第34卷 第9期 物 理 学 报 Vol. 34, No. 9
1985年9月 ACTA PHYSICA SINICA Sept., 1985

Kerr-Newman-De Sitter 时空中的 Dirac 方程的退耦和分离变量

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1984年7月27日收到

提 要

本文给出了 Kerr-Newman-De Sitter 时空中的 Dirac 方程的退耦和分离变量，并在 Kerr-Newman-De Sitter 时空的视界附近通过适当的变换找到了静止质量不为零的 Dirac 方程的有物理意义的解，导出了 Hawking 热谱公式，从而解决了 Dirac 粒子在 Kerr-Newman-De Sitter 黑洞背景下的 Hawking 蒸发问题.

一

Kinnersley 在研究D型时空分类时引入了一套零标架[1]. Chandrasekhar 用 Kinnersley 零标架技巧，导出了 Kerr 背景时空中静止质量不为零的 Dirac 方程的退耦和分离变量的量子方程[2]. Page 把 Chandrasekhar 的工作推广到 Kerr-Newman 背景时空[3]. Khanal 把 Chandrasekhar 的工作推广到 Kerr-De Sitter 背景时空[4]. 本文进一步推广到 Kerr-Newman-De Sitter 时空中.

二

在弯曲时空中，荷电 Dirac 方程的旋表示为

$$(\nabla_{A\dot{B}} + ieA_{A\dot{B}})P^A + \frac{i\mu}{\sqrt{2}}\bar{Q}_{\dot{B}} = 0. \tag{1}$$

$$(\nabla_{A\dot{B}} - ieA_{A\dot{B}})Q^A + \frac{i\mu}{\sqrt{2}}\bar{P}_{\dot{B}} = 0. \tag{2}$$

这里 P^A 和 Q^A 是两个二分量旋量. $\nabla_{A\dot{B}}$ 是旋协变微分. e, μ 分别为 Dirac 粒子的电荷和质量.

在 Boyer-Linaquist 坐标系中 Kerr-Newman-De Sitter 度规为[5]

$$\begin{aligned}-ds^2 &= g_{\mu\nu}d\chi^\mu d\chi^\nu\\ &= \rho^2(\Delta_r^{-1}dr^2 + \Delta_\theta^{-1}d\theta^2) + \rho^{-2}\Xi^{-2}\Delta_\theta[a\,dt\\ &\quad - (r^2 + a^2)a\varphi]^2\sin^2\theta - \Delta_r\Xi^{-2}\rho^{-2}(dt - a\sin^2\theta d\varphi)^2.\end{aligned} \tag{3}$$

式中

$$\bar{\rho} = r + ia\cos\theta, \quad \rho^2 = \bar{\rho}\bar{\rho}^*,$$
$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda}{3} r^2\right) - 2Mr + Q^2,$$
$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta,$$
$$\Xi = 1 + \frac{1}{3}\Lambda a^2. \tag{4}$$

Λ 为宇宙常数. M 为黑洞质量. Q 为黑洞电荷，$a = J/M$ 为比角动量.

而

$$g^{\mu\nu} = \begin{pmatrix} \frac{\Xi^2}{\rho^2}\left[\frac{(r^2+a^2)}{\Delta_r} - \frac{a^2\sin^2\theta}{\Delta_\theta}\right] & 0 & 0 & \frac{a\Xi^2}{\rho^2}\left[\frac{r^2+a^2}{\Delta_r} - \frac{1}{\Delta_\theta}\right] \\ 0 & -\frac{\Delta_r}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{\Delta_\theta}{\rho^2} & 0 \\ \frac{a\Xi^2}{\rho^2}\left[\frac{r^2+a^2}{\Delta_r} - \frac{1}{\Delta_\theta}\right] & 0 & 0 & -\frac{\Xi^2}{\rho^2\sin^2\theta}\left[\frac{1}{\Delta_\theta} - \frac{a^2\sin^2\theta}{\Delta_r}\right] \end{pmatrix}. \tag{5}$$

选取如下零标架：

$$l^\mu = \left[\frac{(r^2+a^2)\Xi}{\Delta_r}, 1, 0, \frac{a\Xi}{\Delta_r}\right],$$
$$n^\mu = \frac{1}{2\rho^2}[\Xi(r^2+a^2), -\Delta_r, 0, a\Xi],$$
$$m^\mu = \frac{1}{\sqrt{2\Delta_\theta}\,\bar{\rho}}\left[ia\Xi\sin\theta, 0, \Delta_\theta, \frac{i\Xi}{\sin\theta}\right]. \tag{6}$$

显见(6)式是零矢量，满足伪正交关系，满足度规条件. 则 Dirac 方程(1),(2)化为四个耦合方程：

$$(D + \epsilon - \rho + ieA_\mu l^\mu)F_1 + (\bar{\delta} + \pi - \alpha + ieA_\mu \bar{m}^\mu)F_2 = \frac{i\mu G_1}{\sqrt{2}},$$
$$(\Delta + \mu - \gamma + ieA_\mu n^\mu)F_2 + (\delta + \beta - \tau + ieA_\mu m^\mu)F_1 = \frac{i\mu G_2}{\sqrt{2}},$$
$$(D + \epsilon^* - \rho^* + ieA_\mu l^\mu)G_2 - (\delta + \pi^* - \alpha^* + ieA_\mu m^\mu)G_1 = \frac{i\mu F_2}{\sqrt{2}},$$
$$(\Delta + \mu^* - \gamma^* + ieA_\mu n^\mu)G_1 - (\bar{\delta} + \beta^* - \tau^* + ieA_\mu \bar{m}^\mu)G_2 = \frac{i\mu F_1}{\sqrt{2}}. \tag{7}$$

式中

$$F_1 = P^0, \quad F_2 = P^1, \quad G_1 = \bar{Q}^{\dot{1}}, \quad G_2 = -\bar{Q}^{\dot{0}},$$
$$D \equiv l^\mu\partial_\mu = \partial_{0\dot{0}}, \quad \Delta \equiv n^\mu\partial_\mu = \partial_{1\dot{1}},$$
$$\delta \equiv m^\mu\partial_\mu = \partial_{0\dot{1}}, \quad \bar{\delta} \equiv \bar{m}^\mu\partial_\mu = \partial_{1\dot{0}},$$

$$\rho = -\frac{1}{\rho^*}, \qquad \tau = -\frac{ia\sqrt{\Delta_\theta}\sin\theta}{\sqrt{2}\cdot\rho^2},$$

$$\mu = -\frac{\Delta_r}{2\rho^2\bar{\rho}^*}, \qquad \beta = \frac{1}{2\sqrt{2}\,\bar{\rho}\sin\theta}\frac{d(\sqrt{\Delta_\theta}\sin\theta)}{d\theta},$$

$$r = \frac{1}{4\rho^2}\frac{d\Delta_r}{dr} + \mu, \qquad \pi = \frac{ia\sqrt{\Delta_\theta}\sin\theta}{\sqrt{2}(\bar{\rho}^*)^2},$$

$$\alpha = \pi - \rho^*.$$

A_μ 是 Kerr-Newman-De Sitter 时空电磁势[6]

$$A_0 = \frac{Qr}{\rho^2\Xi}, \quad A_1 = A_2 = 0,$$

$$A_3 = -\frac{Qra\sin^2\theta}{\rho^2\Xi}, \tag{8}$$

则

$$A_\mu l^\mu = \frac{Qr}{\Delta_r}, \qquad A_\mu n^\mu = \frac{Qr}{2\rho^2},$$

$$A_\mu m^\mu = A_\mu \bar{m}^\mu = 0. \tag{9}$$

利用 Teukolsky 给出的旋系数的显示表式[7],并设

$$F_1 = e^{-i\omega t}e^{im\varphi}f_1(r,\theta),$$
$$F_2 = e^{-i\omega t}e^{im\varphi}f_2(r,\theta)$$
$$G_1 = e^{-i\omega t}e^{im\varphi}g_1(r,\theta),$$
$$G_2 = e^{-i\omega t}e^{im\varphi}g_2(r,\theta). \tag{10}$$

则(7)式为

$$\mathscr{D}_0 f_1 + \sqrt{\frac{\Delta_\theta}{2}}\,\mathscr{L}_{1/2} f_2 = \frac{\mu}{\sqrt{2}}(ir + a\cos\theta)g_1 \tag{11}$$

$$\Delta_r\mathscr{D}^+_{1/2} f_2 - \sqrt{2\Delta_\theta}\,\mathscr{L}^+_{1/2} f_1 = -\sqrt{2}\,\mu(ir + a\cos\theta)g_2, \tag{12}$$

$$\mathscr{D}_0 g_2 - \sqrt{\frac{\Delta_\theta}{2}}\,\mathscr{L}^+_{1/2} g_1 = \frac{\mu}{\sqrt{2}}(ir - a\cos\theta)f_2, \tag{13}$$

$$\Delta_r\mathscr{D}^+_{1/2} g_1 + \sqrt{2\Delta_\theta}\,\mathscr{L}_{1/2} g_2 = -\sqrt{2}\,\mu(ir - a\cos\theta)f_1. \tag{14}$$

式中

$$\mathscr{D}_n = \partial_r - \frac{i\Xi K}{\Delta_r} + \frac{n}{\Delta_r}\frac{d\Delta_r}{dr},$$

$$\mathscr{D}^+_n = \partial_r + \frac{i\Xi K}{\Delta_r} + \frac{n}{\Delta_r}\frac{d\Delta_r}{dr},$$

$$\mathscr{L}_n = \partial_\theta - \frac{\Xi K}{\Delta_\theta} + \frac{n}{\sqrt{\Delta_\theta}\sin\theta}\frac{d(\sqrt{\Delta_\theta}\sin\theta)}{d\theta},$$

$$\mathscr{L}^+_n = \partial_\theta + \frac{\Xi K}{\Delta_\theta} + \frac{n}{\sqrt{\Delta_\theta}\sin\theta}\frac{d(\sqrt{\Delta_\theta}\sin\theta)}{d\theta}.$$

$$K = (r^2 + a^2)\omega - am + \frac{e\theta r}{\Xi},$$

$$H = a\omega \sin\theta - \frac{m}{\sin\theta}. \tag{15}$$

作变换

$$(f_1, f_2, g_1, g_2) = (R_{-1/2(\phi)}S_{-1/2(\theta)},\ R_{1/2(r)}S_{1/2(\theta)},\ R_{1/2(r)}S_{-1/2(\theta)},\ R_{-1/2(r)}S_{1/2(\theta)}). \tag{16}$$

则最后可得如下分离变量方程：

$$\left[\Delta_r \mathscr{D}^{+}_{1/2}\mathscr{D}_0 - \frac{i\mu\Delta_r}{\lambda + i\mu r}\mathscr{D}_0 - (\lambda^2 + \mu^2 r^2)\right]R_{-1/2} = 0. \tag{17}$$

$$\left[\Delta_\theta \mathscr{L}_{1/2}\sqrt{\Delta_\theta}\,\mathscr{L}^{+}_{1/2} + \frac{a\mu\sin\theta\Delta_\theta}{\lambda + a\mu\cos\theta}\mathscr{L}^{+}_{1/2} + (\lambda^2 + a^2\mu^2\cos^2\theta)\right]S_{-1/2} = 0. \tag{18}$$

λ 是分离变量中引入的常数，类似可得关于 $R_{1/2(r)}$ 和 $S_{1/2(\theta)}$ 的方程。

当 $\lambda = 0$，$Q = 0$ 时，即文献[2]中结论。 当 $\lambda = 0$，$Q \neq 0$ 时，即文献[3]中结论。当 $\lambda \neq 0$，$Q = 0$ 时，即文献[4]中结论。

三

我们在 Kerr-Newman-De Sitter 宇宙黑洞外视界 ($r = r_+$) 附近求解方程(17)，为此引入缓变坐标

$$\hat{r} = \ln(r - r_+). \tag{19}$$

其中 r_{++}，r_+，r_-，r_{--} 为 $\Delta_r = 0$ 的四个实根。

由(19)式我们可得如下关系：

$$\frac{d\hat{r}}{dr} = \frac{1}{r - r_+}, \tag{20}$$

$$\frac{d}{dr} = \frac{1}{r - r_+}\frac{d}{d\hat{r}}, \tag{21}$$

$$\frac{d^2}{dr^2} = \frac{1}{(r - r_+)^2}\frac{d^2}{d\hat{r}^2} - \frac{1}{(r - r_+)^2}\frac{d}{d\hat{r}}. \tag{22}$$

由(15)，(17)式有

$$\begin{aligned}
&\Delta_r \frac{d^2 R_{-1/2}}{dr^2} + \left[(r - M) - \frac{i\mu\Delta_r}{\lambda + i\mu r} - \frac{\Lambda}{3} r(2r^2 + a^2)\right]\frac{d}{dr}R_{-1/2} \\
&+ \left[\frac{K^2 + iK(r - M)\Xi}{\Delta_r} - i\left(2\omega r + \frac{eQ}{\Xi}\right) - \frac{\mu K}{\lambda + i\mu r} - (\lambda^2 + \mu^2 r^2)\right. \\
&- \frac{i\Lambda K r(2r^2 + a^2)\Xi}{3\Delta_r} - \frac{i\Lambda a^2}{3}\left(2\omega r + \frac{eQ}{\Xi}\right) - \frac{\Lambda a^2 K\mu}{3(\lambda + i\mu r)} \\
&\left. + \frac{\Lambda a^2 K^2}{\Delta_r} + \frac{\Lambda^2 a^4 K^2}{9\Delta_r}\right] R_{-1/2} = 0.
\end{aligned} \tag{23}$$

将(20)—(22)式代入(23)式，并乘 $1/\Delta_r$，取极限 $r \to r_+$，则(23)式化为

$$\frac{d^2}{d\hat{r}^2}R_{-1/2}+\left[\frac{r_+-M}{(r_+-r_{++})(r_+-r_-)(r_+-r_{--})}\right.$$
$$\left.-\frac{\Lambda r_+}{3}\frac{(2r_+^2+a^2)}{(r_+-r_{++})(r_+-r_-)(r_+-r_{--})}-1\right]\frac{d}{d\hat{r}}R_{-1/2}$$
$$+\left[\frac{K_c^2+iK_c(r_+-M)\Xi}{(r_+-r_{++})^2(r_+-r_-)^2(r_+-r_{--})^2}-\frac{i\Lambda\Xi K_c r_+(2r_+^2+a^2)}{3(r_+-r_{++})^2(r_+-r_-)^2(r_+-r_{--})^2}\right.$$
$$\left.+\frac{\Lambda a^2K_c^2}{(r_+-r_{++})^2(r_+-r_-)^2(r_+-r_{--})^2}+\frac{\Lambda^2a^4K_c^2}{9(r_+-r_{++})^2(r_+-r_-)^2(r_+-r_{--})^2}\right]R_{-1/2}$$
$$=0. \tag{24}$$

式中

$$K_c=(r_+^2+a^2)\omega-am+\frac{eQr_+}{\Xi}$$
$$=(r_+^2+a^2)(\omega-\omega_0), \tag{25}$$
$$\omega_0=m\Omega+eV, \tag{26}$$
$$\Omega=\frac{a}{r_+^2+a^2},\qquad V=\frac{Qr_+}{\Xi(r_+^2+a^2)}, \tag{27}$$

(24)式是一个二阶常系数微分方程,特征根为

$$p_1=\frac{1}{2}(-c_1+\sqrt{c_1^2-4c_2}), \tag{28}$$
$$p_2=\frac{1}{2}(-c_1-\sqrt{c_1^2-4c_2}). \tag{29}$$

而 c_1 为 (24) 式中 $\frac{d}{d\hat{r}}R_{-1/2}$ 前面的系数, c_2 为(24)式中 $R_{-1/2}$ 前面的系数.

对应的通解为

$$R_{-1/2}=A_1e^{p_1\hat{r}}+A_2e^{p_2\hat{r}}. \tag{30}$$

对于 Hawking 蒸发只要考虑 $\omega>\omega_0$ 的出射波,则(30)式中 $A_1=0$.

计入含时间的因子,则视界向外的出射波波函数为

$$\psi_\omega^{\rm out}\sim A_2e^{p_2\hat{r}}e^{-i\omega t}. \tag{31}$$

对入射波 ($A_2=0$) 同样有

$$\psi_\omega^{\rm in}\sim A_1e^{p_1\hat{r}}e^{-i\omega t}. \tag{32}$$

类似于文献[8],用解析延拓方法可得 Kerr-Newman-De Sitter 宇宙中黑洞的 Dirac 粒子的 Hawking 热谱公式:

$$N_\omega^2=\frac{1}{1+e^{(\omega-\omega_0)/kT}}. \tag{33}$$

式中

$$T=\frac{\eta}{2\pi k} \tag{34}$$

是 Kerr-Newman-De Sitter 宇宙中黑洞视界的辐射温度. 当 $\Lambda=0$ 时,回到 Kerr-Newman 黑洞的情况[8]. k 为玻耳兹曼常数.

$$\eta = \frac{\Lambda(r_{++} - r_+)(r_+ - r_-)(r_+ - r_{--})}{6\Xi(r_+^2 + a^2)} \tag{35}$$

为黑洞的表面重力[6]。

完全类似我们可以得到 Kerr-Newman-De Sitter 宇宙视界的辐射温度

$$T' = \frac{\eta'}{2\pi k}, \tag{36}$$

而

$$\eta' = \frac{\Lambda(r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--})^{[6]}}{6\Xi(r_+^2 + a^2)}, \tag{37}$$

当 $M \to 0$, $Q \to 0$, $a \to 0$ 时，回到 de Sitter 宇宙的情况。

工作中得到北京师范大学刘辽教授，赵峥老师和北京大学许殿彦副教授的热情帮助，谨致谢意。

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DECOUPLED DIRAC EQUATIONS IN KERR-NEWMAN-DE SITTER SPACE-TIME

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ABSTRACT

In this work, we solved the decoupling problem of the Dirac equation with non-zero rest mass in a Kerr-Newman-De Sitter background. The physical solutions of the Dirac equation with non-zero rest mass are found outside the event horizon of Kerr-Newman-De Sitter space-time by a proper transformation. The Hawking thermal spectra formula is also dirived. The problem of the Hawking evaporation of Dirac particles in Kerr-Newman-De Sitter black hole background is thus solved.