第 37 卷 第 3 期
1988 年 3 月

物 理 学 报
ACTA PHYSICA SINICA

Vol. 37, No. 3
Mar., 1988

具有零或负扩散系数的 Fokker-Planck 方程的形式解及其在量子光学中的应用

谭维翰 李宇舫 张卫平
（中国科学院上海光学精密机械研究所）

1986 年 12 月 30 日收到

提 要

本文给出量子光学中具有零或负扩散系数的 Fokker-Planck 方程的形式解，并讨论了在利用简并参量放大或简并四波混频过程产生压缩态中的应用及一些其它应用.

一、引 言

一般来说非线性随机系统的 Langevin 方程要比相应的 Fokker-Planck 方程容易处理. 特别是象产生压缩态的量子光学系统中遇到的 Fokker-Planck 方程，扩散系数为负或零，解有可能发散. 本文先求得这种情形的形式解，并发现在求物理量的统计平均时，发散困难可以避免. 在此基础上，我们研究了该解在利用简并参量放大或简并四波混频过程产生压缩态中的应用及一些其它应用[2-5].

二、双光子跃迁的 Fokker-Planck 方程

简并参量放大过程为一双光子过程. 其哈密顿量可写为[2]

$$H = H_0 + V + W, \tag{1}$$

式中

$$\begin{aligned} H_0 &= \hbar\omega_c a^+ a + \sum \hbar\omega_j b_j^+ b_j, \\ V &= \hbar(\sum k_j b_j a^+ + \sum k_j^* b_j^+ a), \\ W &= -\frac{i\hbar}{2}(\varepsilon a^{+2} - \varepsilon^* a^2). \end{aligned} \tag{2}$$

简并四波混频的哈密顿量有类似形式，只是[3]

$$W = \hbar c k (b_1^+ b_2^+ a_1 a_2 + b_1 b_2 a_1^+ a_2^+). \tag{3}$$

在(2)式中，H_0 为阻尼振子和热库的自由哈密顿量，V 为阻尼振子和热库的相互作用能，W 为外场与阻尼振子的相互作用能. 由(2)式可导出简并参量放大情形的约化密度矩阵的运动方程，其 P 表示中的 c 数方程为一Fokker-Planck 方程.

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*,t)=k\left(\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right)P-\left(\varepsilon\alpha^*\frac{\partial}{\partial\alpha}+\varepsilon^*\alpha\frac{\partial}{\partial\alpha^*}\right)P$$
$$+\frac{1}{2}\left(\frac{\partial}{\partial\alpha}\varepsilon\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\alpha^*}\varepsilon^*\frac{\partial}{\partial\alpha^*}\right)P+2k\bar{n}\frac{\partial^2P}{\partial\alpha\partial\alpha^*}, \tag{4}$$

式中 $k=\gamma/2$，γ 为原子横向弛豫时间. $\bar{n}$ 为热库的平均光子数. 通常，(4)式中的最后一项因认为 $\bar{n}\ll 1$ 而略去，而实际上在泵浦场中，除了由W描述的相干相互作用(即与阻尼振子相位匹配的那部分相干场)外，还有位相不匹配的非相干相互作用. 这部分等同于场与热库的相互作用，可包括在 $\bar{n}$ 中，故 $\bar{n}$ 不能略去. 此外，(4)式中 ε 可写为 $\varepsilon=|\varepsilon|e^{i\phi}$，只要 ϕ 为常数，作变换 $\alpha\to\alpha e^{i\frac{\phi}{2}}$，$\alpha^*\to\alpha^*e^{-i\frac{\phi}{2}}$，则相角 ϕ 可被消去. 故不失一般性，在解方程(4)时 ε 可取为实数.

三、简并参量放大的 Fokker-Planck 方程的解及其应用

设

$$P(\alpha,\alpha^*,t)=e^{-a\left(\frac{\alpha^2+\alpha^{*2}}{2}\right)+b\alpha\alpha^*}Q(\alpha,\alpha^*,t). \tag{5}$$

将(5)式代入(4)式，并选定参数 a，b，使得一次导数项$\frac{\partial Q}{\partial\alpha}$，$\frac{\partial Q}{\partial\alpha^*}$ 被消去，于是有

$$a=\frac{1}{2}\left[\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}+\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\right],\quad b=\frac{1}{2}\left[\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}-\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\right], \tag{6}$$

$$\left\{\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\alpha^{*2}}\right)+2k\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{1}{4}\left[\frac{(k+\varepsilon)^2}{\varepsilon-2k\bar{n}}+\frac{(k-\varepsilon)^2}{\varepsilon+2k\bar{n}}\right](\alpha^2+\alpha^{*2})\right.$$
$$\left.+\frac{1}{2}\left[\frac{(k+\varepsilon)^2}{\varepsilon-2k\bar{n}}-\frac{(k-\varepsilon)^2}{\varepsilon+2k\bar{n}}\right]\alpha\alpha^*+k\right\}Q=\frac{\partial}{\partial t}Q(\alpha,\alpha^*,t). \tag{7}$$

又设

$$\alpha=\frac{\beta+i\tilde{\beta}}{\sqrt{2}},\quad\alpha^*=\frac{\beta-i\tilde{\beta}}{\sqrt{2}}, \tag{8}$$

将(8)式代入(7)式，便得

$$\left\{\left(\frac{\varepsilon}{2}+k\bar{n}\right)\frac{\partial^2}{\partial\beta^2}-\left(\frac{\varepsilon}{2}-k\bar{n}\right)\frac{\partial^2}{\partial\tilde{\beta}^2}-\frac{(k-\varepsilon)^2\beta^2}{2(\varepsilon+2k\bar{n})}\right.$$
$$\left.+\frac{(k+\varepsilon)^2\tilde{\beta}^2}{2(\varepsilon-2k\bar{n})}\right\}Q=\frac{\partial Q}{\partial t}. \tag{9}$$

由(9)式等号左端第二项看出，当 $-\left(\frac{\varepsilon}{2}-k\bar{n}\right)\leqslant 0$ 时，$\tilde{\beta}$ 的扩散系数为负或零，如果泵浦场干扰很大，使 $k\bar{n}>\varepsilon/2$，扩散系数为正，则不能实现压缩. 本文主要讨论 $\frac{\varepsilon}{2}-k\bar{n}>0$，$\frac{\varepsilon}{2}+k\bar{n}>0$ 的情形.

现设

$$c=\left|\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\right|,\quad\tilde{c}=\left|\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}\right|,$$

$$Q_{mn}=\exp\left(-\lambda_m\left|\frac{\varepsilon}{2}+k\bar{n}\right|t-\tilde{\lambda}_n\left|\frac{\varepsilon}{2}-k\bar{n}\right|t\right)Q_m(\beta)\tilde{Q}_n(\tilde{\beta}) \tag{10}$$

将(10)式代人(9)式，便得

$$\begin{aligned}&\left(\frac{\partial^2}{\partial\beta^2}-c^2\beta^2\pm\lambda_m\right)Q_m(\beta)=0 \quad 当 \quad \frac{\varepsilon}{2}+k\bar{n}\gtrless 0,\\&\left(-\frac{\partial^2}{\partial\tilde{\beta}^2}+\tilde{c}^2\tilde{\beta}^2\pm\tilde{\lambda}_n\right)\tilde{Q}_n(\tilde{\beta})=0 \quad 当 \quad \frac{\varepsilon}{2}-k\bar{n}\gtrless 0.\end{aligned} \tag{11}$$

(11)式的解为

$$\lambda_m=(2m+1)c,\ \tilde{\lambda}_n=(2n+1)\tilde{c},$$

$$N_m=\left(\frac{\sqrt{c}}{\sqrt{\pi}\ 2^m m!}\right)^{\frac{1}{2}},\quad N_n=\left(\frac{\sqrt{\tilde{c}}}{\sqrt{\pi}\ 2^n n!}\right)^{\frac{1}{2}},$$

$$Q_m(\beta)=\begin{cases}N_m e^{-c\frac{\beta^2}{2}}H_m(\sqrt{c}\,\beta) & \frac{\varepsilon}{2}+k\bar{n}>0,\\ N_m e^{c\beta^2/2}H_m(\sqrt{c}\,i\beta) & \frac{\varepsilon}{2}+k\bar{n}<0;\end{cases} \tag{12}$$

$$\tilde{Q}_n(\tilde{\beta})=\begin{cases}N_n e^{\frac{\tilde{c}\tilde{\beta}^2}{2}}H_n(\sqrt{\tilde{c}}\,i\tilde{\beta}) & \frac{\varepsilon}{2}-k\bar{n}>0,\\ N_n e^{-\tilde{c}\tilde{\beta}^2/2}H_n(\sqrt{\tilde{c}}\,\tilde{\beta}) & \frac{\varepsilon}{2}-k\bar{n}<0,\end{cases}$$

$$Q_{mn}=\exp\left[-\left(m+\frac{1}{2}\right)|k-\varepsilon|t-\left(n+\frac{1}{2}\right)|k+\varepsilon|t\right]Q_m(\beta)\tilde{Q}_n(\tilde{\beta}). \tag{13}$$

当 $\frac{\varepsilon}{2}+k\bar{n}>0$，$\frac{\varepsilon}{2}-k\bar{n}<0$ 时，(12)，(13) 式给出收敛的非压缩态解．但当 $\frac{\varepsilon}{2}-k\bar{n}>0$ $\left(或\ \frac{\varepsilon}{2}+k\bar{n}<0\right)$ 时，$\tilde{Q}_n(\tilde{\beta})$ (或 $Q_m(\beta)$) 当 $\tilde{\beta}$ (或 β) $\to\infty$ 时是发散的．下面将看到这个发散困难在求物理量的统计平均时可以避免．这里仍用 Q_{mn} 作格林函数，参照文献[1]，并按(5)式得

$$\begin{aligned}P_{mn}=\exp\Big[&-\frac{1}{4}\left(\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}+\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\right)(\beta^2-\tilde{\beta}^2)\\&+\frac{1}{4}\left(\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}-\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\right)(\beta^2+\tilde{\beta}^2)\Big]Q_{mn}.\end{aligned} \tag{14}$$

现设 $k-\varepsilon\geqslant 0$ (对 $k-\varepsilon<0$ 的情形见附录)，则上式可写为

$$P_{mn}=\exp\left(-\frac{1}{2}c\beta^2+\frac{1}{2}\tilde{c}\tilde{\beta}^2\right)Q_{mn}. \tag{14$'$}$$

格林函数 $P(\beta,\tilde{\beta},t;\beta_0,\tilde{\beta}_0)$ 为

$$P(\beta,\tilde{\beta},t;\beta_0,\tilde{\beta}_0)=N\sum_{mn}\varepsilon_{mn}P_{mn}, \tag{15}$$

$$\varepsilon_{mn}=N_mN_nH_m(\sqrt{c}\,\beta_0)H_n(\sqrt{\tilde{c}}\,i\tilde{\beta}_0), \tag{16}$$

式中 N 为归一化系数．为求出和式(15)，利用如下等式[6]：

$$\sum \frac{\left(\frac{e^{-|\varepsilon-k|t}}{2}\right)^n}{n!} H_n(\sqrt{c}\,\beta)H_n(\sqrt{c}\,\beta_0)$$

$$= \frac{\exp\left\{\frac{2c\beta\beta_0 e^{-|\varepsilon-k|t} - c(\beta^2+\beta_0^2)e^{-2|\varepsilon-k|t}}{1-e^{-2|\varepsilon-k|t}}\right\}}{(1-e^{-2|\varepsilon-k|t})^{\frac{1}{2}}}, \tag{17}$$

$$P(\beta, \tilde{\beta}, t; \beta_0, \tilde{\beta}) = N\sum_m N_m^2 e^{-m|\varepsilon-k|t - c\beta^2 - \frac{|\varepsilon-k|}{2}t} H_m(\sqrt{c}\,\beta)H_m(\sqrt{c}\,\beta_0)$$

$$\times \sum N_n^2 e^{-n|\varepsilon+k|t + \tilde{c}\tilde{\beta}^2 - \frac{|\varepsilon+k|}{2}t} H_n(\sqrt{\tilde{c}}\, i\tilde{\beta})H_n(\sqrt{\tilde{c}}\, i\tilde{\beta}_0)$$

$$= \frac{N\sqrt{c\tilde{c}}}{\sqrt{\pi}\sqrt{1-e^{-2|\varepsilon-k|t}}}\exp\left[-\frac{|\varepsilon-k|t}{2} + \frac{2c\beta\beta_0 e^{-|\varepsilon-k|t} - c(\beta^2+\beta_0^2)e^{-2|\varepsilon-k|t}}{1-e^{-2|\varepsilon-k|t}} - c\beta^2\right]$$

$$\times \frac{1}{\sqrt{\pi}\sqrt{1-e^{-2|\varepsilon+k|t}}}\exp\left[-\frac{|\varepsilon+k|}{2}t + \frac{-2\tilde{c}\tilde{\beta}\tilde{\beta}_0 e^{-|\varepsilon+k|t} + \tilde{c}(\tilde{\beta}^2+\tilde{\beta}_0^2)e^{-2|\varepsilon+k|t}}{1-e^{-2|\varepsilon+k|t}}\right.$$

$$\left. + \tilde{c}\tilde{\beta}^2\right] = \frac{N\sqrt{c\tilde{c}}}{\sqrt{\pi}\sqrt{1-e^{-2|k-\varepsilon|t}}}\exp\left[-\frac{|\varepsilon-k|}{2}t - \frac{c(\beta-\beta_0 e^{-|\varepsilon-k|t})^2}{1-e^{-2|\varepsilon-k|t}}\right]$$

$$\times \frac{1}{\sqrt{\pi}\sqrt{1-e^{-2|\varepsilon+k|t}}}\exp\left[-\frac{|\varepsilon+k|}{2}t + \frac{\tilde{c}(\tilde{\beta}-\tilde{\beta}_0 e^{-|\varepsilon+k|t})^2}{1-e^{-2|\varepsilon+k|t}}\right]. \tag{18}$$

在(18)式中将归一化系数形式地取为

$$N = \frac{\pi}{\sqrt{c\tilde{c}}}\left\{\int \exp\left[-\frac{|\varepsilon-k|t}{2} - \frac{c(\beta-\beta_0 e^{-|\varepsilon-k|t})^2}{1-e^{-2|\varepsilon-k|t}}\right]\frac{d\beta}{\sqrt{1-e^{-2|\varepsilon-k|t}}}\right.$$

$$\left.\times \int \exp\left[-\frac{|\varepsilon+k|}{2}t + \frac{\tilde{c}(\tilde{\beta}-\tilde{\beta}_0 e^{-|\varepsilon+k|t})^2}{1-e^{-2|\varepsilon+k|t}}\right]\frac{d\tilde{\beta}}{\sqrt{1-e^{-2|\varepsilon+k|t}}}\right\}^{-1}. \tag{19}$$

很明显,这样取定N后,有

$$\iint P(\beta, \tilde{\beta}, t; \beta_0, \tilde{\beta}_0)\, d\beta\, d\tilde{\beta} = 1. \tag{20}$$

令

$$x = (\beta - \beta_0 e^{-|\varepsilon-k|t})/\sqrt{1-e^{-2|\varepsilon-k|t}},$$
$$y = (\tilde{\beta} - \tilde{\beta}_0 e^{-|\varepsilon+k|t})/\sqrt{1-e^{-2|\varepsilon+k|t}}. \tag{21}$$

由上述方程可求得量子起伏如下:

$$\langle(\beta - \beta_0 e^{-|k-\varepsilon|t})^2\rangle = (1-e^{-2|\varepsilon-k|t})\frac{\int x^2 e^{-cx^2}dx}{\int e^{-cx^2}dx}$$

$$= (1-e^{-2|k-\varepsilon|t})\left(-\frac{\partial}{\partial c}\ln\int e^{-cx^2}dx\right)$$

$$= \frac{1}{2c}(1-e^{-2|k-\varepsilon|t}), \tag{22}$$

$$\langle(\tilde{\beta} - \tilde{\beta}_0 e^{-|\varepsilon+k|t})^2\rangle = (1-e^{-2|k+\varepsilon|t})\frac{\int y^2 e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy}$$

$$= (1 - e^{-2|\varepsilon+k|t})\left(\frac{\partial}{\partial\tilde{c}}\ln\int e^{\tilde{c}y^2}dy\right)$$

$$= (1 - e^{-2|\varepsilon+k|t})\left(\frac{\partial}{\partial\tilde{c}}\ln\left(\tilde{c}^{-\frac{1}{2}}\int e^{z^2}dz\right)\right)$$

$$= -\frac{1}{2\tilde{c}}(1 - e^{-2|\varepsilon+k|t}). \tag{23}$$

由(22)和(23)式给出 α 的实部 x_1，虚部 x_2 的正常顺序方差为

$$\langle:\Delta x_1^2:\rangle = \left\langle\left(\frac{\beta - \beta_0 e^{-|k-\varepsilon|t}}{\sqrt{2}}\right)^2\right\rangle = \frac{1}{4}\frac{|\varepsilon + 2k\bar{n}|}{|k - \varepsilon|}(1 - e^{-2|k-\varepsilon|t}),$$

$$\langle:\Delta x_2^2:\rangle = \left\langle\left(\frac{\tilde{\beta} - \tilde{\beta}_0 e^{-|k+\varepsilon|t}}{\sqrt{2}}\right)^2\right\rangle = -\frac{1}{4}\frac{|\varepsilon - 2k\bar{n}|}{|k + \varepsilon|}(1 - e^{-2|k+\varepsilon|t}). \tag{24}$$

实际量子起伏应为[7]

$$\langle\Delta x_1^2\rangle = \frac{1}{4} + \langle:\Delta x_1^2:\rangle = \frac{1}{4} + \frac{1}{4}\frac{|\varepsilon + 2k\bar{n}|}{|k - \varepsilon|}(1 - e^{-2|k-\varepsilon|t}),$$

$$\langle\Delta x_2^2\rangle = \frac{1}{4} + \langle:\Delta x_2^2:\rangle = \frac{1}{4} - \frac{1}{4}\frac{|\varepsilon - 2k\bar{n}|}{|k + \varepsilon|}(1 - e^{-2|k+\varepsilon|t}). \tag{25}$$

这与文献[8]和[9]的结果是一致的。(25)式是在 $\frac{\varepsilon}{2} + k\bar{n} > 0$，$\frac{\varepsilon}{2} - k\bar{n} > 0$ 的情况下导出的。若换为条件 $\frac{\varepsilon}{2} + k\bar{n} > 0$，$\frac{\varepsilon}{2} - k\bar{n} < 0$，则有

$$\langle\Delta x_1^2\rangle = \frac{1}{4} + \frac{1}{4}\frac{|\varepsilon + 2k\bar{n}|}{|k - \varepsilon|}(1 - e^{-2|k-\varepsilon|t}),$$

$$\langle\Delta x_2^2\rangle = \frac{1}{4} + \frac{1}{4}\frac{|\varepsilon - 2k\bar{n}|}{|k + \varepsilon|}(1 - e^{-2|k+\varepsilon|t}). \tag{26}$$

由(25)和(26)式还可看到因子 $|\varepsilon + 2k\bar{n}|$，$|\varepsilon - 2k\bar{n}|$ 即 β，$\tilde{\beta}$ 的扩散系数的绝对值。当扩散系数为零，则均方误差取相干态的方差值 1/4，当然这不是相干态，因为其正交相分量的方差不为 1/4。如在(26)式中，取 $\frac{\varepsilon}{2} + k\bar{n} = 0$，则 $\langle\Delta x_1^2\rangle = 1/4$，而

$$\langle\Delta x_2^2\rangle = \frac{1}{4} + \frac{1}{2}\frac{|\varepsilon|}{|\varepsilon + k|}(1 - e^{-2|k+\varepsilon|t}).$$

(22)和(23)式求方差的方法，还可以推广到求 $2m$ 差的情形。为此定义 m 阶矩 $M^{(m)}$

$$M^{(m)} = \frac{\int y^{2m}e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy}. \tag{27}$$

$$\frac{\partial M^{(m-1)}}{\partial\tilde{c}} = \frac{\partial}{\partial\tilde{c}}\frac{\int y^{2(m-1)}e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy}$$

$$= \frac{\int y^{2m}e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy} - \frac{\int y^2 e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy}\frac{\int y^{2(m-1)}e^{\tilde{c}y^2}dy}{\int e^{\tilde{c}y^2}dy}. \tag{28}$$

即
$$M^{(m)} = M^{(m-1)}M^{(1)} + \frac{\partial}{\partial \tilde{c}} M^{(m-1)}. \tag{29}$$

由(23)式知 $M^{(1)} = -1/2\tilde{c}$，则应用递推关系(29)式，可求得所有的 $M^{(m)}$. 由(29)式，并设 $M^{(m)} \propto (\tilde{c})^{-m}$，得

$$\frac{M^{(m)}}{M^{(m-1)}} = M^{(1)} + \frac{\partial}{\partial \tilde{c}} \ln M^{(m-1)} = -\frac{1}{2\tilde{c}} - \frac{(m-1)}{\tilde{c}} = -\frac{\left(m-\frac{1}{2}\right)}{\tilde{c}},$$

$$M^{(m)} = \frac{M^{(m)}}{M^{(m-1)}} \cdot \frac{M^{(m-1)}}{M^{(m-2)}} \cdots M^{(1)} = \frac{(-1)^m \left(m-\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)}{(\tilde{c})^m}. \tag{30}$$

由(23)式知方差即1阶矩 $M^{(1)}$ 与 $(1-e^{-2|k+\varepsilon|t})$ 的乘积，而 $2m$ 差 $\langle(\beta-\beta_0 e^{-|k-\varepsilon|t})^{2m}\rangle$ 亦即m阶矩 $M^{(m)}$ 与 $(1-e^{-2|k+\varepsilon|t})^m$ 的乘积. 应用这些关系容易计算表现光强起伏的二阶相关函数 g_2 及光子偏离泊松分布的正规排列方差 $\langle\Delta n^2\rangle - \langle n\rangle$ 等量.

将场算符 a, a^+ 在P表示的 c 数 α, α^* 用平均值 α_0, α_0^* 表示，并设 $\alpha_0 = |\alpha_0| e^{i\theta}$,
$$\alpha = \alpha_0 + \Delta\alpha, \quad \alpha^* = \alpha_0^* + \Delta\alpha^*, \tag{31}$$
于是有
$$\begin{aligned}\langle a^+a^+aa\rangle &= \langle(\alpha_0^* + \Delta\alpha^*)(\alpha_0^* + \Delta\alpha^*)(\alpha_0 + \Delta\alpha)(\alpha_0 + \Delta\alpha)\rangle \\ &= \langle(\Delta\alpha^*)^2(\Delta\alpha)^2\rangle + \alpha_0^{*2}\langle\Delta\alpha^2\rangle + \langle\Delta\alpha^{*2}\rangle\alpha_0^2 + \alpha_0^{*2}\alpha_0^2.\end{aligned} \tag{32}$$
又注意到
$$\begin{aligned}\langle\Delta\alpha^2\rangle = \langle\Delta\alpha^{*2}\rangle &= \frac{\langle\Delta\beta^2 - \Delta\tilde{\beta}^2\rangle}{2}, \\ \langle\Delta\alpha^2\Delta\alpha^{*2}\rangle &= \langle(\Delta\beta + i\Delta\tilde{\beta})^2(\Delta\beta - i\Delta\tilde{\beta})^2\rangle/4 \\ &= \langle(\Delta\beta^2 + \Delta\tilde{\beta}^2)^2\rangle/4.\end{aligned} \tag{33}$$
于是有
$$\begin{aligned}g_2 = \frac{\langle a^+a^+aa\rangle}{\langle a^+a\rangle^2} &= 1 + \frac{\cos 2\theta\langle\Delta\beta^2 - \Delta\tilde{\beta}^2\rangle}{|\alpha_0|^2} \\ &+ \frac{\langle(\Delta\beta)^4 + 2(\Delta\beta)^2(\Delta\tilde{\beta})^2 + (\Delta\tilde{\beta})^4\rangle}{4|\alpha_0|^4}.\end{aligned} \tag{34}$$

上式等号右端第三项就可用(30)式计算. 第二项仅涉及一阶矩即方差，与压缩密切相关. 当 α_0 很大第三项的影响可略去时，第二项大于零或小于零决定了 g_2 大于1或小于1，亦即聚束或反聚束. 这样就将聚束、反聚束通过方差 $\langle\Delta\beta^2 - \Delta\tilde{\beta}^2\rangle$ 与压缩态联系起来.

偏离泊松分布的量 $\langle\Delta n^2\rangle - \langle n\rangle$ 可计算如下. 按文献[10]及(32)和(33)式
$$\begin{aligned}\langle(\Delta n)^2\rangle - \langle n\rangle &= \langle a^+a^+aa\rangle - \langle a^+a\rangle^2 \\ &= |\alpha_0|^2\cos 2\theta\langle(\Delta\beta)^2 - (\Delta\tilde{\beta})^2\rangle + \langle(\Delta\beta)^4 + 2(\Delta\beta)^2(\Delta\tilde{\beta})^2 + (\Delta\tilde{\beta})^4\rangle.\end{aligned} \tag{35}$$

上面结果表明，负扩散系数的 Fokker-Planck 方程的解虽然在 $\tilde{\beta}$ (或 β) $\to \infty$ 时是发散的，但并不妨碍利用它来求物理量的平均值. 正象(21)和(23)式所表现的那样，那无限大已被微分掉，不影响最后计算结果.

四、简并四波混频的 Fokker-Planck 方程的解

产生压缩态的另一重要方案即后向简并四波混频(见图 1)。参照文献[3],其 Langevin 方程为

$$\frac{da_1}{dt} = -ka_1 + i\varepsilon a_2^+ + F_1,$$
$$\frac{da_2^+}{dt} = -ka_2^+ + i\varepsilon a_1 + F_2^+. \tag{36}$$

令
$$a = \frac{a_2^+ e^{i\frac{\pi}{4}} + a_1 e^{-i\frac{\pi}{4}}}{\sqrt{2}}, \quad a^+ = \frac{a_2^+ e^{i\frac{\pi}{4}} - a_1 e^{-i\frac{\pi}{4}}}{\sqrt{2}}, \tag{37}$$

则 $[a, a^+] = [a_1, a_2^+] = 1$. 但 a, a^+ 不是厄密共轭的. 由(36),(37)式并略去无规力得

$$\frac{d}{dt}\begin{pmatrix} a \\ a^+ \end{pmatrix} = \begin{pmatrix} -k & \varepsilon \\ -\varepsilon & -k \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix}. \tag{38}$$

对应的H及 Fokker-Planck 方程为

$$H = i\hbar k a^+ a + \hbar\left\{ i\left(-\frac{\varepsilon}{2}\right)a^{+2} - i\left(\frac{\varepsilon}{2}\right)a^2 \right\},$$

$$\frac{\partial}{\partial t}P(\alpha, \alpha^*, t) = \left\{ -\left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \alpha^*}\right)\begin{pmatrix} -k & \varepsilon \\ -\varepsilon & -k \end{pmatrix} \right.$$
$$\left. + \left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \alpha^*}\right)\begin{pmatrix} \frac{\varepsilon}{2} & k\bar{n} \\ k\bar{n} & -\frac{\varepsilon}{2} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial \alpha^*} \end{pmatrix} \right\} P(\alpha, \alpha^*, t). \tag{39}$$

对于前向简并四波混频[8](见图 2),有

$$\frac{da_1}{dt} = -ka_1 + i\varepsilon a_2^+ + F_1,$$
$$\frac{da_2^+}{dt} = -ka_2^+ - i\varepsilon a_1 + F_2^+. \tag{40}$$

定义
$$a^+ = \frac{a_1 + a_2^+}{\sqrt{2}}, \quad a = \frac{a_1 - a_2^+}{\sqrt{2}}, \tag{41}$$

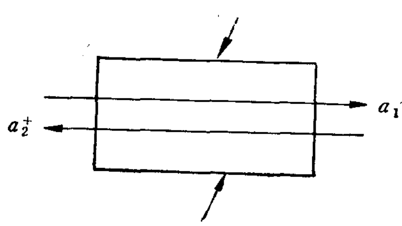

图 1

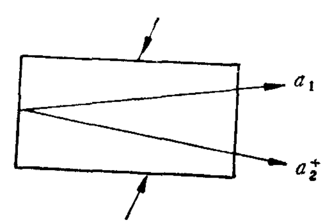

图 2

易证
$$[a, a^+] = [a_1, a_2^+] = 1,$$
$$\frac{d}{dt}\begin{pmatrix} a \\ a^+ \end{pmatrix} = \begin{pmatrix} -k & -i\varepsilon \\ i\varepsilon & -k \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix}. \tag{42}$$

与(42)式相应的 Fokker-Planck 方程形式同(4)式. 故只须讨论后向简并四波混频情形,讨论(39)式的解即可.

(39)式可写作
$$\frac{\partial P}{\partial t} = \left\{ k\left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^*\right) - \varepsilon\left(\alpha^*\frac{\partial}{\partial \alpha} - \alpha\frac{\partial}{\partial \alpha^*}\right) + \frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \alpha^{*2}}\right) + 2k\bar{n}\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right\} P. \tag{43}$$

作变换 $\alpha \to \alpha,\ \alpha^* \to i\alpha^*$,
得
$$\frac{\partial P}{\partial t} = \left\{ k\left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^*\right) - i\varepsilon\left(\alpha^*\frac{\partial}{\partial \alpha} + \alpha\frac{\partial}{\partial \alpha^*}\right) + \frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}}\right) - 2ik\bar{n}\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right\} P. \tag{43$'$}$$

仍按上节方法求解
$$P = e^{-a(\alpha^2+\alpha^{*2})/2 + b\alpha\alpha^*} Q(\alpha, \alpha^*, t),$$
$$a = \frac{1}{2}\left[\frac{k+i\varepsilon}{\varepsilon+2ik\bar{n}} + \frac{k-i\varepsilon}{\varepsilon-2ik\bar{n}}\right],\quad b = \frac{1}{2}\left[\frac{k+i\varepsilon}{\varepsilon+2ik\bar{n}} - \frac{k-i\varepsilon}{\varepsilon-i2k\bar{n}}\right],$$
$$\left\{\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}}\right) - 2k\bar{n}i\frac{\partial^2}{\partial \alpha \partial \alpha^*} - \frac{1}{4}\left(\frac{(k+i\varepsilon)^2}{\varepsilon+2ik\bar{n}} + \frac{(k-i\varepsilon)^2}{\varepsilon-2ik\bar{n}}\right)(\alpha^2+\alpha^{*2}) + \frac{1}{2}\left(\frac{(k+i\varepsilon)^2}{\varepsilon+2ik\bar{n}} - \frac{(k-i\varepsilon)^2}{\varepsilon-2ik\bar{n}}\right)\alpha\alpha^* + k\right\} Q$$
$$= \frac{\partial}{\partial t} Q(\alpha, \alpha^*, t). \tag{44}$$

注意到定义(37)式
$$a^+ = \frac{a_2^+(1+i) - a_1(1-i)}{2} = i\frac{(a_1+a_2^+) + i(a_1-a_2^+)}{2},$$
$$a = \frac{a_2^+(1+i) + a_1(1-i)}{2} = \frac{a_1 + a_2^+ - i(a_1-a_2^+)}{2}. \tag{45}$$

又注意到在得出(43)′式时,已作了变换 $\alpha \to \alpha,\ \alpha^* \to i\alpha^*$, 故在 P 表示中, α, α^* 的含义为
$$\alpha = \frac{\alpha_1 + \alpha_2^* - i(\alpha_1 - \alpha_2^*)}{2} = \frac{\beta + i\tilde{\beta}}{\sqrt{2}},$$
$$\alpha^* = \frac{\alpha_1 + \alpha_2^* + i(\alpha_1 - \alpha_2^*)}{2} = \frac{\beta - i\tilde{\beta}}{\sqrt{2}}, \tag{46}$$

其中
$$\beta = \frac{\alpha_1 + \alpha_2^*}{\sqrt{2}},\quad \tilde{\beta} = \frac{-\alpha_1 + \alpha_2^*}{\sqrt{2}}.$$

定义
$$c = \frac{k - i\varepsilon}{\varepsilon - 2ik\bar{n}}, \quad \tilde{c} = \frac{k + i\varepsilon}{\varepsilon + 2ik\bar{n}},$$

则
$$Q_{mn} = \exp\left[-\left(m + \frac{1}{2}\right)(k - i\varepsilon)t - \left(n + \frac{1}{2}\right)(k + i\varepsilon)t\right] Q_m(\beta)\tilde{Q}_n(\tilde{\beta}),$$

$$\begin{aligned} Q_m(\beta) &= N_m e^{-\frac{c\beta^2}{2}} H_m(\sqrt{c}\,\beta), \\ \tilde{Q}_n(\tilde{\beta}) &= N_n e^{\tilde{c}\tilde{\beta}^2/2} H_n(\sqrt{\tilde{c}}\, i\tilde{\beta}). \end{aligned} \tag{47}$$

令
$$x_1 = \frac{\alpha + \alpha^*}{2} = \frac{\beta}{\sqrt{2}}, \quad x_2 = \frac{\alpha - \alpha^*}{2i} = \frac{\tilde{\beta}}{\sqrt{2}},$$

则

$$\begin{aligned} \langle :\Delta x_1^2: \rangle &= \left\langle \left(\frac{\beta - \beta_0 e^{-(k - i\varepsilon)t}}{\sqrt{2}} \right)^2 \right\rangle = \frac{1}{4c}\left(1 - e^{-2(k - i\varepsilon)t}\right), \\ \langle :\Delta x_2^2: \rangle &= \left\langle \left(\frac{\tilde{\beta} - \tilde{\beta}_0 e^{-(k - i\varepsilon)t}}{\sqrt{2}} \right)^2 \right\rangle = -\frac{1}{4\tilde{c}}\left(1 - e^{-2(k + i\varepsilon)t}\right). \end{aligned} \tag{48}$$

上面已提到由(37)式定义的 a, a^+ 不是厄密的,故 α, α^* 并非互共轭量, x_1, x_2 也不是实数,因此其方差亦是复数.

五、压缩态的传播

上几节只讨论了在一个非线性介质元件中由相干光产生压缩态的问题. 如果在一个腔中包含了多种线性与非线性介质元件，那么就有由一个元件产生的压缩态传播到另一元件，经过非线性相互作用输出继续往下传播的问题. 如何计算在传播过程中的物理量的平均值与方差呢? 这就是我们所要讨论的.

物理量 β 在 $t = 0$ 时的取值 β_0 与 t 时的取值 β 按(22),(23)式有如下关系:

$$\begin{aligned} \beta &= (\bar{\beta}_0 + \beta_0 - \bar{\beta}_0)e^{-|k-\varepsilon|t} + \beta - \beta_0 e^{-|k-\varepsilon|t} \\ &= (\bar{\beta}_0 + \delta\beta_0)e^{-|k-\varepsilon|t} + \Delta\beta. \\ \delta\beta &= \beta - \bar{\beta}_0 e^{-|k-\varepsilon|t} = \delta\beta_0 e^{-|k-\varepsilon|t} + \Delta\beta, \end{aligned} \tag{49}$$

式中 $\delta\beta_0 = \beta_0 - \bar{\beta}_0$ 为输入讯号所包含的总误差, $\bar{\beta}_0$ 为输入讯号的平均值, $\delta\beta = \beta - \bar{\beta}_0 e^{-|k-\varepsilon|t}$ 为输出讯号的总误差, $\bar{\beta}_0 e^{-|k-\varepsilon|t}$ 为输出讯号的平均值. $\Delta\beta = \beta - \beta_0 e^{-|k-\varepsilon|t}$ 为在传播过程中产生的新误差. (49) 式表明输入讯号的总误差 $\delta\beta_0$ 经传递后叠加在新产生的误差 $\Delta\beta$ 上便得出总的误差 $\delta\beta$. 由此容易得出方差的传递关系

$$\begin{aligned} \langle \delta\beta^2 \rangle &= \langle \delta\beta_0^2 \rangle e^{-2|k-\varepsilon|t} + \langle (\Delta\beta)^2 \rangle \\ &= \langle \delta\beta_0^2 \rangle e^{-2|k-\varepsilon|t} + \frac{1}{2}\left[1 + \frac{|\varepsilon + 2\bar{n}k|}{|\varepsilon - k|}\left(1 - e^{-2|k-\varepsilon|t}\right)\right]. \end{aligned} \tag{50}$$

要计算方差,一般需要知道格林函数. 下面给出两个串接非线性元件(见图3)的格林函数的计算. 对于多个元件组成的系统,可参照进行.

$$G(\beta,\tilde{\beta},t;\beta_0,\tilde{\beta}_0) = \iint G_1(\beta,\tilde{\beta},t;\beta',\tilde{\beta}',t') \times G_2(\beta',\tilde{\beta}',t';\beta_0,\tilde{\beta}_0)d\beta' d\tilde{\beta}'. \tag{51}$$

若不计方差,即令 $\langle \delta\beta_0^2 \rangle = \langle (\Delta\beta)^2 \rangle = 0$, 由(50)式得

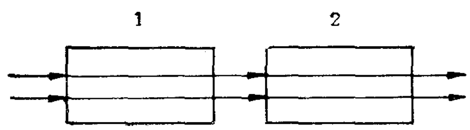

图 3

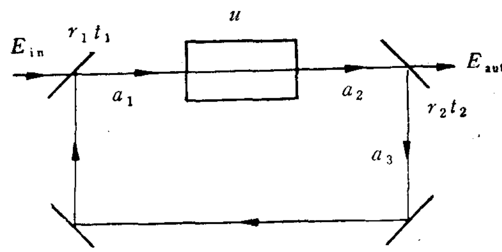

图 4

$$\delta\beta = \beta - \bar{\beta}_0 e^{-(k-\varepsilon)t} = 0,$$
$$\delta\tilde{\beta} = \tilde{\beta} - \bar{\tilde{\beta}}_0 e^{-(k+\varepsilon)t} = 0. \tag{52}$$

去掉 $\bar{\beta}_0$, $\bar{\tilde{\beta}}_0$ 上面的一横,并注意到

$$\beta = \frac{\alpha + \alpha^*}{\sqrt{2}}, \ \tilde{\beta} = \frac{\alpha - \alpha^*}{\sqrt{2}\,i},$$

便得

$$\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix} = e^{-kt}\begin{pmatrix}\cosh\varepsilon t & \sinh\varepsilon t \\ \sinh\varepsilon t & \cosh\varepsilon t\end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_0^*\end{pmatrix}. \tag{53}$$

对于四波混频情形,由(48)式及与上面同样的推导得

$$\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix} = e^{-kt}\begin{pmatrix}\cos\varepsilon t & i\sin\varepsilon t \\ -i\sin\varepsilon t & \cos\varepsilon t\end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_0^*\end{pmatrix}. \tag{54}$$

这相当于(53)式中的 ε 用 $\pm\ i|\varepsilon|$ 代替. 当 $k=0$ 便过渡到文献[11,12]的结果.

(53)和(54)式均可写成如下形式:

$$\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix} = \lambda\begin{pmatrix}\mu & \nu \\ \nu^* & \mu^*\end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_0^*\end{pmatrix} \quad \mu^2 - \nu^2 = 1. \tag{55}$$

将(53)和(54)式中的 t 改为传输距离 $L(=ct)$,

则
$$\lambda = e^{-\frac{kL}{c}}, \ \mu = \cosh\frac{\varepsilon L}{c}, \ \nu = \sinh\frac{\varepsilon L}{c} \tag{56}$$

或
$$\lambda = e^{-\frac{kL}{c}}, \ \ \mu = \cos\frac{\varepsilon L}{c}, \ \ \nu = i\sin\frac{\varepsilon L}{c}. \tag{57}$$

(55)式表明由 $\begin{pmatrix}\alpha_0 \\ \alpha_0^*\end{pmatrix}$ 产生 $\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix}$ 的过程可用传输矩阵 $\begin{pmatrix}\mu & \nu \\ \nu^* & \mu^*\end{pmatrix}$ 来表述. 并且它满足 $ABCD$ 定理.

将此关系用于分析复合腔[13],可使得分析步骤简化. 作为例子,下面计算含非线性元件的环行腔对相干态的传播.

如图 4 所示,入射相干光 E_{in} 经环行腔输出 E_{out}, 其中

$$a_1 = t_1E_{in} + r_1a_3 = t_1E_{in} + r_1r_2a_2 = t_1E_{in} + r_1r_2ua_1. \tag{58}$$
$$(1 - r_1r_2u)a_1 = t_1E_{in},$$
$$a_1 = (u - s)^{-1}TE_{in}, \quad s = \frac{1}{r_1r_2}, \ T = -\frac{t_1}{r_1r_2},$$

故 $$E_{\rm out} = t_2 a_2 = u(u-s)^{-1} t_2 T E_{\rm in}.$$

式中 u 为传输矩阵，$(u-s)^{-1}$ 为 $(u-s)$ 的逆矩阵. 设

$$u = \begin{pmatrix} \mu & \nu \\ \nu^* & \mu^* \end{pmatrix},$$

则 $$\begin{pmatrix} E_{\rm out} \\ E_{\rm out}^{+} \end{pmatrix} = \frac{Tt_2[(1-\mu s)^2 - \nu^2 s^2]}{[(\mu - s)^2 - \nu^2]^2} \begin{pmatrix} \dfrac{1-\mu s}{\sqrt{(1-\mu s)^2 - \nu^2 s^2}} & \dfrac{\nu s}{\sqrt{(1-\mu s)^2 - \nu^2 s^2}} \\ \dfrac{\nu^* s}{\sqrt{(1-\mu s)^2 - \nu^2 s^2}} & \dfrac{1-\mu s}{\sqrt{(1-\mu s)^2 - \nu^2 s^2}} \end{pmatrix} \begin{pmatrix} E_{\rm in} \\ E_{\rm in}^{+} \end{pmatrix}. \tag{59}$$

改变端面反射率 r_1, r_2, 使得 $s \simeq \dfrac{1}{\mu \pm \nu}$ 时，含非线性元件环行腔将给出最大压缩[3].

对于复合系统，如图 3 所示的非线性元件串接系统的传播矩阵，很明显有

$$\lambda \begin{pmatrix} \mu & \nu \\ \nu^* & \mu^* \end{pmatrix} = \lambda_1 \lambda_2 \begin{pmatrix} \mu_1 & \nu_1 \\ \nu_1^* & \mu_1^* \end{pmatrix} \begin{pmatrix} \mu_2 & \nu_2 \\ \nu_2^* & \mu_2^* \end{pmatrix}. \tag{60}$$

对于多个元件串接的系统应有类似结果.

附 录

$k - \varepsilon < 0$ 情形 Fokker-Planck 方程的解

因 $k - \varepsilon < 0$, c, $\tilde{c}$ 仍按(10)式定义，则(14)式可写为

$$P_{mn} = \exp\left(\frac{c}{2}\beta^2 + \frac{\tilde{c}}{2}\tilde{\beta}^2\right) Q_{mn}. \tag{A.1}$$

格林函数 $P(\beta, \tilde{\beta}, t; \beta_0, \tilde{\beta}_0)$ 的表式(18)中与 β 有关的因子应换为

$$\begin{aligned} &\exp\left[-\frac{|\varepsilon - k|}{2} t + \frac{2c\beta\beta_0 e^{-|\varepsilon-k|t} - c(\beta^2 + \beta_0^2) e^{-2|\varepsilon-k|t}}{1 - e^{-2|\varepsilon-k|t}}\right] \\ &= \exp\left[-\frac{|\varepsilon - k|}{2} t - \frac{c(\beta e^{-|\varepsilon-k|t} - \beta_0)^2}{1 - e^{-2|\varepsilon-k|t}} + c\beta_0^2\right]. \end{aligned} \tag{A.2}$$

根据归一化条件 $\iint P(\beta, \tilde{\beta}, t; \beta_0, \tilde{\beta}_0) d\beta d\tilde{\beta} = 1$, 并令

$$x = (\beta e^{-|\varepsilon-k|t} - \beta_0)/\sqrt{1 - e^{-2|\varepsilon-k|t}}, \tag{A.3}$$

$$y = (\tilde{\beta} - \tilde{\beta}_0 e^{-|\varepsilon+k|t})/\sqrt{1 - e^{-2|\varepsilon+k|t}}, \tag{A.4}$$

则得

$$P(\beta, \tilde{\beta}, t; \beta_0, \tilde{\beta}_0) d\beta d\tilde{\beta} = \frac{e^{-cx^2 + \tilde{c}y^2} dx dy}{\int e^{-cx^2} dx \int e^{\tilde{c}y^2} dy}. \tag{A.5}$$

将这结果与(21)式进行比较，主要是 x 的定义 (A.3) 式与 (21)式不一样，其他均与(18)—(21)式同. 按 (A.3)—(A.5) 式，可求得方差

$$\langle (\beta e^{-|k-\varepsilon|t} - \beta_0)^2 \rangle = \frac{1}{2c}(1 - e^{-2|k-\varepsilon|t}), \tag{A.6}$$

$$\langle (\tilde{\beta} - \tilde{\beta}_0 e^{-|k+\varepsilon|t})^2 \rangle = \frac{-1}{2\tilde{c}}(1 - e^{-2|k+\varepsilon|t}). \tag{A.7}$$

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THE FORMAL SOLUTIONS FOR FOKKER-PLANCK EQUATION WITH ZERO OR NEGATIVE DIFFUSION COEFFICIENTS AND THEIR APPLICATIONS TO QUANTUM OPTICS

TAN WEI-HAN LI YU-FANG ZHANG WEI-PING

(Shanghai Institute of Optics and Fine Mechanics, Academia Sinica)

ABSTRACT

In this paper, the formal solutions for Fokker-Planck equation with zero or negative diffusion coefficients in quantum optics are given. Their applications to the generation of squeezed states via degenerate parametric amplification or degenerate four wave mixing and other problems are discussed.