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等离子体中大幅波与低频振荡粒子非线性相互作用效应

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提 要

本文从 Vlasov-Maxwell 方程组出发,保留高频场到四级项,导得了纵、横波大幅高频场与低频振荡粒子拍频作用的动力学方程组.结果表明,在空间一维情况下,对亚声波,高阶大幅高频场的存在使形成 Soliton 的有质动力凝聚作用削弱,而对超声波则更不存在 Soliton 解;在三维情况下,似乎使 Caviton 形成变慢.

一、引 言

等离子体中不稳定性发展的后阶段,常常伴随着增大了振幅的高频波与低频振荡粒子的非线性拍频作用.例如,波与粒子的非线性散射产生了场能在长波长区的积累[1],导致了调制不稳定性[2]开始,其结果是增大了振幅的场,改变了波的传播特性,产生了明显的非线性色散效应.波与低频振荡粒子的这种非线性耦合可能导致强湍流过程的开始[3].在文献[4]中,我们从 Vlasov 方程和电磁场的 Maxwell 方程组出发,讨论了两个高频场与低频振荡粒子的拍频作用,导得了一组包含阻尼的有质动力作用下广义的 Zakharov 方程组,它可以导致 Soliton 和 Caviton 等现象的出现[5].但是,随着场振幅的增大,应当考虑更高阶场与粒子运动的耦合,并研究它们对二级场产生效应的影响.本文仍然从 Vlasov-Maxwell 方程组出发,取自洽场到四级项,导得了低频振荡粒子与四级场非线性作用的粒子微观分布和宏观流体密度的动力学方程组.结果表明,无论纵波或横波,四级场的出现都可明显影响二级场的非线性作用,使一维 Soliton 粒子密度抽空只能具有比初始密度小得多的量级,而使超声一维 Soliton 却变得更加不稳定;对于三维 Caviton 运动,四级场的作用可能使 Caviton 形成变慢.

二、Fourier 表象中粒子微观速度分布与大振幅波的耦合

为了叙述方便,我们重复文献[4]中的某些考虑和表示.引入单位偏振矢量 $\boldsymbol{e}_{\boldsymbol{k}}^{\sigma}$,$\boldsymbol{k}$ 为波矢,σ 为波符号.对纵波 $\sigma=l$,$\boldsymbol{e}_{\boldsymbol{k}}^{l}=\boldsymbol{k}/|\boldsymbol{k}|$;对横波 $\sigma=t$,$\boldsymbol{e}_{\boldsymbol{k}}^{t}$ 已对两个偏振方向求和.写 F 空间的电场矢量为

$$\boldsymbol{E}(k) = \sum_{\sigma=l,t} \boldsymbol{e}_{\boldsymbol{k}}^{\sigma} E^{\sigma}(k). \tag{1}$$

这里 $k \equiv \{\omega, \boldsymbol{k}\}$ 为频率 ω 和波矢 $\boldsymbol{k}$ 的缩写，以下同. 对于每个 σ 波 $\boldsymbol{k}$ 有两个传播方向，我们约定

$$\boldsymbol{e}_{-\boldsymbol{k}}^{\sigma} = -\boldsymbol{e}_{\boldsymbol{k}}^{\sigma}, \quad E^{\sigma}(-k) = -E^{\sigma}(k)^{*}. \tag{2}$$

$*$ 为复数记号. 现在可把电场力与 Lorentz 力写成

$$\boldsymbol{G}(k) = \sum_{\sigma} \boldsymbol{e}_{\boldsymbol{k}}^{\prime\sigma} E^{\sigma}(k), \tag{3}$$

其中

$$\boldsymbol{e}_{\boldsymbol{k}}^{\prime\sigma} = \left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)\boldsymbol{e}_{\boldsymbol{k}}^{\sigma} + \frac{\boldsymbol{k}\boldsymbol{v}\cdot\boldsymbol{e}_{\boldsymbol{k}}^{\sigma}}{\omega}, \tag{4}$$

$\boldsymbol{v}$ 为粒子速度，显然

$$\boldsymbol{e}_{\boldsymbol{k}}^{\prime\sigma}\cdot\boldsymbol{v} = \boldsymbol{e}_{\boldsymbol{k}}^{\sigma}\cdot\boldsymbol{v}. \tag{5}$$

有了上述表示后，就可从 Vlasov-Maxwell 方程组得到 F 表象中高频 (h) 和低频 (d) 的粒子速度分布函数及场的表达式如下[4]:

粒子速度分布函数的低频部分

$$\delta F_{\alpha}(q) = \frac{e_{\alpha}}{im_{\alpha}(\Omega - \boldsymbol{q}\cdot\boldsymbol{v})}\left\{\boldsymbol{G}_{d}(q)\cdot\frac{\partial F_{0\alpha}}{\partial\boldsymbol{v}} + \int_{(d)} \boldsymbol{G}_{h}(k_{1})\cdot\frac{\partial f_{\alpha}(k_{2})}{\partial\boldsymbol{v}}\cdot\delta^{4}(q - k_{1} - k_{2})\frac{dk_{1}dk_{2}}{(2\pi)^{4}}\right\}. \tag{6}$$

上式中已略去了两个低频波拍频产生的新低频波项，因为只感兴趣高频波拍频为低频的效应. (6) 式中右端第一项为低频场力作用使初态分布 $F_{0\alpha}$ 变化对粒子低频扰动 $\delta F_{\alpha}(q)$ 的贡献，第二项为高频分布与高频场作用对 δF_{α} 的贡献. 这里低频分布函数已用了表示

$$F_{\alpha}(q) = F_{0\alpha}(\boldsymbol{v})(2\pi)^{4}\delta^{4}(q) + \delta F_{\alpha}(q), \tag{7}$$

其中 $\delta F_{\alpha}(q) \equiv \delta F_{\alpha}(q, \boldsymbol{v})$, $f_{\alpha}(k) \equiv f_{\alpha}(k, \boldsymbol{v})$, $\alpha = i$ (离子), e (电子). m_{α} 为粒子质量，e_{α} 为电荷，$q \equiv \{\Omega, \boldsymbol{q}\}$ 为低频波的频率和波矢，k_1, k_2 为高频波缩写，$dk \equiv d^3k d\omega$, $dq \equiv d^3q d\Omega$, $\delta^4(k - k_1 - k_2) \equiv \delta(\omega - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)$. 积分号 $\int_{(d)}$ 表示只取高频向低频拍频过程.

粒子的高频分布

$$f_{\alpha}(k) = \frac{e_{\alpha}}{im_{\alpha}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}\left\{\boldsymbol{G}_{h}(k)\cdot\frac{\partial}{\partial\boldsymbol{v}}F_{0\alpha}(\boldsymbol{v}) + \int \boldsymbol{G}_{h}(k_{1})\cdot\frac{\partial\delta F_{\alpha}(q)}{\partial\boldsymbol{v}} \times\delta^{4}(k - k_{1} - q)\frac{dk_{1}dq}{(2\pi)^{4}} + \int \boldsymbol{G}_{d}(q)\cdot\frac{\partial f_{\alpha}(k_{1})}{\partial\boldsymbol{v}}\delta^{4}(k - k_{1} - q)\frac{dk_{1}dq}{(2\pi)^{4}}\right\}. \tag{8}$$

这里已略去了两个高频波拍频产生的更高频波项，因为只感兴趣频率接近等离子体频率的一类高频波. (8) 式中右端第一项来自高频场与初态分布粒子相互作用对高频分布的贡献，第二、三两项分别为高频场与低频振荡粒子作用以及低频场与高频粒子作用引起高频振荡粒子分布的变化.

由电磁场 Maxwell 方程，得低频场和高频场表达式分别为

$$\varepsilon_d^{\sigma_q}(q)E_d^{\sigma_q}(q) = -\frac{4\pi}{\Omega}\sum_{\alpha,\sigma_1}\frac{e_\alpha^2}{m_\alpha}\int_{(d)}\frac{\boldsymbol{e}_q\cdot\boldsymbol{v}}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}+i\delta}\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}f_\alpha(k_2)E_h^{\sigma_1}(k_1)\cdot\delta^4(q-k_1-k_2)\frac{dk_1dk_2d^3v}{(2\pi)^4},\tag{9}$$

$$\varepsilon_h^\sigma(k)E_h^\sigma(k) = -\frac{4\pi}{\omega}\sum_\alpha\frac{e_\alpha^2}{m_\alpha}\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\delta}\left[\sum_{\sigma_1}\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_\alpha(q)E_h^{\sigma_1}(k_1)+\sum_{\sigma_q}\boldsymbol{e}_q'\cdot\frac{\partial f_\alpha(k_1)}{\partial\boldsymbol{v}}E_d^{\sigma_q}(q)\right]\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4}d^3v.\tag{10}$$

(9)，(10) 式右边分别是低频和高频非线性电流部分，被积函数中 $\delta\to 0_+$，积分线路在复平面上沿实轴上岸向下绕过极点，为了方便，以后将省略 $i\delta$ 不写．式中还应用了简写 $\boldsymbol{e}_k'\equiv\boldsymbol{e}_k'^\sigma$，$\boldsymbol{e}_1'\equiv\boldsymbol{e}_{k_1}'^\sigma\cdots$ 等等，以下同．

对 $\sigma=l$，高频纵波线性介质函数为

$$\varepsilon_h^l(k) = 1+\sum_{\alpha=i,e}\frac{\omega_{p\alpha}^2}{k^2}\int\frac{\boldsymbol{k}\cdot\dfrac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}d^3v = 1-\sum_{\alpha=i,e}\left(\frac{\omega_{p\alpha}^2}{\omega^2}+\frac{3k^2v_{T_\alpha}^2}{\omega_{p\alpha}^2}+\cdots\right)+\mathrm{Im}\,\varepsilon_h^l(k).\tag{11}$$

对 $\sigma=t$，高频横波线性介质函数为

$$\varepsilon_h^t(k)+\frac{c^2k^2}{\omega^2}\doteq 1-\frac{\omega_{pe}^2}{\omega^2}.\tag{12}$$

(11) 和 (12) 式中已令 $F_{0\alpha}=n_0f_{0\alpha}$，其中 $n_{0e}=n_{0i}=n_0$ 为初始单位体积粒子数． $v_{T_\alpha}^2=T_\alpha/m_\alpha$，$T_\alpha$ 为粒子温度，以能量为单位． $\omega_{p\alpha}^2=4\pi n_0e_\alpha^2/m_\alpha$．对于 $\varepsilon_d^l(q)$ 和 $\varepsilon_d^t(q)$ 亦有类似表达式．

从 (8) 和 (9) 式可知，低频场 $E_d^{\sigma_q}(q)$ 系两个高频场拍频产生的次级场，因此，粒子低频分布 (6) 式最低阶也是高频场的二级拍频效应，其次才是四级场、六级场…效应．在文献 [4] 中处理时，仅保留了二级场的有质动力作用，所以对低频分布的贡献完全来自有质动力与初态粒子分布的相互作用．但是，当场幅进一步增大时，四级场的拍频效应不能忽略，这时除了波场对初态分布的作用外，还进一步作用于粒子速度分布的扰动态，导致了粒子在速度空间的扩散过程．因此，叠代过程必须保留四级场的存在．展开粒子高频分布至场的三级项，得

$$f_\alpha(k) = \frac{e_\alpha}{im_\alpha(\omega-\boldsymbol{k}\cdot\boldsymbol{v})}\left\{n_0\sum_{\sigma_k}\boldsymbol{e}_k'\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}E_h^{\sigma_k}(k)+\sum_{\sigma_1}\int\boldsymbol{e}_1'\cdot\frac{\partial\delta F_\alpha(q)}{\partial\boldsymbol{v}}E_h^{\sigma_1}(k_1)\times\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4}+\sum_{\beta\sigma_q\sigma_1\sigma_2\sigma_3}n_0\omega_{p\beta}^2\frac{e_\alpha e_\beta}{m_\alpha m_\beta}\iint_{(d)}\boldsymbol{e}_q'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_1'\cdot\dfrac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}}{\omega_1\boldsymbol{k}_1\cdot\boldsymbol{v}}\times\frac{H_\beta^{\sigma_q\sigma_2\sigma_3}(q,k_2,k_3)}{\Omega\varepsilon_d^{\sigma_q}(q)}E_h^{\sigma_1}(k_1)E_h^{\sigma_2}(k_2)E_h^{\sigma_3}(k_3)\delta^4(k-k_1-q)\delta^4\times(q-k_2-k_3)\frac{dk_1dk_2dk_3dq}{(2\pi)^8}\right\}.\tag{13}$$

同样，取至四级场可得低频场

$$
\begin{aligned}
E_{a}^{\sigma_{\boldsymbol{q}}}(q) = & -\sum_{\alpha\sigma_1\sigma_2} \frac{\omega_{p\alpha}^2 e_\alpha}{im_\alpha}\left\{\int_{(d)} \frac{H_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_2}(q, k_1, k_2)}{\Omega\epsilon_d^{\sigma_{\boldsymbol{q}}}(q)} E_{k}^{\sigma_1}(k_1) E_{h}^{\sigma_2}(k_2)\right. \\
& \times \delta^4(q-k_1-k_2)\frac{dk_1dk_2}{(2\pi)^4} + \frac{1}{n_0}\int_{(d)} \frac{Z_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_2}(q, k_1, k_2+q', k_2, q')}{\Omega\epsilon_d^{\sigma_{\boldsymbol{q}}}(q)} \\
& \times E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)\delta^4(q-k_1-k_2-q')\frac{dk_1dk_2dq'}{(2\pi)^8} \\
& + \sum_{\beta\sigma_{\boldsymbol{q}'}\sigma_3\sigma_4}\omega_{p\beta}^2\frac{e_\alpha e_\beta}{m_\alpha m_\beta}\int_{(d)}\frac{K_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_{\boldsymbol{q}'}\sigma_2}(q, k_1, k_2+q', q', k_2)}{\Omega\epsilon_k^{\sigma_{\boldsymbol{q}}}(q)} \\
& \times \frac{H_\beta^{\sigma_{\boldsymbol{q}'}\sigma_3\sigma_4}(q', k_3, k_4)}{\Omega'\epsilon_d^{\sigma_{\boldsymbol{q}'}}(q')} E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)\cdot E_{h}^{\sigma_3}(k_3)E_{h}^{\sigma_4}(k_4) \\
& \left.\times \delta^4(q-k_1-k_2-q')\delta^4(q'-k_3-k_4)\frac{dk_1\cdots dk_4dq'}{(2\pi)^{12}}\right\}.
\end{aligned} \tag{14}
$$

(13) 及 (14) 式中

$$
H_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_2}(q, k_1, k_2) = \int \frac{\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{v}}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}\mathscr{S}_\alpha^{\sigma_1\sigma_2}(k_1, k_2, \boldsymbol{v})d^3\boldsymbol{v}, \tag{15}
$$

$$
\mathscr{S}_\alpha^{\sigma_1\sigma_2}(k_1, k_2, \boldsymbol{v}) \equiv \frac{1}{2}\left(\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_2'\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}} + 1\longleftrightarrow 2\right); \tag{16}
$$

$$
Z_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_2}(q, k_1, k_2+q', k_2, q') = \int\frac{\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{v}}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}\mathscr{R}_\alpha^{\sigma_1\sigma_2}(k_1, k_2+q', k_2, q', \boldsymbol{v})d^3v, \tag{17}
$$

$$
\begin{aligned}
\mathscr{R}_\alpha^{\sigma_1\sigma_2}(k_1, k_2+q', k_2, q', \boldsymbol{v}) \equiv & \frac{1}{2}\left(\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2+\Omega'-(\boldsymbol{k}_2+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_2'\right. \\
& \left.\times\frac{\partial\delta F_\alpha(q')}{\partial\boldsymbol{v}} + 1\longleftrightarrow 2\right);
\end{aligned} \tag{18}
$$

$$
K_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_{\boldsymbol{q}'}\sigma_2}(q, k_1, k_2+q', q', k_2) = \int\frac{\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{v}}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}\mathscr{T}_\alpha^{\sigma_1\sigma_{\boldsymbol{q}'}\sigma_2}(k_1, k_2+q', q', k_2, \boldsymbol{v})d^3v, \tag{19}
$$

$$
\begin{aligned}
\mathscr{T}_\alpha^{\sigma_1\sigma_{\boldsymbol{q}'}\sigma_2}(k_1, k_2+q', q', k_2, \boldsymbol{v}) \equiv & \frac{1}{2}\left(\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2+\Omega'-(\boldsymbol{k}_2+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_{\boldsymbol{q}'}'\right. \\
& \left.\times\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_2'\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}} + 1\longleftrightarrow 2\right).
\end{aligned} \tag{20}
$$

H_α, Z_α 与 K_α 各自对 k_1, k_2 对称．此外，可以证明 H_α 尚具有下列性质：

$$
\frac{1}{\Omega}H_\alpha^{\sigma_{\boldsymbol{q}}\sigma_1\sigma_2}(q, k_1, k_2) = \frac{1}{\omega_2}H_\alpha^{\sigma_2\sigma_{\boldsymbol{q}}\sigma_1}(k_2, q, -k_1). \tag{21}
$$

(21) 式只对 $\Omega-\boldsymbol{q}\cdot\boldsymbol{v}=0$ 无奇点条件下成立．

在以下的书写中，除特殊情况外，将省去 H_α, Z_α, K_α 和 ϵ_d 上角标波符号不写．现在写出低频分布 (6) 式的进一步形式为(至四级场)

$$\delta F_{\alpha}(q)=\frac{e_{\alpha}}{m_{\alpha}(\Omega-\boldsymbol{q}\cdot\boldsymbol{v})}\sum_{\sigma_1\sigma_2}n_0\int_{(d)}\Bigg[\sum_{\beta\sigma_{\boldsymbol{q}}}\omega_{p\beta}^2\frac{e_{\beta}}{m_{\beta}}\boldsymbol{e}_{\boldsymbol{q}}'\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{H_{\beta}(q,k_1,k_2)}{\Omega\epsilon_d(q)}$$
$$-\frac{e_{\alpha}}{m_{\alpha}}\mathscr{S}_{\alpha}(k_1,k_2,\boldsymbol{v})\Bigg]E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)\delta^4(q-k_1-k_2)\frac{dk_1dk_2}{(2\pi)^4}$$
$$+\frac{e_{\alpha}}{m_{\alpha}(\Omega-\boldsymbol{q}\cdot\boldsymbol{v})}\Bigg\{\sum_{\sigma_1\sigma_2}\iint_{(d)}\Bigg[\sum_{\beta\sigma_{\boldsymbol{q}}}\omega_{p\beta}^2\frac{e_{\beta}}{m_{\beta}}\boldsymbol{e}_{\boldsymbol{q}}'$$
$$\times\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{Z_{\beta}(q,k_1,k_2+q_1'k_2,q')}{\Omega\epsilon_d(q)}-\frac{e_{\alpha}}{m_{\alpha}}\boldsymbol{e}_1'$$
$$\times\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2+\Omega'-(\boldsymbol{k}_2+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_2'\cdot\frac{\partial\delta F_{\alpha}(q)}{\partial\boldsymbol{v}}\Bigg]E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)$$
$$\times\delta^4(q-k_1-k_2-q')\frac{dk_1dk_2dq'}{(2\pi)^8}+\sum_{\gamma\sigma_{\boldsymbol{q}'}}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\omega_{p\gamma}^2\frac{e_{\gamma}}{m_{\gamma}}$$
$$\times\iint_{(d)}\int_{(d')}\Bigg[\sum_{\beta\sigma_{\boldsymbol{q}}}\frac{\omega_{p\beta}^2}{4\pi m_{\beta}}\boldsymbol{e}_{\boldsymbol{q}}'\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{K_{\beta}(q,k_1,k_2+q',q',k_2)}{\Omega\epsilon_d(q)}$$
$$-\frac{\omega_{p\alpha}^2}{4\pi m_{\alpha}}\mathscr{T}_{\alpha}(k_1,k_2+q',q',k_2)\Bigg]\cdot\frac{H_{\gamma}(q',k_3,k_4)}{\Omega'\epsilon_d(q')}$$
$$\times E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)E_{h}^{\sigma_3}(k_3)E_{h}^{\sigma_4}(k_4)\delta^4(q-k_1-k_2-q')\delta^4(q'-k_3-k_4)$$
$$\times\frac{dk_1\cdots dk_4dq'}{(2\pi)^{12}}\Bigg\}.\tag{22}$$

(22) 式右端第一项为二级场效应，其余均为四级场贡献.

现在进一步化简 (22) 式. 低频波 $\sigma_{\boldsymbol{q}}$ 可以是迴旋波[1)]和纵波，在以下讨论中，为方便只取 $\sigma_{\boldsymbol{q}}$ 为纵波（例如离子声波），这时 $\boldsymbol{e}_{\boldsymbol{q}}=\boldsymbol{q}/|\boldsymbol{q}|$. 同时，由于离子质量 m_i 远大于电子 m_e，在 (22) 式的低频分布中可略去离子响应，即取 $\alpha=e$. 这样得到

$$\delta F_e(q)=\frac{1}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}\Bigg\{\sum_{\sigma_1\sigma_2}\int_{(d)}B_e(q,k_1,k_2,\boldsymbol{v})E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)\delta^4(q-k_1-k_2)\frac{dk_1dk_2}{(2\pi)^4}$$
$$+\frac{e^2}{m_e^2}\sum_{\sigma_1\sigma_2}\iint_{(d)}\Bigg[\omega_{pe}^2\boldsymbol{e}_{\boldsymbol{q}}\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}\frac{Z_e(q,k_1,k_2+q',k_2,q')}{\Omega\epsilon_d(q)}$$
$$-\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2+\Omega'-(k_2+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_2'\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_e(q')\Bigg]E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)$$
$$\times\delta^4(q-k_1-k_2-q')\cdot\frac{dk_1dk_2dq'}{(2\pi)^4}+\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}$$
$$\times\iint_{(d)}\int_{(d')}C_e(q,k_1,k_2+q',k_2,q',k_3,k_4,\boldsymbol{v})E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)$$
$$\times E_{h}^{\sigma_3}(k_3)E_{h}^{\sigma_4}(k_4)\delta^4(q-k_1-k_2-q')\delta^4(q'-k_3-k_4)\frac{dk_1\cdots dk_4dq'}{(2\pi)^{12}}\Bigg\},\tag{23}$$

式中

1) 例如：当有自发磁场.

$$B_e(q, k_1, k_2, \boldsymbol{v}) = \frac{\omega_{pe}^4}{4\pi m_e} \boldsymbol{e}_q \cdot \frac{\partial f_{0e}}{\partial \boldsymbol{v}} \frac{H_e(q, k_1, k_2)}{\Omega \varepsilon_d(q)} - \frac{\omega_{pe}^2}{4\pi m_e} \mathscr{S}_e(k_1, k_2, \boldsymbol{v}), \tag{24}$$

$$C_e(q, k_1, k_2+q', q', k_3, k_4, \boldsymbol{v}) = \frac{\omega_{pe}^6}{(4\pi m_e)^2 n_0} \Big\{ \boldsymbol{e}_q \cdot \frac{\partial f_{0e}}{\partial \boldsymbol{v}} \omega_{pe}^2 \frac{K_e(q, k_1, k_2+q', q', k_2)}{\Omega \varepsilon_d(q)}$$
$$- \mathscr{T}_e(k_1, k_2+q', q', k_2) \Big\} \frac{H_e(q', k_3, k_4)}{\Omega' \varepsilon_d(q')}, \tag{25}$$

其中 B_e 对脚标 1, 2, C_e 对脚标 3, 4 各为对称.

前面已经指出，积分号 $\int_{(d)}$ 只是对两个相反符号的频率及波矢发生作用，也即只感兴趣频率接近 ω_{pe} 的两个高频波拍频为低频波的过程. 这样，从现在起就可把高频波的求和号表示为对每两个相反符号波求和，即

$$\sum_{\sigma_1\sigma_2} \to \sum_{\substack{\{\sigma_1^+\sigma_2^-\}\\ \{\sigma_1^-\sigma_2^+\}}}, \quad \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \to \sum_{\substack{\{\sigma_1^+\sigma_2^-\}\\ \{\sigma_1^-\sigma_2^+\}}} \sum_{\substack{\{\sigma_3^+\sigma_4^-\}\\ \{\sigma_3^-\sigma_4^+\}}}.$$

经过这样处理后，利用关系式 (2)，便可写 (22) 式为

$$\delta F_e(q) = \frac{1}{\Omega - \boldsymbol{q}\cdot\boldsymbol{v}} \Big\{ -2\int B_e(q, k_1, -k_2) E_h^{\sigma_1^+}(k_1) E_h^{\sigma_2^-}(k_2)^* \delta^4(q-k_1+k_2) \frac{dk_1 dk_2}{(2\pi)^4}$$
$$- \frac{e^2}{m_e^2} \int \Big[2\omega_{pe}^2 \boldsymbol{e}_q \cdot \frac{\partial f_{0e}}{\partial \boldsymbol{v}} \frac{1}{\Omega \varepsilon_d(q)} Z_e(q, k_1, -k_2+q', -k_2, q')$$
$$- 2\mathscr{R}_e(k_1, -k_2+q', -k_2, q') \Big] E_h^{\sigma_1^+}(k_1) E_h^{\sigma_2^-}(k_2)^* \delta^4(q-k_1+k_2-q')$$
$$\times \frac{dk_1 dk_2 dq'}{(2\pi)^8} + 4\int C_e(q, k_1, -k_2+q', -k_2, q', k_3, -k_4, \boldsymbol{v})$$
$$\times E_h^{\sigma_1^+}(k_1) E_h^{\sigma_2^-}(k_2)^* E_h^{\sigma_3^+}(k_3) E_h^{\sigma_4^-}(k_4)^* \cdot \delta^4(q-k_1+k_2-q')$$
$$\times \delta^4(q'-k_3+k_4) \frac{dk_1\cdots dk_4 dq'}{(2\pi)^{12}} \Big\}. \tag{26}$$

现在处理高频场方程. 利用 (13) 和 (14) 式，取 (10) 式到四级场，可得

$$\boldsymbol{e}_h^{\sigma}(k) E_h^{\sigma}(k) = -\frac{1}{\omega} \sum_{\sigma_1} \frac{\omega_{pe}^2}{n_0} \int \frac{\boldsymbol{e}_k \cdot \boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \boldsymbol{e}_1' \cdot \frac{\partial \delta F_e(q)}{\partial \boldsymbol{v}} E_h^{\sigma_1}(k_1)$$
$$\times \delta^4(k-k_1-q) \frac{dk_1 dq}{(2\pi)^4} d^3v - \frac{1}{\omega} \sum_{\sigma_5\sigma_6} \omega_{pe}^4 \frac{e^2}{m_e^2}$$
$$\times \int_{(d)} \Big\{ \sum_{\sigma_1} \int \widetilde{H}_e(k, q, k_1) E_h^{\sigma_1}(k_1) \delta^4(k-k_1-q) dk_1$$
$$+ \sum_{\sigma_2} \frac{1}{n_0} \int \widetilde{Z}_e(k, q, k_2+q', k_2, q') E_h^{\sigma_2}(k_2) \delta^4$$
$$\times (k-k_2-q-q') \frac{dk_2 dq'}{(2\pi)^4} + \sum_{\sigma_2\sigma_3\sigma_4} \omega_{pe}^2 \frac{e^2}{m_e^2}$$
$$\times \iint_{(d)'} \widetilde{K}_e(k, q, k_2+q', q', k_2) \frac{H_e(q', k_3, k_4)}{\Omega' \varepsilon_d(q')} E_h^{\sigma_2}(k_2) E_h^{\sigma_3}(k_3) E_h^{\sigma_4}(k_4)$$
$$\times \delta^4(k-k_2-q-q') \delta^4(q'-k_3-k_4) \frac{dk_2 dk_3 dk_4}{(2\pi)^8} \Big\} \frac{H_e(q, k_5, k_6)}{\Omega \varepsilon_d(q)}$$

$$\times E_{h}^{\sigma_5}(k_5)E_{h}^{\sigma_6}(k_6)\cdot\delta^4(q-k_5-k_6)\frac{dk_5dk_6dq}{(2\pi)^8}-\frac{1}{\omega}\sum_{\sigma_1}\omega_{pe}^4\frac{e^2}{m_e^2}$$

$$\times\int\frac{\widetilde{H}_e(k,q,k_1)}{\Omega\epsilon_d(q)}E_{h}^{\sigma_1}(k_1)\cdot\left\{\sum_{\sigma_2\sigma_3}\frac{1}{n_0}\int_{(d)}Z_e(q,k_2,k_3+q',k_3,q')\right.$$

$$\times E_{h}^{\sigma_2}(k_2)E_{h}^{\sigma_3}(k_3)\delta^4(q-k_2-k_3)\frac{dk_2dk_3dq'}{(2\pi)^8}$$

$$+\sum_{\sigma_2\sigma_3\sigma_4\sigma_5}\omega_{pe}^2\frac{e^2}{m_e^2}\int_{(d)}\int_{(d')}K_e(q,k_2,k_3+q',q',k_2)$$

$$\times\frac{H_e(q',k_4,k_5)}{\Omega'\epsilon_d(q')}E_{h}^{\sigma_2}(k_2)E_{h}^{\sigma_3}(k_3)\cdot E_{h}^{\sigma_4}(k_4)E_{h}^{\sigma_5}(k_5)\delta^4(q-k_2-k_3-q')$$

$$\left.\times\delta^4(q'-k_4-k_5)\frac{dk_2\cdots dk_5dq'}{(2\pi)^{12}}\right\}\delta^4(k-k_1-q)\cdot\frac{dk_1dq}{(2\pi)^4},\tag{27}$$

式中

$$\widetilde{H}_e(k,q,k_1)=\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_q\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_1'\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}d^3v,\tag{28}$$

$$\widetilde{K}_e(k,q,k_2+q',q',k_2)=\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_q\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2+\Omega'-(\boldsymbol{k}_2+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_{q'}$$

$$\times\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_2'\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}d^3v,\tag{29}$$

$$\widetilde{Z}_e(k,k_2+q',k_2,q')=\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_q\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_2'\cdot\frac{\partial\delta F_e(q')}{\partial\boldsymbol{v}}}{(\omega_2+\Omega')-(\boldsymbol{k}_2+\boldsymbol{q}')\cdot\boldsymbol{v}}d^3v,\tag{30}$$

$\widetilde{H}_e$, $\widetilde{K}_e$ 及 $\widetilde{Z}_e$ 没有对称性.

为了化简 (27) 式，将对 (23) 式作速度积分以消去 (27) 式等号右边最后一个大括号项. 注意到

$$\epsilon_{d\alpha}(q)=\frac{\omega_{p\alpha}^2}{|\boldsymbol{q}|}\int\frac{\boldsymbol{e}_q\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}d^3v+1\quad\alpha=i,e,$$

得到

$$n_e(q)=\int\delta F_e(q)d^3v=-\frac{e^2}{m_e^2}\frac{|\boldsymbol{q}|}{\Omega}\frac{\epsilon_{di}(q)}{\epsilon_d(q)}\left\{\sum_{\sigma_1\sigma_2}n_0\int_{(d)}H_e(q,k_1,k_2)\right.$$

$$\times E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)\delta^4(q-k_1-k_2)\frac{dk_1dk_2}{(2\pi)^4}$$

$$+\sum_{\sigma_1\sigma_2}\iint_{(d)}Z_e(q,k_1,k_2+q',k_2,q')\cdot E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)$$

$$\times\delta^4(q-k_1-k_2-q')\frac{dk_1dk_2dq'}{(2\pi)^8}+\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\frac{\omega_{pe}^4}{4\pi m_e}$$

$$\times\iint_{(d)}\int_{(d')}K_e(q,k_1,k_2+q',q',k_2)\frac{H_e(q',k_3,k_4)}{\Omega'\epsilon_d(q')}E_{h}^{\sigma_1}(k_1)E_{h}^{\sigma_2}(k_2)E_{h}^{\sigma_3}(k_3)E_{h}^{\sigma_4}(k_4)$$

$$\times \delta^4(q-k_1-k_2-q')\delta^4(q'-k_3-k_4)\frac{dk_1\cdots dk_4dq'}{(2\pi)^{12}}\Big\}. \tag{31}$$

用(31)式代入(27)式,整理后得

$$\begin{aligned}\epsilon_h^\sigma(k)E_h^\sigma(k) = &-\frac{1}{\omega}\sum_{\sigma_1}\frac{\omega_{pe}^2}{n_0}\int\Big[\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_1'\cdot\frac{\partial\delta F_e(q)}{\partial\boldsymbol{v}}d^3v \\ &-\omega_{pe}^2\frac{\widetilde{H}_e(k,q,k_1)}{|\boldsymbol{q}|\epsilon_{di}(q)}n_e(q)\Big]\cdot E_h^{\sigma_1}(k_1)\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4} \\ &-\frac{1}{\omega}\sum_{\sigma_1\sigma_2\sigma_3}\omega_{pe}^2\frac{e^4}{n_0m_e^2}\iint_{(d)}\int_{(d')}\widetilde{Z}_e(k,q,k_1+q',k_1,q') \\ &\times\frac{H_e(q,k_2,k_3)}{\Omega\epsilon_d(q)}E_h^{\sigma_1}(k_1)E_h^{\sigma_2}(k_2)E_h^{\sigma_3}(k_3)\delta^4(k-k_1-q-q') \\ &\times\delta^4(q-k_2-k_3)\frac{dk_1dk_2dk_3dq'dq}{(2\pi)^{12}}-\frac{1}{\omega}\sum_{\sigma_1\cdots\sigma_5}\omega_{pe}^6\frac{e^4}{m_e^4} \\ &\times\iint_{(d)}\int_{(d')}\widetilde{K}_e(k,q;k_1+q',q',k_1)\frac{H_e(q,k_4,k_5)}{\Omega\epsilon_d(q)} \\ &\times\frac{H_e(q',k_2,k_3)}{\Omega'\epsilon_d(q')}E_h^{\sigma_1}(k_1)E_h^{\sigma_2}(k_2)E_h^{\sigma_3}(k_3)E_h^{\sigma_4}(k_4) \\ &\times E_h^{\sigma_5}(k_5)\delta^4(k-k_1-q-q')\delta^4(q'-k_2-k_3) \\ &\times\delta^4(q-k_4-k_5)\frac{dk_1\cdots dk_5dqdq'}{(2\pi)^{16}}.\end{aligned} \tag{32}$$

经过对向低频拍频波求和后,注意到 H_e 的对称性,从(32)式得到

$$\begin{aligned}\epsilon_h^\sigma(k)E_h^\sigma(k) = &-\frac{1}{\omega}\sum_{\sigma_1}\frac{\omega_{pe}^2}{n_0}\int\Big[\int\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_e(q)d^3v \\ &-\omega_{pe}^2\frac{\widetilde{H}_e(k,q,k_1)}{|\boldsymbol{q}|\epsilon_{di}(q)}n_e(q)\Big]E_h^{\sigma_1}(k_1)\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4} \\ &+\frac{2}{\omega}\sum_{\sigma_1}\omega_{pe}^4\frac{e^2}{n_0m_e^2}\cdot\iint\widetilde{Z}_e(k,q,k_1+q',k_1,q')\frac{H_e(q,k_2,-k_3)}{\Omega\epsilon_d(q)} \\ &\times E_h^{\sigma_1}(k_1)E_h^{\sigma_2^+}(k_2)E_h^{\sigma_3^-}(k_3)^*\delta^4(k-k_1-q-q') \\ &\times\delta^4(q-k_2+k_3)\frac{dk_1dk_2dk_3dqdq'}{(2\pi)^{12}}-\frac{4}{\omega}\sum_{\sigma_1}\omega_{pe}^6\frac{e^4}{m_e^4} \\ &\times\iint\widetilde{K}_e(k,q,k_1+q',q',k_1)\cdot\frac{H_e(q,k_4,-k_5)}{\Omega\epsilon_d(q)} \\ &\times\frac{H_e(q',k_2,-k_3)}{\Omega'\epsilon_d(q')}E_h^{\sigma_1}(k_1)E_h^{\sigma_2^+}(k_2)E_h^{\sigma_3^-}(k_3)^*E_h^{\sigma_4^+}(k_4)E_h^{\sigma_5^-}(k_5)^* \\ &\times\delta^4(k-k_2-q-q')\cdot\delta^4(q-k_2+k_3) \\ &\times\delta^4(q-k_4+k_5)\frac{dk_1\cdots dk_5dqdq'}{(2\pi)^{16}}.\end{aligned} \tag{33}$$

三、时空表象中动力学方程形式

方程(26)和(33)是F表象中粒子分布函数与高频场耦合的闭合方程组，现在我们着手在不同近似下把它们变换为时空表象中动力学方程组.

1. 二级场近似下的动力学方程组

这时方程(26)只保留大括号内头一项，二级场拍频形成有质动力，驱动着粒子低频振荡密度和分布函数的变化. 注意到非线性色散关系和非线性作用占优势过程中，高频波线性 Landau 阻尼不重要，这样可略去高频波虚部项，同时，在跃迁概率中高频波频率的 Doppler 加宽效应亦可取低阶近似. 经这样处理后，关系式(15)和(21)便为

$$
\begin{aligned}
H_e(q, k_1, k_2) &= \frac{\Omega}{\omega_2} H_e(k_2, q, -k_1) \\
&= \frac{1}{2}\frac{\Omega}{\omega_2}\left[\frac{|\boldsymbol{q}|}{\omega_{pe}^2}\frac{\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\omega_2}(\varepsilon_{de}(q)-1)+o\left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)_{1,2}\right].
\end{aligned} \tag{34}
$$

于是(24)式便为

$$
\begin{aligned}
B_e(q, k_1, -k_2) = -\frac{\omega_{pe}^2}{4\pi m_e}\Bigg[&\frac{|\boldsymbol{q}|\boldsymbol{e}_q\cdot\dfrac{\partial f_{0e}}{\partial \boldsymbol{v}}(\varepsilon_{de}(q)-1)}{2\omega_2^2\varepsilon_d(q)}(\boldsymbol{e}_1\cdot\boldsymbol{e}_2) \\
&+\mathscr{S}_e(k_1, -k_2, \boldsymbol{v})\Bigg].
\end{aligned} \tag{35}
$$

(34)式中$\Omega/\omega \ll 1$, $\boldsymbol{k}\cdot\boldsymbol{v}/\omega \ll 1$. 精确到同样阶

$$
\begin{aligned}
\mathscr{S}_e(k_1, -k_2) &= \frac{1}{2}\left[\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_2'\cdot\dfrac{\partial f_{0e}}{\partial\boldsymbol{v}}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}-\boldsymbol{e}_2'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\boldsymbol{e}_1'\cdot\dfrac{\partial f_{0e}}{\partial\boldsymbol{v}}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}\right] \\
&= \frac{1}{2}\left[\frac{\boldsymbol{e}_1'}{\omega_2}\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{e}_2\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}+\frac{\boldsymbol{e}_1'\cdot\boldsymbol{e}_2'}{(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})^2}\boldsymbol{k}_2\right. \\
&\qquad\left.\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}+\frac{\boldsymbol{e}_2\cdot\boldsymbol{v}}{\omega_2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})}\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{k}_2\times\frac{\partial f_{0e}}{\partial\boldsymbol{v}}-1\longleftrightarrow 2\right] \\
&= \frac{1}{2}\left[\frac{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}{\omega_1\omega_2}\boldsymbol{e}_1\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{e}_2\cdot\frac{\partial}{\partial\boldsymbol{v}}f_{0e}-\frac{\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\omega_2^2}|\boldsymbol{q}|\boldsymbol{e}_q\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}\right] \\
&\qquad+o\left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)_{1,2}^2.
\end{aligned} \tag{36}
$$

前面已经提到，我们只感兴趣频率接近等离子体频率的一类高频，所以可取 $\omega_1 \approx \omega_2 = \omega_{pe}$，于是在二级场拍频近似下，(26)式便为

$$
\begin{aligned}
\delta F_e(q) = \frac{1}{4\pi m_e(\Omega-\boldsymbol{q}\cdot\boldsymbol{v})}\int\Bigg[&-|\boldsymbol{q}|\boldsymbol{e}_q\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}}\frac{\varepsilon_{di}(q)}{\varepsilon_d(q)}(\boldsymbol{e}_1\cdot\boldsymbol{e}_2) \\
&+(\Omega-\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{e}_1\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{e}_2\cdot\frac{\partial}{\partial\boldsymbol{v}}f_{0e}\Bigg] \\
&\times E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*\delta^4(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^4}.
\end{aligned} \tag{37}
$$

下面我们专门研究低频波为离子声波的情况．如果不考虑电子、离子的阻尼效应，则

$$\frac{\epsilon_{di}(q)}{\epsilon_d(q)} \doteq -\frac{q^2v_s^2}{\Omega^2 - q^2v_s^2} \qquad 对 \quad qv_{T_i} \ll \Omega \ll qv_{T_e};$$

$$\doteq \frac{T_e}{T_e + T_i} \qquad 对 \quad \Omega \ll qv_{T_i}. \tag{38}$$

这里 $v_s = (T_e/m_i)^{1/2}$ 为离子声速度．

(37) 式两边乘 $(\Omega - \boldsymbol{q}\cdot\boldsymbol{v})$ 因子，然后作 Fourier 逆变换，注意到

$$\delta F_e(\boldsymbol{x}, \boldsymbol{v}, t) = \int \frac{d\Omega}{2\pi} \int \frac{d^3q}{(2\pi)^3} \delta F_e(q) e^{i(\boldsymbol{q}\cdot\boldsymbol{x} - \Omega t)},$$

$$\boldsymbol{E}^\sigma(\boldsymbol{x}, t) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \boldsymbol{e}_k E_k^\sigma(k) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{39}$$

便得到在 $qv_{T_i} \ll \Omega \ll qv_{T_e}$ 近似下时空表象中粒子低频分布函数动力学方程

$$\left(\frac{\partial^2}{\partial t^2} - v_s^2\nabla^2\right)\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\left(F_e(\boldsymbol{x}, \boldsymbol{v}, t) - \frac{\partial}{\partial \boldsymbol{v}}\cdot \mathbf{D}^\sigma \cdot \frac{\partial}{\partial \boldsymbol{v}} F_{0e}\right)$$

$$= -\frac{v_s^2}{8\pi m_e n_0} \nabla^2 \left(\frac{\partial F_{0e}}{\partial \boldsymbol{v}} \cdot \nabla |\boldsymbol{E}^\sigma(\boldsymbol{x}, t)|^2\right). \tag{40}$$

这里

$$F_{0e}(\boldsymbol{v}) = n_0 f_{0e}(\boldsymbol{v}), \quad F_e(\boldsymbol{x}, \boldsymbol{v}, t) = F_{0e}(\boldsymbol{v}) + \delta F_e(\boldsymbol{x}, \boldsymbol{v}, t), \tag{41}$$

$$\mathbf{D}^\sigma = \frac{1}{8\pi n_0 m_e} (\boldsymbol{E}^\sigma \boldsymbol{E}^{\sigma *})_s, \tag{42}$$

$$(\boldsymbol{E}^\sigma \boldsymbol{E}^{\sigma *})_s = 2\boldsymbol{E}^{\sigma +}(\boldsymbol{E}^{\sigma -})^*, \tag{42'}$$

$\mathbf{D}^\sigma$ 为对称张量．(40) 和 (42) 式中已用了关系式

$$\boldsymbol{E}^\sigma = \boldsymbol{E}^{\sigma +} + (\boldsymbol{E}^{\sigma -})^*. \tag{43}$$

在 $\Omega \ll qv_{T_i}$ 的静态近似下，(37) 式便为

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\left(F_e(\boldsymbol{x}, \boldsymbol{v}, t) - \frac{\partial}{\partial \boldsymbol{v}}\cdot \mathbf{D}^\sigma \cdot \frac{\partial}{\partial \boldsymbol{v}} F_{0e}\right)$$

$$= \frac{T_e}{8\pi n_0 m_e (T_i + T_e)} \frac{\partial F_{0e}}{\partial \boldsymbol{v}} \cdot \nabla |\boldsymbol{E}^\sigma|^2. \tag{44}$$

取 F_{0e} 为 Maxwell 分布，稳定态下 (44) 式就为

$$F_e(\boldsymbol{x}, \boldsymbol{v}) = \frac{\partial}{\partial \boldsymbol{v}}\cdot \mathbf{D}^\sigma \cdot \frac{\partial}{\partial \boldsymbol{v}} F_{0e} - \frac{|\boldsymbol{E}^\sigma|^2}{8\pi n_0 (T_i + T_e)} F_{0e}(\boldsymbol{v}). \tag{45}$$

(40) 或 (44) 式就是二级场近似下低频粒子分布函数(包含初态)所服从的动力方程．这里，除了有质动力与等离子体背景粒子分布函数相互作用驱动 F_e 的时空变化外，还附加了二级场作用下背景粒子在速度空间扩散过程的贡献．

当 Ω 靠近 qv_{T_i} 或 qv_{T_e} 端时，离子或电子的 Landau 阻尼起重要作用，本文不作进一步讨论，详见文献 [4]．

保留 (33) 式到二级场，得高频场方程

$$\epsilon_h^\sigma(k)E_h^\sigma(k) = -\frac{1}{\omega}\sum_{\sigma_1}\frac{\omega_{pe}^2}{n_0}\int\left[\frac{\boldsymbol{e}_k\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\boldsymbol{e}_1'\cdot\frac{\partial\delta F_e(q)}{\partial\boldsymbol{v}}d^3v - \frac{\omega_{pe}^2\widetilde{H}_e(k,q,k_1)}{|\boldsymbol{q}|\epsilon_{di}(q)}n_e(q)\right]$$
$$\times E_h^{\sigma_1}(k_1)\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4}. \tag{46}$$

由 (28) 式可知，方程右边第二项中被积函数较第一项至少小一个因子 $\boldsymbol{k}_1\cdot\boldsymbol{v}/\omega_1$，故可略去．于是就有

$$\epsilon_h^\sigma(k)E_h^\sigma(k) = \frac{\omega_{pe}^2}{\omega^2}\sum_{\sigma_1}\int(\boldsymbol{e}_k\cdot\boldsymbol{e}_1)\frac{\delta F_e(q)}{n_0}E_h^{\sigma_1}(k_1)\delta^4(k-k_1-q)\frac{dk_1dq}{(2\pi)^4}d^3v. \tag{47}$$

利用 (11) 和 (12) 式对上式作 Fourier 逆变换后得高频场方程为

$$\frac{\partial^2\boldsymbol{E}^\sigma}{\partial t^2}-(u^\sigma)^2\nabla^2\boldsymbol{E}^\sigma = -\omega_{pe}^2\boldsymbol{E}^\sigma-\frac{\omega_{pe}^2\boldsymbol{E}^\sigma}{n_0}\int\delta F_e(\boldsymbol{x},\boldsymbol{v},t)d^3v, \tag{48}$$

$$\begin{aligned} u^\sigma &= c\ (\text{光速}) \quad &\text{对}\quad \sigma = t;\\ &= \sqrt{3}\,v_{T_e} \quad &\text{对}\quad \sigma = l. \end{aligned} \tag{49}$$

(40) 或 (44) 与 (48) 式就为二级场近似下的闭合方程组．

如果对 (37) 式进行速度积分，便得

$$n_e(q) = -\frac{|\boldsymbol{q}|^2}{4\pi m_e\omega_{pe}^2}\frac{\epsilon_{di}(q)}{\epsilon_d(q)}(\epsilon_{de}(q)-1)\int(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*$$
$$\times\delta^4(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^4}. \tag{50}$$

对 (50) 式作 Fourier 逆变换，展开 $\epsilon_{di}(\epsilon_{de}-1)/\epsilon_d$ 因子，便可得到不同近似下密度时空变化方程，此方程与高频场 (47) 式耦合构成了广义的 Zakharov 方程组[4]．

2. 四级场近似下的动力学方程组

先求扰动密度 $n_e(q)$ 的方程．对 (26) 式进行速度积分后得到

$$\begin{aligned} n_e(q) = \frac{\omega_{pe}^2}{2\pi m_e}\frac{|\boldsymbol{q}|}{\Omega}\frac{\epsilon_{di}(q)}{\epsilon_d(q)}\Bigg\{&\iint H_e(q,k_1,-k_2)E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*\\ &\times\delta^4(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^4}+\frac{1}{n_0}\int Z_e(q,k_1,-k_2+q',-k_2,q')\\ &\times E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*\delta^4(q-k_1+k_2-q')\frac{dk_1dk_2dq'}{(2\pi)^8}\\ &-\frac{\omega_{pe}^4}{2\pi n_0m_e}\int K_e(q,k_1,-k_2+q',q',-k_2)\frac{H_e(q',k_3,-k_4)}{\Omega'\epsilon_d(q')}\\ &\times E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*E_h^{\sigma_3^+}(k_3)E_h^{\sigma_4^-}(k_4)^*\\ &\times\delta^4(q-k_1+k_2-q')\delta^4(q'-k_3+k_4)\frac{dk_1\cdots dk_4dq'}{(2\pi)^{12}}\Bigg\}. \end{aligned} \tag{51}$$

这里 Z_e 见 (17) 式，其中

$$2\mathscr{R}_e(k_1,-k_2+q',-k_2,q') = \boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_2-\Omega'-(\boldsymbol{k}_2-\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_2'\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_e(q')$$
$$-\boldsymbol{e}_2'\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{1}{\omega_1+\Omega'-(\boldsymbol{k}_1+\boldsymbol{q}')\cdot\boldsymbol{v}}\boldsymbol{e}_1'\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_e(q')$$

$$= \left(e_1' \cdot \frac{\partial}{\partial v} \frac{1}{\omega_2 - k_2 \cdot v} e_2' \cdot \frac{\partial}{\partial v} \delta F_e(q') - 1 \longleftrightarrow 2 \right)$$

$$+ \left(e_1 \cdot \frac{\partial}{\partial v} \frac{\Omega' - q' \cdot v}{(\omega_2 - k_2 \cdot v)^2} e_2 \cdot \frac{\partial}{\partial v} \delta F_e(q') + 1 \longleftrightarrow 2 \right) + o \left(\frac{\Omega' - q' \cdot v}{(\omega - k \cdot v)_{1,2}} \right)^2$$

$$= \frac{\Omega - \Omega' - (q - q') \cdot v}{\omega_1 \omega_2} e_1 \cdot \frac{\partial}{\partial v} e_2 \cdot \frac{\partial}{\partial v} \delta F_e(q') - \frac{e_1 \cdot e_2}{\omega_{pe}^2} (q - q') \cdot \frac{\partial}{\partial v} \delta F_e(q')$$

$$- \left(\frac{e_1 \cdot q' e_2 \cdot \frac{\partial}{\partial v} \delta F_e(q')}{(\omega_2 - k_2 \cdot v)^2} + 1 \longleftrightarrow 2 \right) + o \left(\frac{\Omega' - q' \cdot v}{(\omega - k \cdot v)_{1,2}} \right)^2$$

$$= \frac{\Omega + \Omega' - (q + q') \cdot v}{\omega_{pe}^2} e_1 \cdot \frac{\partial}{\partial v} e_2 \cdot \frac{\partial}{\partial v} \delta F_e(q') - \frac{e_1 \cdot e_2}{\omega_{pe}^2} (q - q') \cdot \frac{\partial \delta F_e(q')}{\partial v}$$

$$- \frac{1}{\omega_{pe}^2} \left(e_1 \cdot q' e_2 \cdot \frac{\partial}{\partial v} \delta F_e(q') + 1 \longleftrightarrow 2 \right) + o \left(\frac{\Omega' - q' \cdot v}{(\omega - k \cdot v)_{1,2}} \right)^2. \tag{52}$$

为了进一步求得 Z_e 具体表达式，(52) 式中的 $\delta F_e(q')$ 以 (26) 式中的二级场部分代入，即应用 (37) 式，经过对 Maxwell 分布背景粒子积分，取 $\Omega' \ll |q'| v_{T_e}$ 近似，便得

$$Z_e(q, k_1, -k_2 + q', -k_2, q') = \frac{\Omega}{8\pi T_e |q| v_{T_e}^2 \omega_{pe}^2} \int \Big\{ -(e_3 \cdot e_4) \frac{A(1,2)}{q^2} \frac{\varepsilon_{di}(q')}{\varepsilon_d(q')}$$

$$- 2B(1,2,3,4) + (e_1 \cdot e_2)(e_3 \cdot e_4) \frac{\varepsilon_{di}(q')}{\varepsilon_d(q')} \left[q^2 \lambda_e^2 (\varepsilon_{de}(q) - 1) - \frac{q \cdot q'}{q^2} \right]$$

$$+ 2(e_1 \cdot e_2)(q - q') \cdot q \frac{e_3 \cdot q e_4 \cdot q}{q^4} + \frac{1}{q^2} (e_1 \cdot q e_2 \cdot q' + 1 \longleftrightarrow 2)$$

$$\times \left(e_3 \cdot e_4 \frac{\varepsilon_{di}(q')}{\varepsilon_d(q')} + \frac{2}{q^2} e_3 \cdot q e_4 \cdot q \right) \Big\} E_k^{\sigma_3^+}(k_3) E_k^{\sigma_4^-}(k_4)^*$$

$$\times \delta^4(q' - k_3 + k_4) \frac{dk_3 dk_4}{(2\pi)^4}, \tag{53}$$

其中

$$A(1,2) \equiv (e_1 \cdot q e_2 \cdot q' + 1 \longleftrightarrow 2) - 2 e_1 \cdot q e_2 \cdot q \frac{q \cdot q'}{q^2}, \tag{54}$$

$$B(1,2,3,4) \equiv A(1,2) e_3 \cdot q e_4 \cdot q + \begin{pmatrix} 1 \longleftrightarrow 3 \\ 2 \longleftrightarrow 4 \end{pmatrix}, \tag{55}$$

$$\lambda_e = \left(\frac{T_e}{4\pi n_0 e^2} \right)^{1/2}. \tag{56}$$

关于 K_e 的计算比 Z_e 更复杂，但是比较方程 (51) 式右边第三项和第二项（其中 δF_e 已用表达式 (37) 代入）的被积函数后，很容易看到前者多了一个 $q \cdot v / \omega_{pe}$ 因子，故在四级场近似下 $n_e(q)$ 方程只取右边头两项就够了.

下面在空间为一维情况下讨论从 F 表象到二维时空表象的变换. 这时 (53) 式中 $A = B = 0$，对 $\sigma = t$，

$$Z_e(q, k_1, -k_2 + q', -k_2, q') = \frac{\Omega}{8\pi T_e q_x v_{T_e}^2 \omega_{pe}^2} \int \frac{\varepsilon_{di}(q')}{\varepsilon_d(q')} \left[q_x^2 \lambda_e^2 (\varepsilon_{de}(q) - 1) - \frac{q_x'}{q_x} \right]$$

$$\times E_h^{t_3^+}(k_3)E_h^{t_4^-}(k_4)^*\delta^2(q'-k_3+k_4)\frac{dk_3dk_4}{(2\pi)^2}. \tag{57}$$

这里 $q=\{\Omega, q_x\}$, $k=\{\omega, k_x\}$, $dk=d\omega dk_x$, $\delta^2(q'-k_3+k_4)=\delta(\Omega'-\omega_3+\omega_4)\delta(q'_x-k_{3x}+k_{4x})$, $E_h^t(k)$ 为垂直于 x 轴的 F 表象中量.

对 $\sigma=l$,

$$Z_e(q, k_1, -k_2+q', -k_2, q')=\frac{\Omega}{8\pi T_e q_x v_{T_e}^2\omega_{pe}^2}\int\left\{\frac{\epsilon_{di}(q')}{\epsilon_d(q')}\left[q_x^2\lambda_e^2(\epsilon_{de}(q)-1)+\frac{q'_x}{q_x}\right]+2\left(1+\frac{q'_x}{q_x}\right)\right\}E_h^{l_3^+}(k_3)E_h^{l_4^-}(k_4)^*\delta^2(q'-k_3+k_4)\frac{dk_3dk_4}{(2\pi)^2}. \tag{58}$$

这里的 $E_h^l(k)$ 为平行于 x 轴的 F 表象中量.

于是由 (51) 式可得四级场截断下的动力学方程,对 $\sigma=t$,

$$n_e(q)=-\frac{1}{4\pi T_e}q_x^2\lambda_e^2(\epsilon_{de}(q)-1)\frac{\epsilon_{di}(q)}{\epsilon_d(q)}\left\{\int E_h^{t_1^+}(k_1)E_h^{t_2^-}(k_2)\times\delta^2(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^2}-\frac{1}{4\pi n_0T_eq_x^2\lambda_e^2(\epsilon_{de}(q)-1)}\times\int\frac{\epsilon_{di}(q')}{\epsilon_d(q')}\left[q_x^2\lambda_e^2(\epsilon_{de}(q)-1)-\frac{q'_x}{q_x}\right]E_h^{t_1^+}(k_1)E_h^{t_2^-}(k_2)^*\times E_h^{t_3^+}(k_3)E_h^{t_4^-}(k_4)^*\delta^2(q-k_1+k_2-q')\times\delta^2(q'-k_3+k_4)\frac{dk_1\cdots dk_4dq'_x}{(2\pi)^8}\right\}. \tag{59}$$

当阻尼不重要时, $q_xv_{T_i}\ll\Omega\ll q_xv_{T_e}$,这时

$$n_e(q)=\frac{1}{4\pi T_e}\frac{q_xv_s^2}{\Omega^2-q_x^2v_s^2}\left\{q_x\int E_h^{t_1^-}(k_1)E_h^{t_2^-}(k_2)^*\delta^2(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^2}-\frac{1}{4\pi n_0T_e}\int(q_x-q'_x)\cdot\frac{q_x'^2v_s^2}{q_x'^2v_s^2-\Omega'^2}E_h^{t_1^+}(k_1)E_h^{t_2^-}(k_2)^*E_h^{t_3^+}(k_3)E_h^{t_4^-}(k_4)^*\times\delta^2(q-k_1+k_2-q')\delta^2(q'-k_3+k_4)\frac{dk_1\cdots dk_4dq'}{(2\pi)^8}\right\}. \tag{60}$$

变换到二维时空表象,得

$$\left(\frac{\partial^2}{\partial t^2}-v_s^2\frac{\partial^2}{\partial x^2}\right)n_e(x, t)=\frac{1}{8\pi m_i}\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}|E^t(x, t)|^2-\left[\frac{\partial}{\partial x}(|E^t(x, t)|^2\psi_t(x, t))-|E^t(x, t)|^2\frac{\partial}{\partial x}\psi_t(x, t)\right]\right\},$$

即有

$$\left(\frac{\partial^2}{\partial t^2}-v_s^2\frac{\partial^2}{\partial x^2}\right)n_e(x, t)=\frac{1}{8\pi m_i}\frac{\partial}{\partial x}\left\{\left(1-\frac{\psi_t(x, t)}{8\pi n_0T_e}\right)\frac{\partial}{\partial x}|E^t(x, t)|^2\right\}, \tag{61}$$

其中

$$\psi_t(x, t)=2\int\frac{1}{\left(1-\left(\frac{v_\phi}{v_s}\right)^2\right)}E_h^{t_3^+}(k_3)E_h^{t_4^-}(k_4)^*e^{i(k_3-k_4)_xx-i(\omega_3-\omega_4)t}\frac{dk_3dk_4}{(2\pi)^4},$$

$$v_\phi = \frac{\Omega'}{q'_x}, \quad q'_x = (\boldsymbol{k}_3 - \boldsymbol{k}_4)_x, \quad \Omega' = \omega_3 - \omega_4. \tag{62}$$

在超声情况下 $\phi_t < 0$；亚声情况下 $\phi_t > 0$，在弱亚声时，$\frac{\Omega'}{q'_x} \ll v_s$，$\phi_t = |E^t|^2$.

上述表达式中的 E^t 见 (43) 式表示.

当相速度 v_ϕ 不随频率和波数变化时(例横波 Soliton)，(61) 式便为

$$\left(\frac{\partial^2}{\partial t^2} - v_s^2 \frac{\partial^2}{\partial x^2}\right) n_e(x, t) = \frac{1}{8\pi m_i} \frac{\partial^2}{\partial x^2} \times \left\{ |E^t|^2 \left(1 - \frac{|E^t|^2}{\left[1 - \left(\frac{v_\phi}{v_s}\right)^2\right] 16\pi n_0 T_e}\right)\right\}. \tag{63}$$

在 $\Omega \ll q_x v_{T_i}$ 静态近似下，(51) 式变换结果为

$$n_e(x) = -\frac{|E^t|^2}{8\pi(T_e + T_i)}\left(1 - \frac{|E^t|^2}{16\pi n_0(T_e + T_i)}\right). \tag{64}$$

对 $\sigma = l$，再一次略去 Landau 阻尼，(51) 式变为

$$\begin{aligned} n_e(q) = \frac{1}{4\pi T_e} \frac{q_x v_s^2}{(\Omega^2 - q_x^2 v_s^2)} \Big\{ q_x \int E_h^{l_1^+}(k_1) E_h^{l_2^-}(k_2)^* \delta^2(q - k_1 + k_2) \frac{dk_1 dk_2}{(2\pi)^2} \\ - \frac{1}{4\pi n_0 T_e} \int (q_x + q'_x)\left(2 + \frac{q_x'^2 v_s^2}{q_x'^2 v_s^2 - \Omega'^2}\right) E_h^{l_1^+}(k_1) E_h^{l_2^-}(k_2)^* E_h^{l_3^+}(k_3) E_h^{l_4^-}(k_4)^* \\ \times \delta^2(q - k_1 + k_2 - q')\delta^2(q' - k_3 + k_4) \frac{dk_1 \cdots dk_4 dq'_x}{(2\pi)^8} \Big\}. \end{aligned} \tag{65}$$

变换到二维时空表象，即有

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - v_s^2 \frac{\partial^2}{\partial x^2}\right) n_e(x, t) = \frac{1}{8\pi m_i} \frac{\partial}{\partial x} \Big\{ \left(1 - \frac{\phi_e(x, t)}{8\pi n_0 T_e}\right) \frac{\partial}{\partial x} |E^l(x, t)|^2 \\ - \frac{|E^l(x, t)|^2}{4\pi n_0 T_e} \frac{\partial}{\partial x} \phi_l(x, t) \Big\}. \end{aligned} \tag{66}$$

这里

$$\phi_l(x, t) = 2 \int \beta E_h^{l_3^+}(k_3) E_h^{l_4^-}(k_4)^* \exp[i(\boldsymbol{k}_3 - \boldsymbol{k}_4)_x x - i(\omega_3 - \omega_4)t] \frac{dk_3 dk_4}{(2\pi)^4}. \tag{67}$$

$$\beta = \frac{3 - 2\left(\frac{v_\phi}{v_s}\right)^2}{1 - \left(\frac{v_\phi}{v_s}\right)^2} < 0 \qquad \text{当} \quad 1 < \left(\frac{v_\phi}{v_s}\right)^2 < \frac{3}{2};$$

$$> 0, \qquad \text{其它区间}. \tag{68}$$

当相速度不随频率和波数变化时(例纵波 Soliton)，

$$\phi_l(x, t) = \beta |E^l(x, t)|^2. \tag{69}$$

这时方程 (66) 就为

$$\left(\frac{\partial^2}{\partial t^2} - v_s^2 \frac{\partial^2}{\partial x^2}\right) n_e(x, t) = \frac{1}{8\pi m_i} \frac{\partial^2}{\partial x^2} \left\{ |E^l(x, t)|^2 \left(1 - \frac{3\beta |E^l(x, t)|^2}{16\pi n_0 T_e}\right)\right\}. \tag{70}$$

在 $\Omega \ll q_x v_{T_i}$ 静态近似下，得

$$n_e(\boldsymbol{x}) = -\frac{|E^l(x)|^2}{8\pi(T_e+T_i)}\left(1-\frac{3(2T_i+3T_e)}{16\pi n_0 T_e(T_i+T_e)}|E^l(x)|^2\right). \tag{71}$$

当有阻尼时介质函数的极化项必须保留虚部，这里不作进一步讨论.

在四维时空表象情况下，利用 (53) 式变换会出现结构复杂的项，为了简单起见，假定参与两个高频波拍频的第三个低频波其频率和波数远较新生成的波小，即 $\Omega'\ll\Omega$, $|\boldsymbol{q}'|\ll|\boldsymbol{q}|$，这时 (53) 式成为

$$Z_e = \frac{\Omega(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)}{8\pi T_e|\boldsymbol{q}|v_{T_e}^2\omega_{pe}^2}\int\left\{(\boldsymbol{e}_3\cdot\boldsymbol{e}_4)q^2\lambda_e^2(\varepsilon_{de}(q)-1)\frac{\varepsilon_{di}(q')}{\varepsilon_d(q')}+2\frac{\boldsymbol{e}_3\cdot\boldsymbol{q}\,\boldsymbol{e}_4\cdot\boldsymbol{q}}{q^2}\right\}$$
$$\times E_h^{\sigma_3^+}(k_3)E_h^{\sigma_4^-}(k_4)^*\delta^4(q-k_3+k_4)\frac{dk_3dk_4}{(2\pi)^4}. \tag{72}$$

这样，在忽略阻尼情况下，扰动密度空间三维动力学方程便为

$$\left(\frac{\partial^2}{\partial t^2}-v_s^2\nabla^2\right)n_e(\boldsymbol{x},t)=\frac{1}{8\pi m_i}\nabla\cdot\left\{\left(1-\frac{\psi_\sigma(\boldsymbol{x},t)}{8\pi n_0T_e}\right)\nabla|\boldsymbol{E}^\sigma(\boldsymbol{x},t)|^2\right.$$
$$\left.-\frac{1}{8\pi n_0T_e}(|\boldsymbol{E}^\sigma|^2\nabla\psi_\sigma+2\nabla\cdot[(\boldsymbol{E}^\sigma\boldsymbol{E}^{\sigma*})_s|\boldsymbol{E}^\sigma|^2])\right\}, \tag{73}$$

$$\psi_\sigma(\boldsymbol{x},t)=2\int\frac{1}{1-\left(\dfrac{v_\phi}{v_s}\right)^2}(\boldsymbol{e}_3\cdot\boldsymbol{e}_4)\boldsymbol{E}_h^{\sigma_3^+}(k_3)\boldsymbol{E}_h^{\sigma_4^-}(k_4)^*\exp$$
$$\times[i(\boldsymbol{k}_3-\boldsymbol{k}_4)\cdot\boldsymbol{x}-i(\omega_3-\omega_4)t]\frac{dk_3dk_4}{(2\pi)^8}. \tag{74}$$

在静态近似下，得到

$$\nabla n_e = -\frac{1}{8\pi(T_i+T_e)}\left\{\nabla\left[|\boldsymbol{E}^\sigma|^2\left(1-\frac{|\boldsymbol{E}^\sigma|^2}{8\pi n_0(T_i+T_e)}\right)\right]\right.$$
$$\left.-\frac{1}{4\pi n_0T_e}\nabla\cdot[(\boldsymbol{E}^\sigma\boldsymbol{E}^{\sigma*})_s|\boldsymbol{E}^\sigma|^2]\right\}. \tag{75}$$

关于四级场近似下高频场方程，从 (33) 式可以看到，除了右边方括号内第一项外，其余项的速度空间被积函数中至少多一个 $\boldsymbol{k}\cdot\boldsymbol{v}/\omega_{pe}$ 因子，故可以忽略．因此，高阶场近似下高频场方程形式完全同二级场的 (47) 式一样，所不同的是这时 n_e 中包含了四级场项．这样，方程 (47) 与二维方程 (63) 或 (64)，(70) 或 (71) 以及四维方程 (73) 或 (75) 构成了相应条件下的各个闭合方程组.

现在讨论粒子速度分布所服从的方程，仍从方程 (26) 出发，并略去右边最后一项．为了得到简洁表达式，利用方程 (51) 代入 (26) 式以消去含 Z_e 的项，这样得到

$$\delta F_e(q)=\frac{-1}{\Omega-\boldsymbol{q}\cdot\boldsymbol{v}}\left\{\frac{\omega_{pe}^2\boldsymbol{q}\cdot\dfrac{\partial f_{0e}}{\partial\boldsymbol{v}}}{|\boldsymbol{q}|^2\varepsilon_{di}(q)}\int\delta F_e(q)d^3v-\right.$$
$$\frac{\omega_{pe}^2}{2\pi m_e}\int\mathscr{S}_e(k_1,-k_2,\boldsymbol{v})\ E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*$$
$$\times\delta^4(q-k_1+k_2)\frac{dk_1dk_2}{(2\pi)^4}-\frac{\omega_{pe}^2}{2\pi n_0m_e}\int\mathscr{R}_e(k_1,-k_2+q',-k_2,q')$$

$$\times E_h^{\sigma_1^+}(k_1)E_h^{\sigma_2^-}(k_2)^*\delta^4(q-k_1+k_2-q')\cdot\frac{dk_1dk_2dq'}{(2\pi)^8}\Big\}. \tag{76}$$

应用关系式(36)和(53)，注意到 $qv_{T_i}\ll\Omega\ll qv_{T_e}$ 时，$\epsilon_{di}=-\omega_{pi}^2/\Omega^2$，在忽略阻尼条件下，(76)式变换到时空表象后得

$$\begin{aligned}\nabla^2\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)F_e(\boldsymbol{x},\boldsymbol{v},t)&=-\frac{m_i}{m_e}\frac{\partial^2}{\partial t^2}\left(\frac{\partial F_{0e}}{\partial\boldsymbol{v}}\cdot\frac{\nabla n_e(\boldsymbol{x},t)}{n_0}\right)\\&+\nabla^2\left\{\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{D}^\sigma\cdot\frac{\partial}{\partial\boldsymbol{v}}\nabla F_e(\boldsymbol{x},\boldsymbol{v},t)\right.\\&+\frac{\partial}{\partial\boldsymbol{v}}F_e(\boldsymbol{x},\boldsymbol{v},t)\cdot\nabla\frac{|\boldsymbol{E}^\sigma|^2}{8\pi n_0m_e}+\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{D}^\sigma\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\partial}{\partial t}F_e(\boldsymbol{x},\boldsymbol{v},t)\\&\left.+\left(\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{D}^\sigma\cdot\frac{\partial}{\partial\boldsymbol{v}}\nabla F_e\right)\cdot\boldsymbol{v}+\frac{\partial}{\partial\boldsymbol{v}}\cdot(\boldsymbol{D}^\sigma+\boldsymbol{D}^{\sigma*})\cdot\nabla F_e\right\}.\end{aligned} \tag{77}$$

这里 F_e 和 $\boldsymbol{D}^\sigma$ 的表达式见(41)和(42)式.

(77)式稳定态为

$$\begin{aligned}\boldsymbol{v}\cdot\nabla F_e&=\boldsymbol{v}\cdot\nabla\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{D}^\sigma\cdot\frac{\partial}{\partial\boldsymbol{v}}F_e+\frac{\partial}{\partial\boldsymbol{v}}F_e\cdot\nabla\frac{|\boldsymbol{E}^\sigma|^2}{8\pi n_0m_e}\\&+\left(\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{D}^\sigma\cdot\frac{\partial}{\partial\boldsymbol{v}}\nabla F_e\right)\cdot\boldsymbol{v}+\frac{\partial}{\partial\boldsymbol{v}}\cdot(\boldsymbol{D}^\sigma+\boldsymbol{D}^{\sigma*})\cdot\nabla F_e.\end{aligned} \tag{78}$$

可见 F_e 的变化包含着速度空间的扩散过程.

四、拉氏函数和守恒量

我们讨论二维时空表象中横波或纵波有质动力作用下的密度方程(63),(64)和(70),(71)的拉氏密度函数 $\mathscr{L}$ 和守恒量. 首先讨论静态近似下结果. (64)或(71)式代入(48)式后得高频场方程 ($\sigma=l,t$)

$$\begin{aligned}\frac{\partial^2E^\sigma}{\partial t'^2}-(u^\sigma)^2\frac{\partial^2}{\partial x'^2}E^\sigma&=-\omega_{pe}^2E^\sigma+\omega_{pe}^2E^\sigma\frac{|E^\sigma|^2}{8\pi n_0(T_i+T_e)}\\&\times\left(1-\frac{|E^\sigma|^2}{16\pi n_0(T_e+T_i)}\right).\end{aligned} \tag{79}$$

分离 E^σ 的缓变部分和高频振荡部分，令

$$E^\sigma(x',t')=\tilde{\mathscr{E}}^\sigma(x',t')e^{-i\omega_{pe}t'}. \tag{80}$$

这里 $\tilde{\mathscr{E}}^\sigma(x',t')$ 为随时间缓慢变化振幅，代入方程(79)，注意到

$$\omega_{pe}\tilde{\mathscr{E}}^\sigma\gg\frac{\partial}{\partial t'}\tilde{\mathscr{E}}^\sigma,$$

略去时间的二阶导数项，得到包含四级场作用下的高阶非线性 Schrödinger 方程

$$\begin{aligned}i\frac{\partial}{\partial t}\mathscr{E}^\sigma(x,t)+\frac{\partial^2}{\partial x^2}\mathscr{E}^\sigma(x,t)&=-\mathscr{E}^\sigma(x,t)|\mathscr{E}^\sigma(x,t)|^2\\&\times(1-|\mathscr{E}^\sigma(x,t)|^2).\end{aligned} \tag{81}$$

(81)式中已用了无量纲关系

$$\mathscr{E}^\sigma = \frac{\tilde{\mathscr{E}}_\sigma}{\sqrt{16\pi n_0(T_i + T_e)}}, \quad t = \omega_{pe} t', \quad x = \sqrt{\frac{2\omega_{pe}^2}{(u^\sigma)^2}}\, x'. \tag{82}$$

注意，现在空间座标的标度与波的种类有关。

很容易写出 (81) 式的拉氏密度函数为

$$\mathscr{L}^\sigma = \frac{1}{2} i(\mathscr{E}^{\sigma *}\mathscr{E}_t^\sigma - \mathscr{E}^\sigma \mathscr{E}_t^{\sigma *}) - |\mathscr{E}_x^\sigma|^2 + \frac{1}{2}|\mathscr{E}^\sigma|^4 - \frac{1}{3}|\mathscr{E}^\sigma|^6. \tag{83}$$

这里 $\mathscr{E}_t^\sigma = \dfrac{\partial \mathscr{E}^\sigma}{\partial t}$, $\mathscr{E}_x^\sigma = \dfrac{\partial \mathscr{E}^\sigma}{\partial x}$.

现在给出方程 (81) 式的三个运动积分.

由四度流守恒: $\dfrac{\partial j_\mu}{\partial x_\mu} = 0$, $j_\mu = -i\left(\dfrac{\partial \mathscr{L}}{\partial \mathscr{E}_\mu}\mathscr{E} - \dfrac{\partial \mathscr{L}}{\partial \mathscr{E}_\mu^*}\mathscr{E}^*\right)$, $\mu = 1, 2$, $x_\mu = x_\mu(t, x)$, $\mathscr{E}_\mu = \mathscr{E}_\mu(\mathscr{E}_t, \mathscr{E}_x)$, 得粒子数守恒

$$N = \int j_0 dx = \int |\mathscr{E}^\sigma|^2 dx = \text{const}. \tag{84}$$

由动量-能量张量密度守恒 $\dfrac{\partial T_{\mu\nu}}{\partial x_\nu} = 0$, $T_{\mu\nu} = \mathscr{L}\delta_{\mu\nu} - \mathscr{E}_\mu^{(i)}\dfrac{\partial \mathscr{L}}{\partial \mathscr{E}_\nu^{(i)}}$, $\mathscr{E}^{(i)} = \{\mathscr{E}, \mathscr{E}^*\}$, 得动量守恒

$$P = \int T_{10} dx = \frac{1}{2} i \int (\mathscr{E}^\sigma \mathscr{E}_x^{\sigma *} - \mathscr{E}^{\sigma *}\mathscr{E}_x^\sigma) dx. \tag{85}$$

能量守恒

$$G = -\int T_{00} dx = \int \left(|\mathscr{E}_x^\sigma|^2 - \frac{1}{2}|\mathscr{E}^\sigma|^4 + \frac{1}{3}|\mathscr{E}^\sigma|^6 \right) dx. \tag{86}$$

在非静态近似下，这时 $\mathscr{L}$ 中包含着密度 n_e 与场的相互作用项. 先由 (48) 式写出高频场方程

$$\frac{\partial^2 E^\sigma}{\partial t'^2} - (u^\sigma)^2 \frac{\partial^2}{\partial x'^2} E^\sigma = -\omega_{pe}^2 E^\sigma \left(1 + \frac{\tilde{n}_e}{n_0}\right). \tag{87}$$

为了以后应用方便，这里已把 n_e 改写为 $\tilde{n}_e$. 采用表示式 (80) 后，得

$$\frac{i}{\omega_{pe}} \frac{\partial \tilde{\mathscr{E}}^\sigma}{\partial t'} + \frac{(u^\sigma)^2}{2\omega_{pe}^2} \frac{\partial^2}{\partial x'^2} \tilde{\mathscr{E}}^\sigma = \frac{\tilde{\mathscr{E}}^\sigma}{2} \frac{\tilde{n}_e}{n_0}. \tag{88}$$

其次把方程 (63) 和 (70) 统一写成

$$\left(\frac{\partial^2}{\partial t'^2} - v_s^2 \frac{\partial^2}{\partial x'^2}\right) \tilde{n}_e(x', t') = \frac{1}{8\pi m_i} \frac{\partial^2}{\partial x'^2} \times \left\{ |\tilde{\mathscr{E}}^\sigma|^2 \left(1 - \frac{\gamma_\sigma}{(1 - V_\phi^2)} \frac{|\tilde{\mathscr{E}}^\sigma|^2}{16\pi n_0 T_e}\right)\right\}. \tag{89}$$

这里

$$V_\phi = v_\phi / v_s, \tag{90}$$

$$\gamma_\sigma = 1 \quad 对 \quad \sigma = t;$$
$$= \frac{3(2T_i + 3T_e)}{T_i + T_e} \quad 对 \quad \sigma = l. \tag{91}$$

引入表示：$\xi_\sigma = \frac{v_s}{u^\sigma}$，$t = 2\omega_{pe}\xi_\sigma^2 t'$，$x = \frac{2\omega_{pe}\xi_\sigma^2}{v_s} x'$，

$$n_e = \frac{\tilde{n}_e}{4n_0\xi_\sigma^2}, \quad |\mathscr{E}^\sigma|^2 = \frac{|\tilde{\mathscr{E}}^\sigma|^2}{32\pi n_0\xi_\sigma^2 T_e}. \tag{92}$$

(88) 和 (89) 式就分别写成为

$$i\frac{\partial \mathscr{E}^\sigma}{\partial t} + \frac{\partial^2}{\partial x^2}\mathscr{E}^\sigma = \mathscr{E}^\sigma n_e, \tag{93}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) n_e = \frac{\partial^2}{\partial x^2}\left\{|\mathscr{E}^\sigma|^2\left(1 - \frac{\Gamma_\sigma}{1 - V_\phi^2}|\mathscr{E}^\sigma|^2\right)\right\}, \tag{94}$$

其中

$$\Gamma_\sigma = 2\gamma_\sigma\xi_\sigma^2. \tag{95}$$

对于 (93) 和 (94) 式我们只能构筑一个近似的拉氏密度函数

$$\mathscr{L}^\sigma = \frac{1}{2}i(\mathscr{E}^{\sigma*}\mathscr{E}_t^\sigma - \mathscr{E}^\sigma\mathscr{E}_t^{\sigma*}) - |\mathscr{E}_x^\sigma|^2 + \frac{1}{2}|\mathscr{E}^\sigma|^4 - \frac{\Gamma_\sigma}{3(1 - V_\phi^2)}|\mathscr{E}^\sigma|^6 - \varphi_t|\mathscr{E}^\sigma|^2\left(1 - \frac{\Gamma_\sigma}{(1 - V_\phi^2)}|\mathscr{E}^\sigma|^2\right) + \frac{1}{2}(\varphi_t^2 - \varphi_x^2). \tag{96}$$

这里

$$\varphi_t = n_e + |\mathscr{E}^\sigma|^2\left(1 - \frac{\Gamma_\sigma}{1 - V_\phi^2}|\mathscr{E}^\sigma|^2\right), \tag{97}$$

$$\varphi_{tt} = \varphi_{xx} + |\mathscr{E}^\sigma|^2\left(1 - \frac{\Gamma_\sigma}{1 - V_\phi^2}|\mathscr{E}^\sigma|^2\right). \tag{98}$$

(96) 式可以准确地给出方程 (94)，但是在导得 (93) 式时多了高阶项

$\frac{2\Gamma_\sigma}{1 - V_\phi^2}|\mathscr{E}^\sigma|^2(n_e + |\mathscr{E}^\sigma|^2)\mathscr{E}^\sigma - 2\frac{\Gamma_\sigma^2}{(1 - V_\phi^2)^2}|\mathscr{E}^\sigma|^6\mathscr{E}^\sigma$，除了第二项超过方程本身精度自然可忽略外，对四级场部分，在稳定态下取 $n_e = -|\mathscr{E}^\sigma|^2$，第一项也可近似消去.

从(96)式可以得到类似于 (84)—(86) 式的三个守恒量.

五、结论和讨论

1. 从方程 (48)，(63)，(70) 或 (93)，(94) 可以看到，四级场的出现在某些情况下将明显改变作为有质动力的二级场作用. 例如考虑以下解:

$\xi = x - V_\phi t$，这里 V_ϕ 取用 (90) 式特定形式. 再令

$$\mathscr{E}^\sigma(x, t) = \mathscr{E}_0^\sigma(\xi)e^{iqx - i\theta t}, \quad \mathscr{E}_0 \text{ 为实数}, \tag{99}$$

$$n_e(x, t) = n_e(\xi). \tag{100}$$

由 (93) 式就得到

$$n_e(\xi) = -\frac{(\mathscr{E}_0^\sigma)^2}{1 - V_\phi^2}\left[1 - \frac{\Gamma_\sigma}{(1 - V_\phi^2)}(\mathscr{E}_0^\sigma)^2\right]. \tag{101}$$

(94) 式则为

$$\frac{1}{2}(\mathscr{E}_{0\xi})^2 + b_0\mathscr{E}_0^4 - a_0\mathscr{E}_0^6 - c_0\mathscr{E}_0^2 = 0, \tag{102}$$

$$a_0=\frac{\Gamma_\sigma}{6(1-V_\phi^2)^2},\quad b_0=\frac{1}{4(1-V_\phi^2)},\quad c_0=\frac{1}{2}\left(\frac{V_\phi^2}{4}-\Theta\right). \tag{103}$$

方程(102)是不难解的,但我们直接从相平面上势

$$V=-a_0\mathscr{E}_0^6+b_0\mathscr{E}_0^4-c_0\mathscr{E}_0^2, \tag{104}$$

就可以看到，有了四级场后仍然存在 Soliton 解，但是比起二级场来却有了明显的变化. 在 $V_\phi<1$ 亚声条件下,当 $\frac{\Gamma_\sigma}{1-V_\phi^2}(\mathscr{E}_0^\sigma)^2=\frac{\gamma_\sigma}{1-V_\phi^2}\left(\frac{|E^\sigma|^2}{16\pi}\Big/n_0T_e\right)$ 趋向于1时,n_e 趋向于0,这时四级场的作用完全抵消了二级场效应，使一维 Soliton 不可能形成. 只有在 $\frac{\Gamma_\sigma}{1-V_\phi^2}(\mathscr{E}_0^\sigma)^2\ll 1$ 时,(101)式右边第二项可忽略,才有通常的 Soliton 出现，这说明一维稳定结构解只出现在早期调制不稳定性场幅还较小的发展阶段或 $V_\phi\ll 1$ 时，随着场幅的增加或在 V_ϕ 接近1时,高阶场的作用使非线性凝聚作用开始消失,直至消灭 Soliton. 所以四级场的作用促使密度抽空效应只能具有 $\frac{n_e}{n_0}<1$ 的大小.

另一方面,从色散关系看,方程(81)的色散关系为

$$\Omega_\sigma=\frac{q^2(\mu^\sigma)^2}{2\omega_{pe}}-\omega_{pe}\frac{|E^\sigma|^2}{16\sigma\pi n_0(T_e+T_i)}\left(1-\frac{|E^\sigma|^2}{16\sigma\pi n_0(T_e+T_i)}\right). \tag{105}$$

这里四级场的作用总是抑制着二阶场的非线性色散性，使波的传播在高场能密度时接近线性色散关系特征.

2. 从方程(73)和(74)，可以预料四级场的出现将阻碍 Caviton 的形成过程. 由于含 $(\boldsymbol{EE}^*)$ 项的出现,也不见得出现太强凝聚力而加速"坍塌".

3. 在阻尼效应不能忽略情况下，$\varepsilon_d(q)$ 和 $\varepsilon_{d\alpha}(q)$ 必须保留虚部,这时,在密度方程中就出现阻尼项,详细处理见文献[4].

感谢于敏教授的有益讨论.

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NON-LINEAR EFFECT ON THE LARGE AMPLITUDE WAVES INTERACTION WITH PARTICLES OF LOW FREQUENCY OSCILLATION IN PLASMA

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ABSTRACT

In this paper, retaining high-frequency fields up to fourth order, the kinetic and hydrodynamic equations for the beat frequency interaction between longitudinal, transverse high-frequency fields and particles in low-frequency oscillation were derived from Vlasov-Maxwell equations. The results show that, in the case of one dimension, the large amplitude high order fields in sub-acoustic waves weaken the ponderomotive force which causes solitons. In three dimensions, it possibly makes the formation of cavitons slower.