla_{\mathscr{V}}F_{4_g}dz + t(\nabla_{v_g'}F_{4_g})_{z_a}\right\}. \tag{55}$$

故在象平面 $z = z_b$ 处的三级几何象差为

$$\Delta v_b = \frac{\beta}{\sqrt{V_a}}\int_{z_a}^{z_b}\nabla_{\mathscr{W}}(-F_{4_g})dz. \tag{56}$$

由附录一知任意两个复数 u, v 都满足

$$(u, u)(v, v) - (u, v)^2 = (iu, v)^2.$$

故由 (31) 及 (52) 式得

$$-F_{4C_g} = \frac{A}{4}(\mathscr{V}, \mathscr{V})^2 + \frac{B}{4}(\mathscr{W}, \mathscr{W})^2 + C(\mathscr{V}, \mathscr{W})^2 + \frac{D}{2}(\mathscr{V}, \mathscr{V})(\mathscr{W}, \mathscr{W}) + E(\mathscr{V}, \mathscr{V})(\mathscr{V}, \mathscr{W}) + F(\mathscr{W}, \mathscr{W})(\mathscr{V}, \mathscr{W}) + e(\mathscr{V}, \mathscr{V})(i\mathscr{V}, \mathscr{W}) + f(\mathscr{W}, \mathscr{W})(i\mathscr{V}, \mathscr{W}) + c(\mathscr{V}, \mathscr{W})(i\mathscr{V}, \mathscr{W}), \tag{57}$$

其中

$$A = Ls^4 + 2Ms^2s'^2 + Ns'^4, \quad B = Lt^4 + 2Mt^2t'^2 + Nt'^4,$$

$$
\begin{aligned}
&C = Ls^2t^2 + 2Msts't' + Ns'^2t'^2 - V_aK,\\
&D = Ls^2t^2 + M(s^2t'^2 + s'^2t^2) + Ns'^2t'^2 + 2V_aK,\\
&E = Ls^3t + Mss'(st)' + Ns'^3t', \quad F = Lst^3 + Mtt'(st)' + Ns't'^3,\\
&e = \sqrt{V_a}(Ps^2 + Qs'^2), \quad f = \sqrt{V_a}(Pt^2 + Qt'^2), \quad c = 2\sqrt{V_a}(P_{st} + Qs't').
\end{aligned} \tag{58}
$$

由 (33) 式及 (52) 式得

$$
\begin{aligned}
-F_{4D_g} = I\{&(\mathscr{V}, \mathscr{V})[(a_1\omega, \mathscr{V}) + (b_1\omega, \mathscr{W})] + (\mathscr{W}, \mathscr{W})[(a_2\omega, \mathscr{V})\\
&+ (b_2\omega, \mathscr{W})] + 2(\mathscr{V}, \mathscr{W})[(a_3\omega, \mathscr{V}) + (b_3\omega, \mathscr{W})]\\
&+ 2(i\mathscr{V}, \mathscr{W})[(a_4\omega, \mathscr{V}) + (b_4\omega, \mathscr{W})] - (\alpha\omega^3, \mathscr{V}^3)s^3\\
&- (\alpha\omega^3, \mathscr{W}^3)t^3 - (\alpha\omega^3, \mathscr{V}^2\mathscr{W})3s^2t - (\alpha\omega^3, \mathscr{V}\mathscr{W}^2)3st^2\}\\
&+ I^2\{(\tau\omega, \mathscr{V})^2s^2 + (\tau\omega, \mathscr{W})^2t^2 + (\tau\omega, \mathscr{V})(\tau\omega, \mathscr{W})2st\},
\end{aligned} \tag{59}
$$

其中

$$
\begin{aligned}
&a_1 = \lambda ss'^2 + \mu s^3 + \nu(s^3)', &&b_1 = \lambda s'^2t + \mu s^2t + \nu(s^2t)',\\
&a_2 = \lambda st'^2 + \mu st^2 + \nu(st^2)', &&b_2 = \lambda tt'^2 + \mu t^3 + \nu(t^3)',\\
&a_3 = \lambda ss't' + \mu s^2t + \nu(s^2t)', &&b_3 = \lambda s'tt' + \mu st^2 + \nu(st^2)',\\
&a_4 = \kappa s, &&b_4 = \kappa t.
\end{aligned} \tag{60}
$$

今对任一矩阵 $[A_{ij}]$ 定义 $\{[A_{ij}]\}_{z_a}^{z_b}$ 为[3]　$\{[A_{ij}]\}_{z_a}^{z_b} = \left[\int_{z_a}^{z_b} A_{ij}dz\right]$,　(61)

则由 (56), (57) 及 (59) 式得

$$
\begin{aligned}
\Delta v_b = \frac{\beta}{\sqrt{V_a}} \Bigg\{ &\begin{bmatrix} (\mathscr{W}, \mathscr{W}) \\ 2(\mathscr{V}, \mathscr{W}) \\ (\mathscr{V}, \mathscr{V}) \end{bmatrix}^* \begin{bmatrix} B & 0 & F & 3f \\ F & -f & C & c \\ D & -c & E & e \end{bmatrix} \begin{bmatrix} \mathscr{W} \\ i\mathscr{W} \\ \mathscr{V} \\ i\mathscr{V} \end{bmatrix} \\
&+ I \begin{bmatrix} (\omega, \mathscr{W})^2 \\ (i\omega, \mathscr{W})^2 \\ 2(\omega, \mathscr{W})(i\omega, \mathscr{W}) \\ 2(\omega, \mathscr{W})(\omega, \mathscr{V}) \\ 2(i\omega, \mathscr{W})(i\omega, \mathscr{V}) \\ 2(\omega, \mathscr{W})(i\omega, \mathscr{V}) \\ 2(i\omega, \mathscr{W})(\omega, \mathscr{V}) \\ (\omega, \mathscr{V})^2 \\ (i\omega, \mathscr{V})^2 \\ 2(\omega, \mathscr{V})(i\omega, \mathscr{V}) \end{bmatrix}^* \begin{bmatrix} C_{1r} & C_{2i} \\ C_{2r} & C_{1i} \\ C_{2i} & C_{2r} \\ C_{3r} & C_{4i} \\ C_{4r} & C_{3i} \\ C_{5i} & C_{4r} \\ C_{4i} & C_{5r} \\ C_{6r} & C_{7i} \\ C_{7r} & C_{6i} \\ C_{8i} & C_{8r} \end{bmatrix} \begin{bmatrix} \omega \\ i\omega \end{bmatrix} \\
&+ I^2 \begin{bmatrix} (\omega, \mathscr{W}) \\ (i\omega, \mathscr{W}) \\ (\omega, \mathscr{V}) \\ (i\omega, \mathscr{V}) \end{bmatrix}^* \begin{bmatrix} 2\tau_r^2t^2 & 2\tau_r\tau_it^2 \\ 2\tau_r\tau_it^2 & 2\tau_i^2t^2 \\ 2\tau_r^2st & 2\tau_r\tau_ist \\ 2\tau_r\tau_ist & 2\tau_i^2st \end{bmatrix} \begin{bmatrix} \omega \\ i\omega \end{bmatrix} \Bigg\}_{z_a}^{z_b},
\end{aligned} \tag{62}
$$

其中 $[A_{ij}]^*$ 表示矩阵 $[A_{ij}]$ 的转置矩阵,

$$
C_1 = 3[\lambda tt'^2 + (\mu - \bar{\alpha})t^3 + \nu(t^3)'],
$$

$$
\begin{aligned}
C_2 &= \lambda t t'^2 + (\mu + 3\bar{\alpha})t^3 + \nu(t^3)', \\
C_3 &= \lambda(s t'^2 + 2s' t t') + 3[(\mu - \bar{\alpha}) s t^2 + \nu(s t^2)'], \\
C_4 &= \lambda s' t t' + (\mu + 3\bar{\alpha}) s t^2 + \nu(s t^2)' + i\kappa t, \\
C_5 &= \lambda s t'^2 + (\mu + 3\bar{\alpha}) s t^2 + \nu(s t^2)' - 2i\kappa t, \\
C_6 &= \lambda(t s'^2 + 2t' s s') + 3[(\mu - \bar{\alpha}) t s^2 + \nu(t s^2)'], \\
C_7 &= \lambda t s'^2 + (\mu + 3\bar{\alpha}) t s^2 + \nu(t s^2)' + 2i\kappa s, \\
C_8 &= \lambda t' s s' + (\mu + 3\bar{\alpha}) t s^2 + \nu(t s^2)' - i\kappa s.
\end{aligned} \tag{63}
$$

这里 $\bar{\alpha}$ 为 α 的共轭复数.

由(62)式即得[3]

$$
\Delta v_b = (\Delta v_b)_O + (\Delta v_b)_\Gamma + (\Delta v_b)_\Lambda, \tag{64}
$$

其中 $(\Delta v_b)_O$ 为聚焦场的象差:

$$
(\Delta v_b)_O = \frac{\beta}{\sqrt{V_a}} \begin{bmatrix} (v_a', v_a') \\ 2(v_a, v_a') \\ (v_a, v_a) \end{bmatrix}^* \left\{ \begin{bmatrix} B & O & F & 3f \\ F & -f & C & c \\ D & -c & E & e \end{bmatrix} \right\}_{z_a}^{z_b} \begin{bmatrix} v_a' \\ i v_a' \\ v_a \\ i v_a \end{bmatrix}, \tag{65}
$$

$(\Delta v_b)_\Gamma$, $(\Delta v_b)_\Lambda$ 为聚焦场与偏转场的相互作用象差，$(\Delta v_b)_\Gamma$ 是以聚焦场为主的象差，$(\Delta v_b)_\Lambda$ 是以偏转场为主的象差. 如令

$$
\delta = \mathscr{D}/I, \quad \varepsilon = \mathscr{E}/I, \tag{66}
$$

则[3]

$$
\begin{aligned}
(\Delta v_b)_\Gamma = \frac{\beta}{\sqrt{V_a}} \Big\{ & I^3 \Gamma + I^2[(v_a', \nabla_\varepsilon) + (v_a, \nabla_\delta)]\Gamma \\
& + \frac{I}{2}[(v_a', \nabla_\varepsilon) + (v_a, \nabla_\delta)]^2 \Gamma \Big\}_{z_a}^{z_b},
\end{aligned} \tag{67}
$$

$$
\Gamma = \begin{bmatrix} (\varepsilon, \varepsilon) \\ 2(\delta, \varepsilon) \\ (\delta, \delta) \end{bmatrix}^* \begin{bmatrix} B & O & F & 3f \\ F & -f & C & c \\ D & -c & E & e \end{bmatrix} \begin{bmatrix} \varepsilon \\ i\varepsilon \\ \delta \\ i\delta \end{bmatrix}, \tag{68}
$$

$$
\begin{aligned}
(\Delta v_b)_\Lambda = \frac{\beta}{\sqrt{V_a}} \Big\{ & I^3 \Lambda + I^2[(v_a', \nabla_\varepsilon) + (v_a, \nabla_\delta)]\Lambda \\
& + \frac{I}{2}[(v_a', \nabla_\varepsilon) + (v_a, \nabla_\delta)]^2 \Lambda \Big\}_{z_a}^{z_b},
\end{aligned} \tag{69}
$$

在(65)式中 $\frac{\beta}{\sqrt{V_a}}\{B\}_{z_a}^{z_b}$ 为球差系数，$\frac{\beta}{\sqrt{V_a}}\{F\}_{z_a}^{z_b}$, $\frac{\beta}{\sqrt{V_a}}\{f\}_{z_a}^{z_b}$ 为各向同性和各向异性彗差系数，$\frac{\beta}{\sqrt{V_a}}\{C\}_{z_a}^{z_b}$, $\frac{\beta}{\sqrt{V_a}}\{c\}_{z_a}^{z_b}$ 为各向同性和各向异性象散系数，$\frac{\beta}{\sqrt{V_a}}\{D\}_{z_a}^{z_b}$ 为场曲系数，$\frac{\beta}{\sqrt{V_a}}\{E\}_{z_a}^{z_b}$, $\frac{\beta}{\sqrt{V_a}}\{e\}_{z_a}^{z_b}$ 为各向同性和各向异性畸变系数. 在(67)和(69)式中，与 v_a' 二次方成正比的项为彗差，与 v_a' 一次方成正比的项为场曲象散，与 v_a' 无关的项为畸变.

$$
\Lambda = \begin{bmatrix} (\omega, \varepsilon)^2 \\ (i\omega, \varepsilon)^2 \\ 2(\omega, \varepsilon)(i\omega, \varepsilon) \\ 2(\omega, \varepsilon)(\omega, \delta) \\ 2(i\omega, \varepsilon)(i\omega, \delta) \\ 2(\omega, \varepsilon)(i\omega, \delta) \\ 2(i\omega, \varepsilon)(\omega, \delta) \\ (\omega, \delta)^2 \\ (i\omega, \delta)^2 \\ 2(\omega, \delta)(i\omega, \delta) \\ (\omega, \varepsilon) \\ (i\omega, \varepsilon) \\ (\omega, \delta) \\ (i\omega, \delta) \end{bmatrix}^* \begin{bmatrix} C_{1r} & C_{2i} \\ C_{2r} & C_{1i} \\ C_{2i} & C_{2r} \\ C_{3r} & C_{4i} \\ C_{4r} & C_{3i} \\ C_{5i} & C_{4r} \\ C_{4i} & C_{5r} \\ C_{6r} & C_{7i} \\ C_{7r} & C_{6i} \\ C_{8i} & C_{8r} \\ 2\tau_r^2 t^2 & 2\tau_r \tau_i t^2 \\ 2\tau_r \tau_i t^2 & 2\tau_i^2 t^2 \\ 2\tau_r^2 st & 2\tau_r \tau_i st \\ 2\tau_r \tau_i st & 2\tau_i^2 st \end{bmatrix} \begin{bmatrix} \omega \\ i\omega \end{bmatrix}. \tag{70}
$$

六、结　　论

本文是关于聚焦场与偏转场重迭时复合系统的普遍的相对论三级几何象差理论．在偏转场为纯磁场，并且不考虑相对论修正时，由本文理论可导出文献[3]的三级象差公式．

文献[5]在 $\Phi_3 = \Psi_3 = 0$ 的条件下，讨论了电、磁的聚焦和偏转复合系统的相对论象差理论，但该文最终的象差公式(19)存在着印刷错误．

本文得到西门纪业同志的热情支持，并与他作了有益的讨论，特此致谢．

附　　录　　一

我们知道复数域是实数域上的二维可除结合代数，现在其中定义内积如下：

令 u, v 为任意两个复数，则 u, v 的内积被定义为如下的实数：

$$(u, v) = u_r v_r + u_i v_i.$$

显然它满足(实数域上的)欧氏空间的内积公理，这样就形成了一个新的代数系统，它既是复数域，又是二维欧氏空间．易证下述诸命题：

(1) 对任意复数 u, v, μ, ν 下式都成立：

$$(u, v)(\mu, \nu) - (u, \mu)(v, \nu) = (iu, \nu)(iv, \mu).$$

(2) 对任意复数 u, v, w 下式都成立：

$$(u, wv) = (\bar{w}u, v), \quad \therefore (wu, wv) = |w|^2(u, v).$$

(3) 对任意复数 u, v 下式都成立：

$$(iu, v) = u_r v_i - u_i v_r = -(u, iv), \quad (\bar{u}v)_r = (u, v), \ (\bar{u}v)_i = (iu, v).$$

故对任意三复数 u, v, w 下式必成立：$(uw, v) = u_r(w, v) + u_i(iw, v)$.

如 $|w| = 1$，则 $\quad (u, v) = (w, u)(w, v) + (iw, u)(iw, v).$

(4) 如 u, v 为实变量 z 的复值函数，则

$$(u, v)' = (u', v) + (u, v'), \quad \int_{z_1}^{z_2} (u', v) dz = (u, v)\Big|_{z_1}^{z_2} - \int_{z_1}^{z_2} (u, v') dz.$$

(5) 如 u, v 为复变量，

$$\nabla_v = \frac{\partial}{\partial v_r} + i\frac{\partial}{\partial v_i},$$

则

$$\nabla_v(v, v) = 2v, \quad \nabla_v(u, v^m) = mu\bar{v}^{m-1},$$

如 v 与 u 无关.

$$(u, \nabla_v) = u_r\frac{\partial}{\partial v_r} + u_i\frac{\partial}{\partial v_i}.$$

(6) 如 u, v 为实变量 z 的复值函数，α 为 z 的实值函数，同时 $u = ve^{i\alpha}$，则有

$$(u, u) = (v, v), \quad (u, u') = (v, v'), \quad (iu, u') = (iv, v') + \alpha'(v, v),$$

$$(u', u') = (v', v') + \alpha'^2(v, v) + 2\alpha'(iv, v').$$

附 录 二

易证以下结论：

给定任一右手正交曲线坐标系 (x_1, x_2, x_3) 及磁感强度场 $\boldsymbol{B} = (B_1, B_2, B_3)$，则可选择这磁场的矢势场

$$\boldsymbol{A} = (A_1, A_2, A_3)$$

为

$$h_1A_1 = \int_{x_3^*}^{x_3} h_1h_3B_2dx_3 - \alpha\int_{x_2^*}^{x_2}[h_1h_2B_3]_{x_3=x_3^*}dx_2,$$

$$h_2A_2 = -\int_{x_3^*}^{x_3} h_2h_3B_1dx_3 + (1-\alpha)\int_{x_1^*}^{x_1}[h_1h_2B_3]_{x_3=x_3^*}dx_1,$$

$$A_3 = 0,$$

其中 α 为任一实数，h_1, h_2, h_3 为拉梅系数．矢量 p 的物理分量以 $p_n(n = 1, 2, 3)$ 表示．

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THE RELATIVISTIC ABERRATION THEORY OF A COMBINED ELECTRIC-MAGNETIC FOCUSING-DEFIECTION SYSTEM

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Abstract

The complex number field is an associative division algebra over the real number field. In this paper we define an inner product in this field, and take this new algebraic system as a tool to study a general combined electric-magnetic focussing-deflection system with superimposed fields. The relativistic third-order geometrical aberration formulas for this combined system have been derived.