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相量的 q 形变相干态及其压缩性质

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研究位相正交算符在相量的 q 形变相干态中的压缩性质和二能级系统中粒子数-位相压缩及其不确定关系, 进而得到了一些新的粒子数-位相不确定态.

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1　引　　言

谐振子的振幅和位相这两个物理概念在力学和光学中有着重要的地位, 并且早就被引入量子力学和电磁学理论中[1,2]. 在量子理论中, 场的振幅正比于光子数算符的平方根, 而位相是用 Susskind 和 Glogower 提出的 SG 位相算符来描述的[3—5], 但 SG 位相算符不具有幺正性, 因而不能建立厄密位相算符.

八十年代末, Pegg 和 Barnett[6—8]提出了一种新的位相算符理论, 其核心是在有限(但任意多)维的希耳伯特空间中定义厄密位相算符. 他们把位相算符定义为位相态中投影算符与相应的位相值乘积之和, 所有物理量的期望值都含有空间维数这一参数, 当维数趋于无穷大时, 这些期望值与在无限维空间中得出的结果相同. 在过去的几年中, PB 理论得到了进一步的完善[9,10], 并被应用到量子光学的许多方面, 特别为谐振子相干态的研究提供了坚实的理论基础. 谐振子在粒子数表象和相表象中的相干态[11—13]都得以定义, 其压缩性质也得以详细讨论.

近年来, 在研究量子代数表示的过程中, 引入了 q 形变谐振子系统[14—16]. 同时, q 形变 SG 位相算符和 q 形变相干态[17—19]的建立, 促进了 q 形变谐振子研究的多方面进展[20,21]. 但由于 q 形变 SG 位相算符同样也不具有幺正性, 运用 PB 理论, 由此而定义了有限维空间中 q 形变算符[22,23], 并建立了新的 q 形变相干态[23].

本文将在文献[23]定义的相量的 q 形变算符的基础上, 定义有限维空间中相量的 q 形变相干态, 研究其位相波动、振幅-位相压缩, 并讨论其是否最小不确定态[4,5]或 intelligent 态[24,25].

2　相量的 q 形变算符

在$(s+1)$维希耳伯特空间 Ψ 中, 粒子态$|n\rangle \in \Psi$ 正交归一

$$\langle n | m \rangle = \delta_{nm}, \qquad \sum_{n=0}^{s} | n \rangle \langle n | = 1 \qquad m, n = 0, 1, \cdots, s. \tag{1}$$

Ψ 也可由 $(s+1)$ 个位相态 $|\theta_0\rangle, |\theta_1\rangle, \cdots, |\theta_s\rangle$ 展开[6—8]

$$|\theta_m\rangle = (s+1)^{-1/2} \sum_{n=0}^{s} \exp(\mathrm{i} n \theta_m) | n \rangle, \tag{2}$$

$|\theta_m\rangle$ 同样具有正交归一性

$$\langle \theta_n | \theta_m \rangle = \delta_{nm}, \qquad \sum_{m=0}^{s} | \theta_m \rangle \langle \theta_m | = 1. \tag{3}$$

这里相量 θ_m 定义为

$$\theta_m = \theta_0 + 2\pi \frac{m}{s+1} \qquad m = 0, 1, \cdots, s,$$

其中 θ_0 为任意值，但一旦选定，就组成了一套基矢. 为便于讨论，本文取 $\theta_0 = 0$.

厄密位相算符 $\hat{\Phi}_\theta$ 可由投影算符 $|\theta_m\rangle\langle\theta_k|$ 来定义

$$\hat{\Phi}_\theta = \sum_{m=0}^{s} \theta_m | \theta_m \rangle \langle \theta_m |, \tag{4}$$

由于 $\hat{\Phi}_\theta$ 的厄密性，指数算符 $\exp(\mathrm{i}\hat{\Phi}_\theta)$ 及其共轭算符具有么正性，并以位相态为本征态

$$\exp(\pm \mathrm{i} \hat{\Phi}_\theta) | \theta_m \rangle = \sum_{m=0}^{s} \exp(\pm \mathrm{i} \theta_m) | \theta_m \rangle. \tag{5}$$

q 形变时相量的产生算符 $\hat{\phi}^+$ 和湮没算符 $\hat{\phi}$ 可以定义如下：

$$\hat{\phi}^+ = \sum_{m=1}^{s} \sqrt{[\theta_m]} \, | \theta_m \rangle \langle \theta_{m-1} |,$$

$$\hat{\phi} = \sum_{m=1}^{s} \sqrt{[\theta_m]} \, | \theta_{m-1} \rangle \langle \theta_m |. \tag{6}$$

这里记号 $[x] \equiv \dfrac{q^x - q^{-x}}{q - q^{-1}}$，$x$ 为算符或常量，q 为形变参数，在正实数范围内，当 $q \leftrightarrow 1/q$ 时，$[x]$ 结果不变，故在本文中，取 $0 < q < 1$. $\hat{\phi}^+$，$\hat{\phi}$ 对位相态空间作用如下[23]：

$$\begin{aligned} \hat{\phi}^+ | \theta_m \rangle &= \sqrt{[\theta_{m+1}]} \, | \theta_{m+1} \rangle \qquad \hat{\phi}^+ | s \rangle = 0, \\ \hat{\phi} | \theta_m \rangle &= \sqrt{[\theta_m]} \, | \theta_{m-1} \rangle \qquad \hat{\phi} | 0 \rangle_p = 0. \end{aligned} \tag{7}$$

特别有

$$(\hat{\phi}^+)^k | 0 \rangle_p = \Big(\prod_{j=1}^{k} [\theta_j] \Big)^{1/2} | \theta_k \rangle \qquad (1 \leqslant k \leqslant s);$$

$$(\hat{\phi}^+)^k | 0 \rangle_p = 0 \qquad (k > s), \tag{8}$$

式中真空态 $|0\rangle_p \equiv |\theta_0 = 0\rangle$.

为了便于表达 q 形变算符的性质，引入位相态空间中的算符 $\hat{\mathscr{M}}_{ij}$[13]

$$\hat{\mathscr{M}}_{ij} = | \theta_i \rangle \langle \theta_j | \qquad i, j = 0, 1, \cdots, s.$$

由以上各式可得

$$\hat{\phi}\hat{\phi}^{+}-q^{\frac{2\pi}{s+1}}\hat{\phi}^{+}\ \hat{\phi}=\left[\frac{2\pi}{s+1}\right]q^{-\hat{\Phi}_{\theta}}-[2\pi]\hat{\mathscr{M}}_{ss},$$

$$\hat{\phi}\hat{\phi}^{+}-q^{-\frac{2\pi}{s+1}}\hat{\phi}^{+}\ \hat{\phi}=\left[\frac{2\pi}{s+1}\right]q^{\hat{\Phi}_{\theta}}-[2\pi]\hat{\mathscr{M}}_{ss},$$

$$[\hat{\Phi}_{\theta},\hat{\phi}^{+k}]=\frac{2\pi}{s+1}k\hat{\phi}^{k}\qquad 1\leqslant k\leqslant s,$$

$$[\hat{\Phi}_{\theta},\hat{\phi}^{k}]=-\frac{2\pi}{s+1}k\hat{\phi}^{+k}\qquad 1\leqslant k\leqslant s,$$

$$[\hat{\phi}^{+},\hat{\mathscr{M}}_{ij}]=\sqrt{[\theta_{i+1}]}\mathscr{H}(s-i)\hat{\mathscr{M}}_{(i+1)j}-\sqrt{[\theta_{j}]}\hat{\mathscr{M}}_{i(j-1)},\tag{9}$$

$$[\hat{\phi},\hat{\mathscr{M}}_{ij}]=\sqrt{[\theta_{i}]}\hat{\mathscr{M}}_{(i-1)j}-\sqrt{[\theta_{j+1}]}\mathscr{H}(s-j)\hat{\mathscr{M}}_{i(j+1)},$$

$$[\hat{\Phi}_{\theta},\hat{\mathscr{M}}_{ij}]=(\theta_{i}-\theta_{j})\hat{\mathscr{M}}_{ij},$$

$$[\hat{\mathscr{M}}_{ij},\hat{\mathscr{M}}_{kl}]=\delta_{jk}\hat{\mathscr{M}}_{il}-\delta_{il}\hat{\mathscr{M}}_{kj},$$

其中 $\mathscr{H}(x)$是阶跃函数,定义为

$$\mathscr{H}(x)=\begin{cases}1 & x>0;\\ 0 & x\leqslant 0.\end{cases}$$

很明显,在位相态中,相量的产生算符和湮没算符发生形变,而粒子数算符却没有发生形变.当 $q\to 1$ 时,这些公式简化为普通谐振子相量的算符性质公式[13].

3 相量的 q 形变相干态

有限维希耳伯特空间中 q 形变谐振子相量的 q 形变相干态定义如下:

$$|\alpha,s\rangle=C(\alpha,s)\mathrm{e}^{\alpha\hat{\phi}^{+}}_{q^{\frac{2\pi}{s+1}}}|0\rangle_{p}=C(\alpha,s)\sum_{m=0}^{s}\frac{\tilde{\alpha}^{m}}{\sqrt{[\theta_{m}]!}}|\theta_{m}\rangle.\tag{10}$$

这里利用了(8)式,并引入如下符号:

$$[\theta_{0}]!=1,\qquad [\theta_{m}]!=\prod_{i=1}^{m}[\theta_{i}]\qquad m=1,2,\cdots,s,$$

$\mathrm{e}^{x}_{q^{\frac{2\pi}{s+1}}}$是 q 指数函数

$$\mathrm{e}^{x}_{q^{\frac{2\pi}{s+1}}}=\sum_{m=0}^{\infty}\frac{x^{m}}{[m]_{q^{\frac{2\pi}{s+1}}}!},\qquad [m]_{q^{\frac{2\pi}{s+1}}}=\frac{q^{\frac{2\pi m}{s+1}}-q^{-\frac{2\pi m}{s+1}}}{q^{\frac{2\pi}{s+1}}-q^{-\frac{2\pi}{s+1}}}=\frac{[\theta_{m}]}{\left[\frac{2\pi}{s+1}\right]},$$

$\tilde{\alpha}=\left[\frac{2\pi}{s+1}\right]\alpha$, α 是复变量, $C(\alpha,s)$是归一化系数

$$C(\alpha,s)=\left(\sum_{m=0}^{s}\frac{(\tilde{\alpha}\tilde{\alpha}^{*})^{m}}{[\theta_{m}]!}\right)^{-1/2}=\left(\mathrm{e}^{|\tilde{\alpha}|^{2}}_{q,p,s}\right)^{-1/2},\tag{11}$$

其中 $\tilde{\alpha}^{*}$ 是 $\tilde{\alpha}$ 的复共轭, $\mathrm{e}^{x}_{q,p,s}$定义为

$$e^{x}_{q,\,p,\,s}=\sum_{m=0}^{s}\frac{x^m}{[\theta_m]!}.$$

可以看出，当 $s\to\infty$ 时，$\mathrm{e}^{x}_{q,\,p,\,s}\to\mathrm{e}^{x}_{q,\,p}=\sum_{m=0}^{\infty}\frac{x^m}{[\theta_m]!}$，$|\alpha,s\rangle$ 趋向于无限维空间中相量的 q 形变相干态[17,18]；当 $q\to1$ 时，$\mathrm{e}^{x}_{q,\,p,\,s}\to\sum_{m=0}^{s}\frac{x^m}{\theta_m!}$，$|\alpha,s\rangle$ 趋向于有限维空间中普通谐振子相量相干态[13].

湮没算符 $\hat{\phi}$ 对 $|\alpha,s\rangle$ 的作用如下：

$$\hat{\phi}|\alpha,s\rangle=\tilde{\alpha}\left\{|\alpha,s\rangle-\left(\mathrm{e}^{|\tilde{\alpha}|^2}_{q,\,p,\,s}\right)^{-1/2}\frac{\tilde{\alpha}^s}{\sqrt{[\theta_s]!}}|\theta_s\rangle\right\}. \tag{12}$$

这表明，只要 s 取有限值，相量的 q 形变相干态 $|\alpha,s\rangle$ 就不会是湮没算符 $\hat{\phi}$ 的本征态，上式等号右边括号内的第二项是因有限维空间而存在.

上面定义的相量的 q 形变相干态相互之间并不正交

$$\langle\alpha,s|\alpha',s\rangle=\left(\mathrm{e}^{|\tilde{\alpha}|^2}_{q,\,p,\,s}\mathrm{e}^{|\tilde{\alpha}'|^2}_{q,\,p,\,s}\right)^{-1/2}\mathrm{e}^{\tilde{\alpha}^*\tilde{\alpha}'}_{q,\,p,\,s}, \tag{13}$$

但还是完备的

$$\int|\alpha,s\rangle\langle\alpha,s|\mathrm{d}\mu(\alpha)=\sum_{m=0}^{s}|\theta_m\rangle\langle\theta_m|=1, \tag{14}$$

其中

$$\mathrm{d}\mu(\alpha)=\frac{1}{2\pi}\mathrm{e}^{-|\tilde{\alpha}|^2}_{q,\,p}\mathrm{e}^{|\tilde{\alpha}|^2}_{q,\,p,\,s}\mathrm{d}_{q^{\frac{2\pi}{s+1}}}|\tilde{\alpha}|^2\mathrm{d}\theta,$$

式中对 θ 的积分是普通积分，而对 $|\tilde{\alpha}|^2$ 的积分是 q 积分. q 导数定义如下：

$$\frac{\mathrm{d}}{\mathrm{d}_{q^{\frac{2\pi}{s+1}}}x}f(x)=\frac{f(q^{\frac{2\pi}{s+1}}x)-f(q^{-\frac{2\pi}{s+1}}x)}{(q^{\frac{2\pi}{s+1}}-q^{-\frac{2\pi}{s+1}})x}.$$

q 积分可从 q 导数的定义导出.

由上述性质可知，相量的 q 形变相干态是超完备的. 任意一个相量的 q 形变相干态 $|\alpha',s\rangle$ 可写为

$$|\alpha',s\rangle=\int\left(\mathrm{e}^{|\tilde{\alpha}|^2}_{q,\,p,\,s}\mathrm{e}^{|\tilde{\alpha}'|^2}_{q,\,p,\,s}\right)^{-1/2}\mathrm{e}^{\tilde{\alpha}^*\tilde{\alpha}'}_{q,\,p,\,s}|\alpha,s\rangle\mathrm{d}\mu(\alpha). \tag{15}$$

$|\alpha,s\rangle$ 位相分布

$$P(\theta_m,s)=\left|\langle\theta_m|\alpha,s\rangle\right|^2=\frac{|\tilde{\alpha}|^{2m}}{[\theta_m]!}\left(\mathrm{e}^{|\tilde{\alpha}|^2}_{q,\,p,\,s}\right)^{-1} \tag{16}$$

满足归一化条件

$$\sum_{m=0}^{s}P(\theta_m,s)=1. \tag{17}$$

由 $|\alpha,s\rangle$ 的完备性，位相态 $|\theta_m\rangle$ 可以 $|\alpha,s\rangle$ 为基矢展开

$$|\theta_m\rangle = \frac{1}{2\pi}\int e_{q,p}^{-|\tilde{\alpha}|^2}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{1/2}\frac{(\tilde{\alpha}^*)^m}{\sqrt{[\theta_m]!}}|\alpha, s\rangle d_{q^{\frac{2\pi}{s+1}}}|\tilde{\alpha}|^2 d\theta. \tag{18}$$

将(2)式代入(10)式,可得到$|\alpha, s\rangle$以$|n\rangle$为基矢的展开式

$$|\alpha, s\rangle = (s+1)^{-1/2}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1/2}\sum_{n,m=0}^{s}\frac{\tilde{\alpha}^m}{\sqrt{[\theta_m]!}}\exp(\mathrm{i}n\theta_m)|n\rangle, \tag{19}$$

其粒子数分布

$$P(n, s) = (s+1)^{-1}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1}\left|\sum_{m=0}^{s}\frac{\tilde{\alpha}^m}{\sqrt{[\theta_m]!}}\exp(\mathrm{i}n\theta_m)\right|^2 \tag{20}$$

满足归一化条件

$$\sum_{n=0}^{s}P(n, s) = 1. \tag{21}$$

类似地,粒子态$|n\rangle$也可以$|\alpha, s\rangle$为基矢展开

$$|n\rangle = \frac{1}{2\pi(s+1)^{1/2}}\sum_{m=0}^{s}\int e_{q,p}^{-|\tilde{\alpha}|^2}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{1/2}\exp(\mathrm{i}n\theta_m)\frac{(\tilde{\alpha}^*)^m}{\sqrt{[\theta_m]!}}|\alpha, s\rangle\, d_{q^{\frac{2\pi}{s+1}}}|\tilde{\alpha}|^2 d\theta. \tag{22}$$

4　二阶压缩性质

本节讨论位相正交算符[11]在相量的 q 形变相干态中的压缩性质.

首先引入一对位相正交算符 $\hat{X}_\phi$ 和 $\hat{Y}_\phi$

$$\hat{X}_\phi = \frac{1}{2}(\hat{\phi} + \hat{\phi}^+), \qquad \hat{Y}_\phi = \frac{1}{2\mathrm{i}}(\hat{\phi} - \hat{\phi}^+), \tag{23}$$

其对易关系

$$[\hat{X}_\phi, \hat{Y}_\phi] = \frac{\mathrm{i}}{2}[\hat{\phi}, \hat{\phi}^+], \tag{24}$$

并满足不确定关系

$$\langle(\Delta\hat{X}_\phi)^2\rangle\cdot\langle(\Delta\hat{Y}_\phi)^2\rangle \geqslant \frac{1}{4}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|^2, \tag{25}$$

其中$\langle(\Delta\hat{X}_\phi)^2\rangle = \langle\hat{X}_\phi^2\rangle - (\langle\hat{X}_\phi\rangle)^2$, $\langle(\Delta\hat{Y}_\phi)^2\rangle = \langle\hat{Y}_\phi^2\rangle - (\langle\hat{Y}_\phi\rangle)^2$.如果

$$\langle(\Delta\hat{X}_\phi)^2\rangle < \frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle| \quad 或 \quad \langle(\Delta\hat{Y}_\phi)^2\rangle < \frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|,$$

则称态在 $\hat{X}_\phi$ 或 $\hat{Y}_\phi$ 分量上压缩.

为了度量算符的压缩程度,这里引入两个压缩参数

$$S_X = \frac{\langle(\Delta\hat{X}_\phi)^2\rangle - \frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|}{\frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|}, \qquad S_Y = \frac{\langle(\Delta\hat{Y}_\phi)^2\rangle - \frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|}{\frac{1}{2}|\langle[\hat{X}_\phi, \hat{Y}_\phi]\rangle|}, \tag{26}$$

则压缩条件简化为

$$S_X < 0 \qquad 或 \qquad S_Y < 0. \tag{27}$$

下面是一些算符在相量的 q 形变相干态中的期望值:

$$\begin{aligned}
\langle \alpha, s|\hat{\phi}|\alpha, s\rangle &= \tilde{\alpha}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-1}^{|\tilde{\alpha}|^2},\\
\langle \alpha, s|\hat{\phi}^+|\alpha, s\rangle &= \tilde{\alpha}^*\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-1}^{|\tilde{\alpha}|^2},\\
\langle \alpha, s|\hat{\phi}^2|\alpha, s\rangle &= \tilde{\alpha}^2\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-2}^{|\tilde{\alpha}|^2},\\
\langle \alpha, s|\hat{\phi}^{+2}|\alpha, s\rangle &= \tilde{\alpha}^{*2}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-2}^{|\tilde{\alpha}|^2},\\
\langle \alpha, s|\hat{\phi}^+\hat{\phi}|\alpha, s\rangle &= |\tilde{\alpha}|^2\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-1}^{|\tilde{\alpha}|^2},\\
\langle \alpha, s|\hat{\phi}\hat{\phi}^+|\alpha, s\rangle &= \left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1}\left(q^{\frac{2\pi}{s+1}}|\tilde{\alpha}|^2 e_{q,p,s-2}^{|\tilde{\alpha}|^2} + \left[\frac{2\pi}{s+1}\right] e_{q,p,s-1}^{|\tilde{\alpha}|^2 q^{-\frac{2\pi}{s+1}}}\right).
\end{aligned} \tag{28}$$

由上可得算符 $\hat{X}_\phi$ 和 $\hat{Y}_\phi$ 的方差

$$\begin{aligned}
\langle \alpha, s|(\Delta\hat{X}_\phi)^2|\alpha, s\rangle = {}& \frac{1}{4}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1}\left\{\left(\tilde{\alpha}^2 + \tilde{\alpha}^{*2} + q^{\frac{2\pi}{s+1}}|\tilde{\alpha}|^2\right) e_{q,p,s-2}^{|\tilde{\alpha}|^2}\right.\\
&\left. + \left[\frac{2\pi}{s+1}\right] e_{q,p,s-1}^{|\tilde{\alpha}|^2 q^{-\frac{2\pi}{s+1}}} + |\tilde{\alpha}|^2 e_{q,p,s-1}^{|\tilde{\alpha}|^2}\right\} - \left\{\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-1}^{|\tilde{\alpha}|^2}\mathrm{Re}(\tilde{\alpha})\right\}^2,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\langle \alpha, s|(\Delta\hat{Y}_\phi)^2|\alpha, s\rangle = {}& -\frac{1}{4}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1}\left\{\left(\tilde{\alpha}^2 + \tilde{\alpha}^{*2} - q^{\frac{2\pi}{s+1}}|\tilde{\alpha}|^2\right) e_{q,p,s-2}^{|\tilde{\alpha}|^2}\right.\\
&\left. - \left[\frac{2\pi}{s+1}\right] e_{q,p,s-1}^{|\tilde{\alpha}|^2 q^{-\frac{2\pi}{s+1}}} - |\tilde{\alpha}|^2 e_{q,p,s-1}^{|\tilde{\alpha}|^2}\right\} - \left\{\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1} e_{q,p,s-1}^{|\tilde{\alpha}|^2}\mathrm{Im}(\tilde{\alpha})\right\}^2,
\end{aligned} \tag{30}$$

和$[\hat{X}_\phi, \hat{Y}_\phi]$的期望值

$$\begin{aligned}
\langle \alpha, s|[\hat{X}_\phi, \hat{Y}_\phi]|\alpha, s\rangle = {}& \frac{i}{2}\left(e_{q,p,s}^{|\tilde{\alpha}|^2}\right)^{-1}\left\{\left[\frac{2\pi}{s+1}\right] e_{q,p,s-1}^{|\tilde{\alpha}|^2 q^{-\frac{2\pi}{s+1}}}\right.\\
&\left. + q^{\frac{2\pi}{s+1}}|\tilde{\alpha}|^2 e_{q,p,s-2}^{|\tilde{\alpha}|^2} - |\tilde{\alpha}|^2 e_{q,p,s-1}^{|\tilde{\alpha}|^2}\right\}.
\end{aligned} \tag{31}$$

(29)—(31)式表明,在有限维希耳伯特空间中,相量的 q 形变相干态不是最小不确定态. 当 $s\to\infty$ 时,有

$$\langle \alpha, s|(\Delta\hat{X}_\phi)^2|\alpha, s\rangle = \langle \alpha, s|(\Delta\hat{Y}_\phi)^2|\alpha, s\rangle = \frac{1}{2}\left|\langle \alpha, s|[\hat{X}_\phi, \hat{Y}_\phi]|\alpha, s\rangle\right| = 0 \tag{32}$$

和

$$S_X = S_Y = 0, \tag{33}$$

(25)式等号成立,即$|\alpha, s\rangle$为最小不确定态.

下面以 α 是实数为例讨论 $\hat{X}_\phi$ 和 $\hat{Y}_\phi$ 的压缩性质.

当 $\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}>0$ 时,

$$S_X=\frac{2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}+2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}-4\tilde{\alpha}^2\left(e^{\tilde{\alpha}^2}_{q,p,s}\right)^{-1}\left(e^{\tilde{\alpha}^2}_{q,p,s-1}\right)^2}{\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}},$$

$$S_Y=\frac{2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}-2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}}{\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}}\geqslant 0, \tag{34}$$

因此,这种情况下 $\hat{Y}_\phi$ 不存在压缩,但是,当

$$\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}>0,$$

$$2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}+2\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}-4\tilde{\alpha}^2\left(e^{\tilde{\alpha}^2}_{q,p,s}\right)^{-1}\left(e^{\tilde{\alpha}^2}_{q,p,s-1}\right)^2<0 \tag{35}$$

时, $\hat{X}_\phi$ 分量存在压缩.

当 $\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}<0$ 时,

$$S_X=\frac{2\tilde{\alpha}^2\left(1+q^{\frac{2\pi}{s+1}}\right)e^{\tilde{\alpha}^2}_{q,p,s-2}+2\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-4\tilde{\alpha}^2\left(e^{\tilde{\alpha}^2}_{q,p,s}\right)^{-1}\left(e^{\tilde{\alpha}^2}_{q,p,s-1}\right)^2}{\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}-\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}},$$

$$S_Y=\frac{2\tilde{\alpha}^2\left(q^{\frac{2\pi}{s+1}}-1\right)e^{\tilde{\alpha}^2}_{q,p,s-2}-2\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}}{\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}-\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}}, \tag{36}$$

显而易见,当

$$\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-1}+q^{\frac{2\pi}{s+1}}\tilde{\alpha}^2 e^{\tilde{\alpha}^2}_{q,p,s-2}<0,$$

$$2\tilde{\alpha}^2\left(1+q^{\frac{2\pi}{s+1}}\right)e^{\tilde{\alpha}^2}_{q,p,s-2}+2\left[\frac{2\pi}{s+1}\right]e^{\tilde{\alpha}^2 q^{-\frac{2\pi}{s+1}}}_{q,p,s-1}-4\tilde{\alpha}^2\left(e^{\tilde{\alpha}^2}_{q,p,s}\right)^{-1}\left(e^{\tilde{\alpha}^2}_{q,p,s-1}\right)^2<0 \tag{37}$$

时, $\hat{X}_\phi$ 分量存在压缩.经过大量数值计算,无论 q, s, α 取何值(其中 $0<q<1$, α 为实数, s 为正整数), S_Y 都大于零,故 $\hat{Y}_\phi$ 分量不存在压缩.

图 1 和图 2 分别是 $\alpha=1,4$ 时,$(s+1)$维态空间中对于不同 s 压缩参数 S_X 和 S_Y 与 q 的关系图.由这些图可以看出, S_Y 总是大于零,而 S_X 有可能小于零.

由以上计算分析,可得出如下结论:只要 α 取实数, $\hat{Y}_\phi$ 分量就不存在压缩,但 $\hat{X}_\phi$ 分量的压缩在一定条件仍然存在.同样,当 α 取纯虚数时, $\hat{X}_\phi$ 分量的压缩不存在,而 $\hat{Y}_\phi$ 分量可能存在压缩.

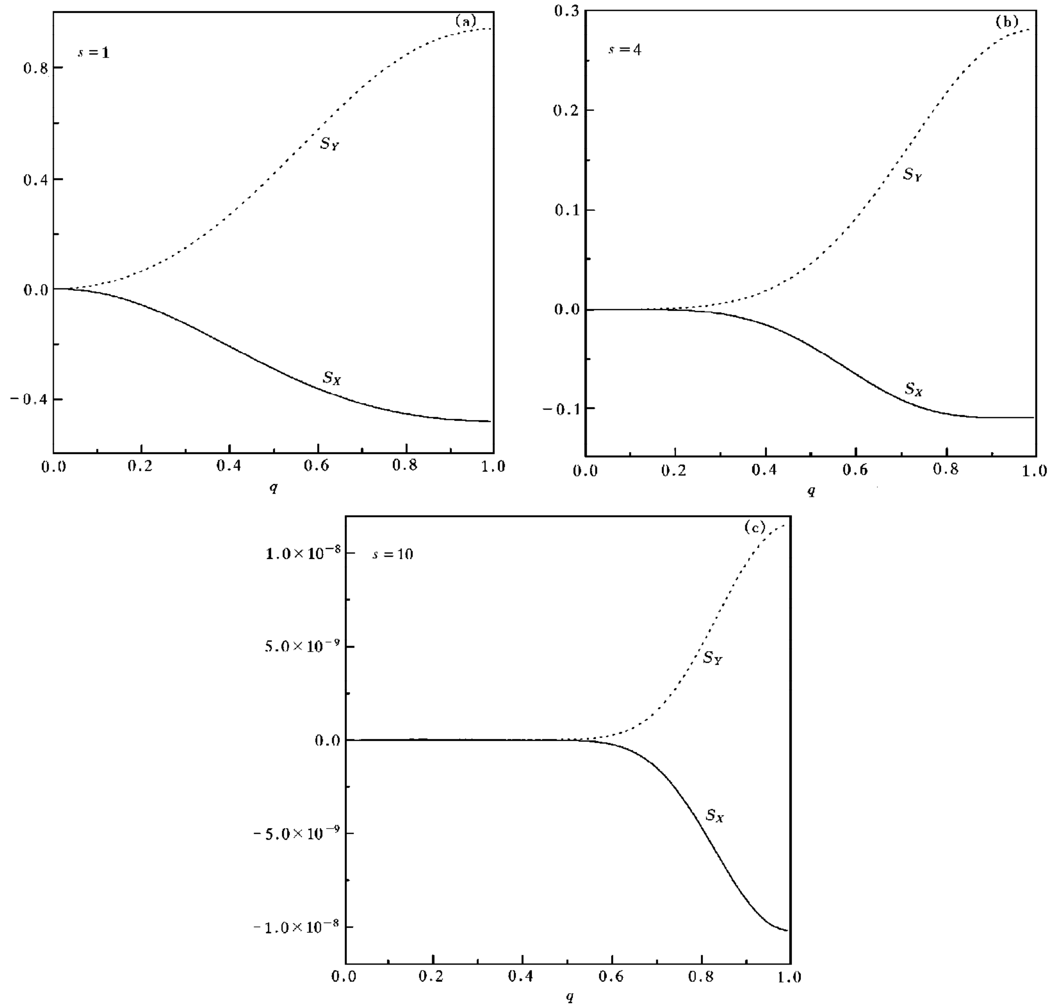

图 1　$\alpha=1$ 时,压缩参数 S_X 和 S_Y 与形变参数 q 的关系图

5　二能级系统粒子数-位相压缩及其不确定关系

众所周知,研究量子光学中的粒子数-位相压缩及其最小不确定态具有重要意义[4,10],但在一个任意的有限维空间中研究这一问题是很复杂的.本节将以二能级系统(即 $s=1$)为例,详细讨论其粒子数-位相压缩及不确定关系,并且寻找其最小不确定态.

在二能级系统中,取 $\theta_0=0$,则 $\theta_1=\pi$.粒子数算符、位相算符和余弦算符分别为[13]

$$\begin{aligned}
\hat{N} &= \frac{1}{2}\left(|0\rangle_{pp}\langle 0| + |\theta_1\rangle\langle\theta_1| - |0\rangle_p\langle\theta_1| - |\theta_1\rangle_p\langle 0|\right),\\
\hat{\Phi}_\theta &= \pi|\theta_1\rangle\langle\theta_1|, \qquad (38)\\
\cos\hat{\Phi}_\theta &= |0\rangle_{pp}\langle 0| - |\theta_1\rangle\langle\theta_1|,
\end{aligned}$$

其对易关系

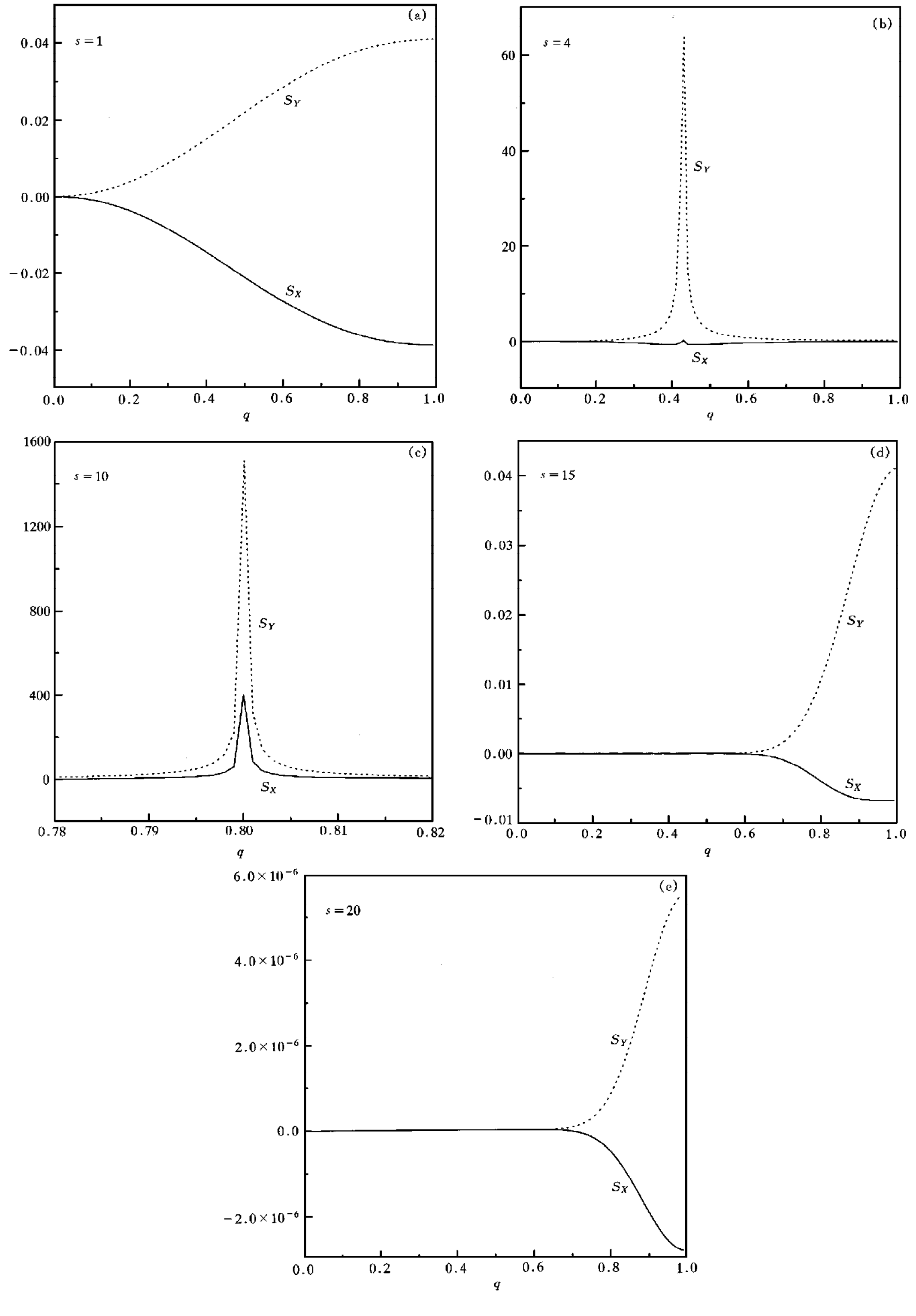

图 2　$\alpha=4$ 时，压缩参数 S_X 和 S_Y 与形变参数 q 的关系图

$$[\hat{N}, \hat{\Phi}_\theta] = \frac{\pi}{2}(|\theta_1\rangle_p\langle 0| - |0\rangle_p\langle\theta_1|),$$
$$[\hat{N}, \cos\hat{\Phi}_\theta] = |0\rangle_p\langle\theta_1| - |\theta_1\rangle_p\langle 0|. \tag{39}$$

$s=1$ 时,相量的 q 形变相干态

$$|\alpha,1\rangle = \left(1+\frac{|\tilde{\alpha}|^2}{[\pi]}\right)^{-1/2}\left(|0\rangle_p + \frac{\tilde{\alpha}}{\sqrt{[\pi]}}|\theta_1\rangle\right)$$
$$= \cos|\tilde{\alpha}||0\rangle_p + \exp(\mathrm{i}\phi)\sin|\tilde{\alpha}||\theta_1\rangle$$
$$= \frac{1}{\sqrt{2}}\{\cos|\tilde{\alpha}| + \exp(\mathrm{i}\phi)\sin|\tilde{\alpha}|\}|0\rangle + \frac{1}{\sqrt{2}}\{\cos|\tilde{\alpha}| - \exp(\mathrm{i}\phi)\sin|\tilde{\alpha}|\}|1\rangle, \tag{40}$$

其中 $\tilde{\alpha}=|\alpha|e^{\mathrm{i}\phi}, \cos|\tilde{\alpha}| \equiv \left(1+\frac{|\tilde{\alpha}|^2}{[\pi]}\right)^{-1/2}, \sin|\tilde{\alpha}| \equiv \frac{|\tilde{\alpha}|}{\sqrt{[\pi]}}\left(1+\frac{|\tilde{\alpha}|^2}{[\pi]}\right)^{-1/2}$,并满足 $\sin(2|\tilde{\alpha}|) = 2\sin|\tilde{\alpha}|\cos|\tilde{\alpha}|, \cos(2|\tilde{\alpha}|) = \cos^2|\tilde{\alpha}| - \sin^2|\tilde{\alpha}|$.

5.1　$\hat{\Phi}_\theta$-$\hat{N}$压缩

粒子数算符和位相算符的方差及其对易关系在 q 形变相干态(40)式中的平均值分别为

$$\langle(\Delta\hat{N})^2\rangle = \frac{1}{4}\{1-\cos^2\phi\sin^2(2|\tilde{\alpha}|)\},$$
$$\langle(\Delta\hat{\Phi}_\theta)^2\rangle = \frac{\pi^2}{4}\sin^2(2|\tilde{\alpha}|), \tag{41}$$
$$\langle[\hat{N}, \hat{\Phi}_\theta]\rangle = -\frac{\mathrm{i}\pi}{2}\sin\phi\sin(2|\tilde{\alpha}|),$$

由此可得其相应的压缩参数

$$S_\Phi = \frac{\pi\sin^2(2|\tilde{\alpha}|) - |\sin\phi\sin(2|\tilde{\alpha}|)|}{|\sin\phi\sin(2|\tilde{\alpha}|)|},$$
$$S_N = \frac{1-\cos^2\phi\sin^2(2|\tilde{\alpha}|) - \pi|\sin\phi\sin(2|\tilde{\alpha}|)|}{\pi|\sin\phi\sin(2|\tilde{\alpha}|)|}. \tag{42}$$

下面将分析在不同条件下 $\hat{N}$或 $\hat{\Phi}_\theta$ 的压缩状况.

1) $\hat{\Phi}_\theta$ 压缩　如果 $\theta_0-\phi=n\pi+\pi/2$(n 为整数),则相量的 q 形变相干态

$$|\alpha,1\rangle = \cos|\tilde{\alpha}||0\rangle_p \pm \mathrm{i}\sin|\tilde{\alpha}||\theta_1\rangle. \tag{43}$$

易证

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\Phi}_\theta)^2\rangle = \frac{1}{4}|\langle[\hat{N}, \hat{\Phi}_\theta]\rangle|^2 = \frac{\pi^2}{16}\sin^2(2|\tilde{\alpha}|), \tag{44}$$

故$|\alpha,1\rangle$是 intelligent 态. 压缩参数

$$S_\Phi = \frac{\pi\sin^2(2[\pi]|\alpha|) - |\sin(2[\pi]|\alpha|)|}{|\sin(2[\pi]|\alpha|)|},$$
$$S_N = \frac{1-\pi|\sin^2(2[\pi]|\alpha|)|}{\pi|\sin(2[\pi]|\alpha|)|}. \tag{45}$$

由这两式可知,当 $\pi|\sin(2[\pi]|\alpha|)|<1$ 时,位相算符 $\hat{\Phi}_\theta$ 下落,(即 $S_\Phi<0$),粒子数算符 $\hat{N}$

上涨(即 $S_N>0$),当 $\pi|\sin(2[\pi]|\alpha|)|>1$ 时,$\hat{\Phi}_\theta$ 上涨,$\hat{N}$下落.当 $\pi|\sin(2[\pi]|\alpha|)|\to 0$(即 $\alpha\to 0$)时,$S_\Phi\to -1$,$S_N\to +\infty$,$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\Phi}_\theta)^2\rangle\to 0$,达到其最小值,得到最小不确定态

$$\lim_{\alpha\to 0}|\alpha,1\rangle = |0\rangle_p. \tag{46}$$

2) $\hat{N}$压缩 如果 $\sin(2|\tilde{\alpha}|)=1$(即 $|\alpha|=1/\sqrt{[\pi]}$),则相量的 q 形变相干态

$$|\alpha,1\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_p + \exp(\mathrm{i}\phi)|\theta_1\rangle\}. \tag{47}$$

易证

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\Phi}_\theta)^2\rangle = \frac{1}{4}|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|^2 = \frac{\pi^2}{16}\sin^2\phi, \tag{48}$$

故 $|\alpha,1\rangle$ 是 intelligent 态.压缩参数

$$S_\Phi = \frac{\pi-|\sin\phi|}{|\sin\phi|} > 0,$$
$$S_N = \frac{\sin^2\phi-\pi|\sin\phi|}{\pi|\sin\phi|} < 0. \tag{49}$$

上两式表明,此时,位相算符 $\hat{\Phi}_\theta$ 上涨,粒子数算符 $\hat{N}$下落.如果 $\phi=n\pi$(n 为整数),$S_N\to -1$,$S_\Phi\to +\infty$,$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\Phi}_\theta)^2\rangle\to 0$,达到其最小值,得到最小不确定态

$$|\alpha,1\rangle = \begin{cases}\frac{1}{\sqrt{2}}(|0\rangle_p + |\theta_1\rangle) & (\phi=0);\\ \frac{1}{\sqrt{2}}(|0\rangle_p - |\theta_1\rangle) & (\phi=\pi).\end{cases} \tag{50}$$

5.2 $\cos\hat{\Phi}_\theta$-$\hat{N}$压缩

$\cos\hat{\Phi}_\theta$ 的方差和对易关系$[\cos\hat{\Phi}_\theta,\hat{N}]$在 q 形变相干态(40)式中的平均值分别为

$$\langle(\Delta\cos\hat{\Phi}_\theta)^2\rangle = \sin^2(2|\tilde{\alpha}|),$$
$$\langle[\hat{N},\cos\hat{\Phi}_\theta]\rangle = \mathrm{i}\sin\phi\sin(2|\tilde{\alpha}|). \tag{51}$$

由此可得其相应的压缩参数

$$S_{\cos\Phi} = \frac{2\sin^2(2|\tilde{\alpha}|)-|\sin\phi\sin(2|\tilde{\alpha}|)|}{|\sin\phi\sin(2|\tilde{\alpha}|)|},$$
$$S_N = \frac{1-\cos^2\phi\sin^2(2|\tilde{\alpha}|)-2|\sin\phi\sin(2|\tilde{\alpha}|)|}{2|\sin\phi\sin(2|\tilde{\alpha}|)|}. \tag{52}$$

如果 $\theta_0-\phi=n\pi+\pi/2$(n 为整数),相量的形变相干态与(43)式相同,易证

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\cos\hat{\Phi}_\theta)^2\rangle = \frac{1}{4}|\langle[\hat{N},\cos\hat{\Phi}_\theta]\rangle|^2 = \frac{1}{4}\sin^2(2|\tilde{\alpha}|), \tag{53}$$

故 $|\alpha,1\rangle$ 是 intelligent 态.压缩参数

$$S_{\cos\Phi} = \frac{2\sin^2(2|\tilde{\alpha}|)-|\sin\phi\sin(2|\tilde{\alpha}|)|}{|\sin\phi\sin(2|\tilde{\alpha}|)|},$$

$$S_N = \frac{1-\cos^2\phi\sin^2(2|\tilde{\alpha}|)-2|\sin\phi\sin(2|\tilde{\alpha}|)|}{2|\sin\phi\sin(2|\tilde{\alpha}|)|}. \tag{54}$$

由这两式可知,当$|\sin(2[\pi]|\alpha|)|<1/2$ 时,$\cos\hat{\Phi}_\theta$ 下落,(即 $S_\Phi<0$),$\hat{N}$上涨(即 $S_N>0$),当$|\sin(2[\pi]|\alpha|)|>1/2$ 时,$\hat{\Phi}_\theta$ 上涨,$\hat{N}$下落.如果$|\sin2[\pi]|\alpha||\to0$(即 $\alpha\to0$),$S_{\cos\Phi}\to-1$,$S_N\to+\infty$,$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\cos\hat{\Phi}_\theta)^2\rangle\to0$,达到其最小值,得到最小不确定态与(46)式相同.

如果 $\sin(2|\tilde{\alpha}|)=1$(即$|\alpha|=1/\sqrt{[\pi]}$),$\hat{N}$的压缩性质与(2)节相似.

6 结　论

本文阐明了 q 形变谐振子在有限维希耳伯特空间中相量方面的一些性质.总结了 q 形变算符的基本性质,由相量的 q 形变相干态的定义出发,研究其基本性质,如超完备性、位相(粒子数)分布函数、不同表象间基矢的变换公式等,并讨论了它的二阶压缩性质.作为特例,详细讨论了二能级系统中粒子数-位相压缩及其不确定关系,得到一些新的最小不确定态.

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q-PHASE-COHERENT STATES AND THEIR SQUEEZING PROPERTIES

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ABSTRACT

Squeezing properties of q-phase-coherent states with respect to the phase quadrature operators are studied. The relations between number-phase squeezing and number-phase uncertainty are also studied in detail for a two-state system. Some new number-phase minimum uncertainty states are found.

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