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与不变量有关的幺正变换方法和含时均匀电场中的量子 Dirac 场的演化*

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在推广的不变量理论的基础上,运用与不变量有关的幺正变换方法研究了含时均匀电场下的量子 Dirac 场的演化,求解了 Dirac 场的泛函 Schrödinger 方程,得到了方程的精确解以及对应的总相位,总相位包括动力学相和几何相(Aharonov-Anandan phase).

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1 引 言

量子不变量理论特别适合于含时系统的研究,它首先由 Lewis 和 Riesenfeld 提出[1],随后我们推广了这一理论[2],提出了基本不变量的思想并运用它研究了一些含时系统的 Schrödinger 方程的精确解及其对应的几何相[3,4].越来越多的人认识到,尽管总相位中的各个部分有着不同的名称和作用,但对于含时系统的 Schrödinger 方程的精确解来说重要的是总相位[5].由于用来求解含时系统精确解的不变量理论[1,2]本身与研究系统的总相位(包括动力学相和几何相)有着密切的联系,因此有时不变量表述也称作相表述.进一步,基本不变量概念的引入[2]使人们有可能找到某些无穷维含时量子系统——含时量子场[6—8]的对易不变量的完全集和广义的产生湮没算符.正是利用了相表述和新发展的与不变量有关的幺正变换方法,得到了含时均匀电场下带荷的 Klein-Gordon 场的精确解及对应的总相位[7],求得了含时均匀电场中的量子 Dirac 场的精确解以及对应的总相位(包括动力学相和几何相).

附录中列出了我们得到的辅助方程,这些辅助方程远比文献[7]中得到的复杂,只能用数值方法进行求解.

2 相表述与不变量有关的幺正变换方法

简单介绍一下 Lewis-Riesenfeld 不变量理论,相表述及与不变量有关的幺正变换方法.考虑一个一维含时系统,其哈密顿量为 $\hat{H}(t)$.系统的厄米不变量 $\hat{I}(t)$满足($\hbar=c=$

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1)

$$\frac{\partial \hat{I}(t)}{\partial t} - \mathrm{i}[\hat{I}(t), \hat{H}(t)] = 0, \tag{1}$$

$\hat{I}(t)$的本征值方程为

$$\begin{aligned} \hat{I}(t)|\lambda_n, t\rangle &= \lambda_n|\lambda_n, t\rangle, \\ \frac{\partial \lambda_n}{\partial t} &= 0. \end{aligned} \tag{2}$$

系统的含时 Schrödinger 方程

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}(t)|\psi(t)\rangle_S \tag{3}$$

的特解$|\lambda_n, t\rangle_S$与不变量$\hat{I}(t)$的本征态$|\lambda_n, t\rangle$相差一个相因子$\exp[\mathrm{i}\varphi_n(t)]$,即

$$|\lambda_n, t\rangle_S = \exp[\mathrm{i}\varphi_n(t)]|\lambda_n, t\rangle, \tag{4}$$

于是方程(3)的通解可写为

$$\begin{aligned} |\psi(t)\rangle_S &= \sum_n C_n \exp[\mathrm{i}\varphi_n(t)]|\lambda_n, t\rangle, \\ \varphi_n(t) &= \int_0^t \langle\lambda_n, t'|\hat{H}(t')|\lambda_n, t'\rangle \mathrm{d}t', \\ C_n &= \langle\lambda_n, 0|\psi(0)\rangle_S. \end{aligned} \tag{5}$$

这里$|\lambda_n, t\rangle_S$, $(n=1,2,\cdots)$构成方程(3)的完全集,值得注意的是,不变量的选择不是唯一的并且随着不变量选择的不同完全集的形式也不同.以上是 Lewis-Riesenfeld 不变量理论的主要内容.

在 Lewis-Riesenfeld 不变量理论的基础上,我们提出了基本不变量的思想[2]:① 方程(1)的形式解为 $\hat{I}(t) = \hat{U}(t)\hat{I}(0)\hat{U}^+(t)$, 这里 $\hat{U}(t) = \mathrm{P}\exp\{-\mathrm{i}\hbar^{-1}\int_0^t \hat{H}(t')\mathrm{d}t'\}$是系统的演化算符,并且 $\hat{I}(0)$的选择是任意,因此可以是厄米的也可以是非厄米的;②有两个基本的不变量 $\hat{x}(t) = \hat{U}(t)\hat{x}\hat{U}^+(t)$, $\hat{p}(t) = \hat{U}(t)\hat{p}\hat{U}^+(t)$,若 $\hat{I}(0)$能表示为 $\hat{x}$, $\hat{p}$的幂级数的形式,则任意的不变量 $\hat{I}(t) = \hat{U}(t)\hat{I}(0)\hat{U}^+(t)$都能表示为基本不变量 $\hat{x}(t)$, $\hat{p}(t)$的幂级数的形式;③ 在某些情形,可以选择某一非厄米不变量作为解的生成算符,利用它可以由含时 Schrödinger 方程(3)的一个解生成出解的完全集;④ 利用这一概念,可以找到不变量的完全集建立不变量表述——相表述,从而用来研究高于一维的含时系统(包括无穷维量子系统——量子场论系统[6—8]).

在基本不变量概念的基础上,进一步发展了与不变量有关的幺正变换方法.在许多的有物理意义的问题中,对于某一选定的不变量 $\hat{I}(t)$可以构造一个含时的幺正变换 $\hat{V}(t)$使得 ① $\hat{I}_0 = \hat{V}^+(t)\hat{I}(t)\hat{V}(t)$为不含时的算符,且

$$\begin{aligned} \hat{I}_0|\lambda_n\rangle &= \lambda_n|\lambda_n\rangle, \\ |\lambda_n\rangle &= \hat{V}^{-1}|\lambda_n, t\rangle. \end{aligned} \tag{6}$$

其中 λ_n 为方程(2)中的本征值. ② 同时哈密顿量 $\hat{H}(t)$变为 $\hat{H}_0(t)$, 即

$$\hat{H}_0(t) = \hat{V}^+(t)\hat{H}(t)\hat{V}(t) - \mathrm{i}\hat{V}^+(t)\frac{\partial\hat{V}(t)}{\partial t}. \tag{7}$$

很容易证明此幺正变换保证与 $\hat{H}_0(t)$对应的含时 Schrödinger 方程

$$\mathrm{i}\frac{\partial}{\partial t}|\lambda_n, t\rangle_{S_0} = \hat{H}_0(t)|\lambda_n, t\rangle_{S_0} \tag{8}$$

的特解 $|\lambda_n, t\rangle_{S_0}$ 与 $\hat{I}_0$ 的本征态 $|\lambda_n\rangle$ 之间仅仅相差一个相因子 $\exp[\mathrm{i}\varphi_n(t)]$, 即

$$|\lambda_n, t\rangle_{S_0} = \exp[\mathrm{i}\varphi_n(t)]|\lambda_n\rangle, \tag{9}$$

值得强调的是此相因子与(4)式中的完全相同. 将(9)式代入(8)式得到

$$-\dot{\varphi}_n(t)|\lambda_n\rangle = \hat{H}_0(t)|\lambda_n\rangle. \tag{10}$$

这意味着 $\hat{H}_0(t)$与不变量 $\hat{I}_0$ 之间仅仅相差一个含时的 c 数. 因此只要能找到幺正变换 $\hat{V}(t)$, 那么含时的 Schrödinger 方程(3)的求解就能转化为较简单的方程(8)的求解. 在解出方程(8)后, 即可得系统含时 Schrödinger 方程(3)的一般解.

值得指出的是① 不变量 $\hat{I}(t)$的选择不是唯一的, 通常可以选择系统的初始态作为 $\hat{I}(0)$的本征态($\hat{I}(0)$可以是 $\hat{H}(t=0)$); ② 对于一选定的不变量 $\hat{I}(t)$, 方程(3)解的完全集是确定的, 但使得 $\hat{I}_0 = \hat{V}(t)\hat{I}(t)\hat{V}^+(t)$不含时的幺正变换 $\hat{V}(t)$并不唯一, 可以相差一个不含时的幺正变换.

3　运用与不变量有关的幺正变换方法研究含时均匀电场中的量子 Dirac 场的演化

考虑含时均匀电场作用下的 Dirac 场, 其 Lagrangian 密度为

$$L(x) = \bar{\psi}(x)[\mathrm{i}\not{\nabla} - e\not{A} - m]\psi(x). \tag{11}$$

其中

$$\bar{\psi}(x) = \psi^+(x)\gamma_0, \quad \not{\nabla} = \gamma^\mu\partial_\mu,$$

$$\not{A} = \gamma^\mu A_\mu, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

这里的 σ^i 为 Pauli 矩阵. 这样, 在 Weyl 规范下 $A^0 = 0$, 利用方程(11)能得到 ψ_α 的正则动量密度 π_α 为

$$\pi_\alpha = \frac{\partial L}{\partial\dot{\psi}_\alpha} = \mathrm{i}\psi_\alpha^+, \tag{12}$$

采用正则量子化方法, $\hat{\psi}$与 $\hat{\pi}$成为一对正则共厄算符, 它们满足等时反对易关系

$$\begin{aligned} &[\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{x}', t)]_+ = \mathrm{i}\delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \\ &[\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{\psi}_\beta(\boldsymbol{x}', t)]_+ = 0, \ [\hat{\pi}_\beta(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{x}', t)]_+ = 0. \end{aligned} \tag{13}$$

我们选择在泛函 Schrödinger 图像中研究[6—8,10,11]. 从方程(11)和(12),可以得到 Schrödinger 图像中的含时量子哈密顿量

$$\begin{aligned}H(t) &= \int \hat{\pi}(x,0)\left\{-\alpha_i\left[\frac{\partial}{\partial x^i} - eA^i(t)\right] - \mathrm{i}\beta m\right\}\hat{\psi}(x,0)\mathrm{d}^3 x \\ &= \int \hat{\psi}^+(x,0)\left\{-\mathrm{i}\alpha_i\left[\frac{\partial}{\partial x^i} - eA^i(t)\right] + \beta m\right\}\hat{\psi}(x,0)\mathrm{d}^3 x.\end{aligned} \tag{14}$$

其中

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \beta = \gamma_0.$$

由于所考虑的电场空间部分是均匀的,可以在"动量表象"将 $\hat{\psi}(x,0)$ 与 $\hat{\psi}^+(x,0)$ 展开

$$\begin{aligned}\hat{\psi}(x,0) &= \sum_{\pm s}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sqrt{\frac{m}{E_p}}\left[\hat{b}_s(p)u_s(p)\mathrm{e}^{\mathrm{i}p\cdot x} + \hat{d}_s^+(p)\nu_s(p)\mathrm{e}^{-\mathrm{i}p\cdot x}\right], \\ \hat{\psi}^+(x,0) &= \sum_{\pm s}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sqrt{\frac{m}{E_p}}\left[\hat{b}_s^+(p)\bar{u}_s(p)\gamma_0\mathrm{e}^{\mathrm{i}p\cdot x} + \hat{d}_s(p)\bar{\nu}_s(p)\gamma_0\mathrm{e}^{-\mathrm{i}p\cdot x}\right].\end{aligned} \tag{15}$$

其中 $E_p = \sqrt{|p|^2 + m^2}$, $u_s(p)$, $\bar{u}_s(p)$, $\nu_s(p)$, $\bar{\nu}_s(p)$ 的定义见文献[9]. 算符 $\hat{b}_s^+(p)$ $(\hat{b}_s(p))$ 是具有动量为 p 自旋为 s 的"电子"的产生(湮没)算符,算符 $\hat{d}_s^+(p)(\hat{d}_s(p))$ 是具有动量为 p 自旋为 s 的"正电子"的产生(湮没)算符,由方程(13)得到产生湮没算符满足的反对易关系为

$$\begin{aligned}&[\hat{b}_s(p), \hat{b}_{s'}^+(p')]_+ = \delta_{ss'}\delta(p - p'), [\hat{d}_s(p), \hat{d}_{s'}^+(p')]_+ = \delta_{ss'}\delta(p - p'), \\ &[\hat{b}_s(p), \hat{b}_{s'}(p')]_+ = [\hat{b}_s^+(p), \hat{b}_{s'}^+(p')]_+ = [\hat{d}_s(p), \hat{d}_{s'}(p')]_+ = [\hat{d}_s^+(p), \hat{d}_{s'}^+(p')]_+ = 0, \\ &[\hat{b}_s(p), \hat{d}_{s'}(p')]_+ = [\hat{b}_s^+(p), \hat{d}_{s'}^+(p')]_+ = [\hat{b}_s(p), \hat{d}_{s'}^+(p')]_+ = [\hat{b}_s^+(p), \hat{d}_{s'}(p')]_+ = 0.\end{aligned} \tag{16}$$

将方程(15)代入方程(14)得到

$$\begin{aligned}H(t) = \sum_{\pm s}\int \mathrm{d}^3 p\Big\{&\sum_{\pm s}\left[E(t)b_s^+(p)b_s(p) - E(t)d_s(p)d_s^+(p)\right] + \lambda_1(t)b_{+s}^{\ +}(p)d_{+s}^{\ +}(p) \\ &+ \lambda_2(t)b_{-s}^{\ +}(p)d_{-s}^{\ +}(p) + \lambda_3(t)b_{+s}^{\ +}(p)d_{-s}^{\ +}(p) + \lambda_4(t)b_{-s}^{\ +}(p)d_{+s}^{\ +}(p) \\ &+ \lambda_1^*(t)d_{+s}(p)b_{+s}(p) + \lambda_2^*(t)d_{-s}(p)b_{-s}(p) + \lambda_3^*(t)d_{-s}(p)b_{+s}(p) \\ &+ \lambda_4^*(t)d_{+s}(p)b_{-s}(p)\Big\}.\end{aligned} \tag{17}$$

其中

$$E(t) = E_p - \frac{e}{E_p}\left[p_x A_x(t) + p_y A_y(t) + p_z A_z(t)\right],$$

$$\lambda_1(t) = eA_z(t) + \frac{e}{(E_p + m)E_p}\left[p_x p_z A_x(t) + p_y p_z A_y(t) - (p_x^2 + p_y^2)A_z(t)\right],$$

$$\lambda_3(t) = eA_x(t) - \mathrm{i}eA_y(t) + \frac{e}{(E_p + m)E_p}\left[p_x p_y A_y(t) + p_z p_x A_z(t) - (p_z^2 + p_y^2)A_x(t)\right]$$

$$- \mathrm{i} \frac{e}{(E_p + m) E_p} [p_x p_y A_x(t) + p_z p_y A_z(t) - (p_x^2 + p_z^2) A_y(t)], \tag{18}$$

$\lambda_2(t) = -\lambda_1(t),\ \lambda_4(t) = \lambda_3^*(t).$

这样得到描述量子 Dirac 场运动的泛函 Schrödinger 方程

$$\mathrm{i} \frac{\partial}{\partial t} \Psi[\psi(\boldsymbol{p}); t] = \hat{H}(t) \Psi[\psi(\boldsymbol{p}); t]. \tag{19}$$

根据不变量理论,可以找到一个满足

$$\frac{\partial \hat{I}(t)}{\partial t} - \mathrm{i}[\hat{I}(t), \hat{H}(t)] = 0 \tag{20}$$

的不变量 $\hat{I}(t)$:

$$\hat{I}(t) = \sum_{\pm s} \int \mathrm{d}^3 \boldsymbol{p} [\hat{B}_s^+(\boldsymbol{p}, t) \hat{B}_s(\boldsymbol{p}, t) - \hat{D}_s(\boldsymbol{p}, t) \hat{D}_s^+(\boldsymbol{p}, t)]. \tag{21}$$

式中

$$\begin{aligned}
\hat{B}_{+s}(\boldsymbol{p}, t) = & (\cos\theta_1 \cos\theta_2 \cos\theta_6 + \sin\theta_1 \sin\theta_3 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3+\phi_6)}) \hat{b}_{+s}(\boldsymbol{p}) \\
& + (\cos\theta_1 \sin\theta_2 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_2+\phi_5)} - \sin\theta_1 \cos\theta_3 \cos\theta_5 \mathrm{e}^{-\mathrm{i}\phi_1}) \hat{b}_{-s}(\boldsymbol{p}) \\
& + (\cos\theta_1 \sin\theta_2 \cos\theta_5 \mathrm{e}^{\mathrm{i}\phi_2} + \sin\theta_1 \cos\theta_3 \sin\theta_5 \mathrm{e}^{-\mathrm{i}(\phi_1+\phi_5)}) \hat{d}_{+s}^+(\boldsymbol{p}) \\
& - (\cos\theta_1 \cos\theta_2 \sin\theta_6 \mathrm{e}^{-\mathrm{i}\phi_6} - \sin\theta_1 \sin\theta_3 \cos\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3)}) \hat{d}_{-s}^+(\boldsymbol{p}),
\end{aligned} \tag{22a}$$

$$\begin{aligned}
\hat{B}_{-s}(\boldsymbol{p}, t) = & (-\cos\theta_1 \sin\theta_3 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_3-\phi_6)} + \sin\theta_1 \cos\theta_3 \cos\theta_6 \mathrm{e}^{\mathrm{i}\phi_1}) \hat{b}_{+s}(\boldsymbol{p}) \\
& + (\cos\theta_1 \cos\theta_3 \cos\theta_5 + \sin\theta_1 \sin\theta_3 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_1+\phi_3+\phi_5)}) \hat{b}_{-s}(\boldsymbol{p}) \\
& + (-\cos\theta_1 \cos\theta_3 \sin\theta_5 \mathrm{e}^{-\mathrm{i}\phi_5} + \sin\theta_1 \sin\theta_2 \cos\theta_5 \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)}) \hat{d}_{+s}^+(\boldsymbol{p}) \\
& - (\cos\theta_1 \sin\theta_3 \cos\theta_6 \mathrm{e}^{-\mathrm{i}\phi_3} + \sin\theta_1 \cos\theta_2 \sin\theta_6 \mathrm{e}^{\mathrm{i}(\phi_1-\phi_6)}) \hat{d}_{-s}^+(\boldsymbol{p}),
\end{aligned} \tag{22b}$$

$$\begin{aligned}
\hat{D}_{-s}^+(\boldsymbol{p}, t) = & (\cos\theta_4 \cos\theta_3 \sin\theta_6 \mathrm{e}^{\mathrm{i}\phi_6} + \sin\theta_4 \sin\theta_2 \cos\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_4)}) \hat{b}_{+s}(\boldsymbol{p}) \\
& + (\cos\theta_4 \cos\theta_3 \cos\theta_5 \mathrm{e}^{\mathrm{i}\phi_3} - \sin\theta_4 \cos\theta_2 \sin\theta_5 \mathrm{e}^{-\mathrm{i}(\phi_4-\phi_5)}) \hat{b}_{-s}(\boldsymbol{p}) \\
& - (\cos\theta_4 \sin\theta_3 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_3-\phi_5)} + \sin\theta_4 \cos\theta_2 \cos\theta_5 \mathrm{e}^{-\mathrm{i}\phi_4}) \hat{d}_{+s}^+(\boldsymbol{p}) \\
& + (\cos\theta_4 \cos\theta_3 \cos\theta_6 - \sin\theta_4 \sin\theta_2 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_6+\phi_4)}) \hat{d}_{-s}^+(\boldsymbol{p}),
\end{aligned} \tag{22c}$$

$$\begin{aligned}
\hat{D}_{+s}^+(\boldsymbol{p}, t) = & (-\cos\theta_4 \sin\theta_2 \cos\theta_6 \mathrm{e}^{-\mathrm{i}\phi_2} + \sin\theta_4 \cos\theta_3 \sin\theta_6 \mathrm{e}^{\mathrm{i}(\phi_4+\phi_6)}) \hat{b}_{+s}(\boldsymbol{p}) \\
& + (\cos\theta_4 \cos\theta_2 \sin\theta_5 \mathrm{e}^{\mathrm{i}\phi_5} + \sin\theta_4 \sin\theta_3 \cos\theta_5 \mathrm{e}^{\mathrm{i}(\phi_4+\phi_3)}) \hat{b}_{-s}(\boldsymbol{p}) \\
& + (\cos\theta_4 \cos\theta_2 \cos\theta_5 - \sin\theta_4 \sin\theta_3 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_4+\phi_3-\phi_5)}) \hat{d}_{+s}^+(\boldsymbol{p}) \\
& + (\cos\theta_4 \sin\theta_2 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_6)} + \sin\theta_4 \cos\theta_3 \cos\theta_6 \mathrm{e}^{\mathrm{i}\phi_4}) \hat{d}_{-s}^+(-\boldsymbol{p}).
\end{aligned} \tag{22d}$$

其中 $\theta_m, \phi_m, (m=1,2,\cdots,6)$ 满足辅助方程(见附录). 值得注意的是, 这里的 $\hat{B}_s(\boldsymbol{p}, t)$, $\hat{B}_s^+(\boldsymbol{p}, t)$, $\hat{D}_s^+(\boldsymbol{p}, t)$, $\hat{D}_s(\boldsymbol{p}, t)$ 既是不变量又可以认为是广义的产生、湮没算符. 它们满足等时反对易关系

$$[\hat{B}_s(\boldsymbol{p},t),\hat{B}_{s'}^{+}(\boldsymbol{p}',t]_{+}=\delta_{ss'}\delta(\boldsymbol{p}-\boldsymbol{p}'),[\hat{D}_s(\boldsymbol{p},t),\hat{D}_{s'}^{+}(\boldsymbol{p}',t)]_{+}=\delta_{ss'}\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{23}$$

根据与不变量有关的幺正变换方法,首先构造幺正变换

$$\begin{aligned}
&\hat{V}(t)=\hat{V}_3(t)\hat{V}_2(t)\hat{V}_1(t),\\
&\hat{V}_1(t)=\exp\int d^3\boldsymbol{p}[(-\theta_1e^{-i\phi_1}\hat{b}_{+s}^{+}\hat{b}_{-s}+\theta_1e^{i\phi_1}\hat{b}_{-s}^{+}\hat{b}_{+s})+(-\theta_4e^{-i\phi_4}\hat{d}_{+s}\hat{d}_{-s}^{+}+\theta_4e^{i\phi_4}\hat{d}_{-s}\hat{d}_{+s}^{+})],\\
&\hat{V}_2(t)=\exp\int d^3\boldsymbol{p}[(-\theta_2e^{-i\phi_2}\hat{b}_{+s}^{+}\hat{d}_{+s}^{+}+\theta_2e^{i\phi_2}\hat{d}_{+s}\hat{b}_{+s})+(-\theta_3e^{-i\phi_3}\hat{b}_{-s}^{+}\hat{d}_{-s}^{+}+\theta_3e^{i\phi_3}\hat{d}_{-s}\hat{b}_{-s})],\\
&\hat{V}_3(t)=\exp\int d^3\boldsymbol{p}[(-\theta_5e^{-i\phi_5}\hat{b}_{-s}^{+}\hat{d}_{+s}^{+}+\theta_5e^{i\phi_5}\hat{d}_{+s}\hat{b}_{-s})+(-\theta_6e^{-i\phi_6}\hat{b}_{+s}^{+}\hat{d}_{-s}^{+}+\theta_6e^{i\phi_6}\hat{d}_{-s}\hat{b}_{+s})].
\end{aligned} \tag{24}$$

这里,为了简化,将 $\hat{b}_s$, $\hat{b}_s^{+}$, $\hat{d}_s$, $\hat{d}_s^{+}$ 的参量 $\boldsymbol{p}$ 省去.很容易得出幺正变换 $\hat{V}(t)$,它将不变量 $\hat{I}(t)$变换为不含时的 $\hat{I}_V$:

$$\hat{I}_V=\hat{V}^{+}(t)\hat{I}(t)\hat{V}(t)=\sum_{\pm S}\int d^3\boldsymbol{p}[\hat{b}_s^{+}(\boldsymbol{p})\hat{b}_s(\boldsymbol{p})+\hat{d}_s(\boldsymbol{p})\hat{d}_s^{+}(\boldsymbol{p})]. \tag{25}$$

利用方程(24)和 Backer-Campbell-Hausdorff 公式[12],可以由 $\hat{H}(t)$得到 $\hat{H}_0(t)$:

$$\hat{H}_0(t)=\hat{V}^{+}(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^{+}(t)\frac{\partial\hat{V}(t)}{\partial t}=\int d^3\boldsymbol{p}\hat{H}_0(\boldsymbol{p},t). \tag{26}$$

先计算(26)式的第一项

$$\hat{V}^{+}(t)\hat{H}(t)\hat{V}(t)=\sum_{\pm s}\int d^3\boldsymbol{p}[\alpha_s^{(d)}(p,t)\hat{b}_s^{+}(\boldsymbol{p})\hat{b}_s(\boldsymbol{p})+\beta_s^{(d)}(p,t)\hat{d}_s(\boldsymbol{p})\hat{d}_s^{+}(\boldsymbol{p})]. \tag{27}$$

这里

$$\begin{aligned}
\alpha_{+s}^{(d)}(p,t)&=\sin^2\theta_6\chi_4^{(d)}(p,t)+\cos^2\theta_6\chi_1^{(d)}(p,t)-\sin2\theta_6[\chi_6^{(d)}(p,t)\sin\phi_6\\
&\quad+\chi_5^{(d)}(p,t)\cos\phi_6],\\
\alpha_{-s}^{(d)}(p,t)&=\sin^2\theta_5\chi_3^{(d)}(p,t)+\cos^2\theta_5\chi_2^{(d)}(p,t)-\sin2\theta_5[\chi_8^{(d)}(p,t)\sin\phi_5\\
&\quad+\chi_7^{(d)}(p,t)\cos\phi_5],\\
\beta_{+s}^{(d)}(p,t)&=\sin^2\theta_5\chi_2^{(d)}(p,t)+\cos^2\theta_5\chi_3^{(d)}(p,t)+\sin2\theta_5[\chi_8^{(d)}(p,t)\sin\phi_5\\
&\quad+\chi_7^{(d)}(p,t)\cos\phi_5],\\
\beta_{-s}^{(d)}(p,t)&=\sin^2\theta_6\chi_4^{(d)}(p,t)+\cos^2\theta_6\chi_1^{(d)}(p,t)+\sin2\theta_6[\chi_6^{(d)}(p,t)\sin\phi_6\\
&\quad+\chi_5^{(d)}(p,t)\cos\phi_6].
\end{aligned} \tag{28}$$

其中 $\chi_m^{(d)}$,($m=1,2,\cdots,8$)见附录.(26)式中的第二项运算更为复杂,利用辅助方程及 Backer-Campbell-Hausdorff 公式[12],经过繁复计算可得

$$-i\hat{V}^{+}(t)\frac{\partial\hat{V}(t)}{\partial t}=\sum_{\pm s}\int d^3\boldsymbol{p}[\alpha_s^{(g)}(p,t)\hat{b}_s^{+}(\boldsymbol{p})\hat{b}_s(\boldsymbol{p})+\beta_s^{(g)}(p,t)\hat{d}_s(\boldsymbol{p})\hat{d}_s^{+}(\boldsymbol{p})]. \tag{29}$$

这里

$$\begin{aligned}
\alpha_{+s}^{(g)}(p,t) &= \sin^2\theta_6\,\chi_4^{(g)}(p,t)+\cos^2\theta_6\,\chi_1^{(g)}(p,t)-\sin 2\theta_6[\chi_6^{(g)}(p,t)\sin\phi_6 \\
&\quad +\chi_5^{(d)}(p,t)\cos\phi_6]-\dot{\phi}_6\sin\theta_6, \\
\alpha_{-s}^{(g)}(p,t) &= \sin^2\theta_5\,\chi_3^{(g)}(p,t)+\cos^2\theta_5\,\chi_2^{(g)}(p,t)-\sin 2\theta_5[\chi_8^{(g)}(p,t)\sin\phi_5 \\
&\quad +\chi_7^{(d)}(p,t)\cos\phi_5]-\dot{\phi}_5\sin\theta_5, \\
\beta_{+s}^{(g)}(p,t) &= \sin^2\theta_5\,\chi_2^{(g)}(p,t)+\cos^2\theta_5\,\chi_3^{(g)}(p,t)+\sin 2\theta_5[\chi_8^{(g)}(p,t)\sin\phi_5 \\
&\quad +\chi_7^{(d)}(p,t)\cos\phi_5]-\dot{\phi}_5\sin\theta_5, \\
\beta_{-s}^{(g)}(p,t) &= \sin^2\theta_6\,\chi_4^{(g)}(p,t)+\cos^2\theta_6\,\chi_1^{(g)}(p,t)+\sin 2\theta_6[\chi_6^{(g)}(p,t)\sin\phi_6 \\
&\quad +\chi_5^{(d)}(p,t)\cos\phi_6]-\dot{\phi}_6\sin\theta_6.
\end{aligned} \tag{30}$$

其中 $\chi_m^{(d)}$,($m=1,2,\cdots,8$)见附录. 从方程(27,29)可以看出, 不变量 $\hat{I}_V$ 是不含时的, 而 $\hat{H}_0(t)$与 $\hat{I}_V$ 之间只相差含时的 c-数. $\hat{I}_V$ 不含时, 在间断的表述中, $\hat{I}_V$ 可看作动量 $\boldsymbol{p}$ 空间无穷多个频率为 1 的费米振子的哈密顿量之和. 对应着不同动量模 $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, …的费米振子的能量本征值分别由整数 n_1, n_2, …(n_1, n_2, …$=0,1$)来表示. 我们用$|0\rangle$表示 $\hat{I}_V(\boldsymbol{p}_m)$的基态, 它满足

$$\begin{aligned}
\hat{b}_s(\boldsymbol{p}_m)|0\rangle &\equiv \hat{b}_{ms}|0\rangle = 0, \\
\hat{d}_s(\boldsymbol{p}_m)|0\rangle &\equiv \hat{d}_{ms}|0\rangle = 0.
\end{aligned} \tag{31}$$

利用基态$|0\rangle$和升算符 $\hat{b}_s^+(\boldsymbol{p}_m)\equiv\hat{b}_{ms}^+$, $\hat{d}_s^+(\boldsymbol{p}_m)\equiv\hat{d}_{ms}^+$, 可以得到 $\hat{I}_V$ 的 N 粒子激发态(定义粒子数算符 $\hat{n}_{mbs}=\hat{b}_{ms}^+\hat{b}_{ms}$, $\hat{n}_{mds}=\hat{d}_{ms}^+\hat{d}_{ms}^+$, $\hat{N}_{bs}=\sum_m\hat{n}_{mbs}$, $\hat{N}_{ds}=\sum_m\hat{n}_{mds}$):

$$\begin{aligned}
|N_{bs}\rangle &= |n_{1bs}, n_{2bs}, \cdots(n_{1bs}+n_{2bs}+\cdots=N_{bs})\rangle \\
&= \prod_m[b_s^+(p_m)]^{n_{mbs}}|0\rangle_{bs}, \\
|N_{ds}\rangle &= |n_{1ds}, n_{2ds}, \cdots(n_{1ds}+n_{2ds}+\cdots=N_{ds})\rangle \\
&= \prod_m[d_s^+(p_m)]^{n_{mds}}|0\rangle_{ds} \qquad (n_{mbs}, n_{mds}=0,1).
\end{aligned} \tag{32}$$

它们满足

$$\hat{I}_V|N_{bs}, N_{ds}\rangle_{I_V} = (N_{b+s}+N_{b-s}+N_{d+s}+N_{d-s})|N_{bs}, N_{ds}\rangle_{I_V}. \tag{33}$$

这里为了方便, 定义$|N_{bs}, N_{ds}\rangle_{I_V}\equiv|N_{b+s}\rangle|N_{b-s}\rangle|N_{d+s}\rangle|N_{d-s}\rangle$. 根据与不变量有关的幺正变换方法[2,8], 从 $\hat{I}_V$ 的本征态很容易得到与 $\hat{H}_0(t)$对应的含时 Schrödinger 方程的解 $|N_{bs}, N_{ds}, t\rangle_{S_0}$:

$$|N_{bs}, N_{ds}, t\rangle_{S_0} = \exp\Big[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\Big]\exp\Big[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\Big]|N_{bs}, N_{ds}\rangle_{I_V}, \tag{34}$$

$$\vartheta_{bs}(t) = -\int_0^t\langle N_{bs}|\hat{H}_0(t')|N_{bs}\rangle\,\mathrm{d}t'$$

$$= -\int_0^t \left[\langle N_{bs}|\ \hat{V}^+(t')\hat{H}(t')\hat{V}(t')|N_{bs}\rangle - \mathrm{i}\langle N_{bs}|\ \mathbf{V}^+(t')\frac{\partial \hat{V}(t')}{\partial t}|N_{bs}\rangle\right]\mathrm{d}t'$$

$$= -\sum_m n_{mbs}\int_0^t [\alpha_s^{(d)}(p_m, t') + \alpha_s^{(g)}(p_m, t')]\mathrm{d}t',$$

$$\vartheta_{ds}(t) = -\int_0^t \langle N_{ds}|\ \hat{H}_0(t')|N_{ds}\rangle \mathrm{d}t'$$

$$= -\int_0^t \left[\langle N_{ds}|\ \hat{V}^+(t')\hat{H}(t')\hat{V}(t')|N_{ds}\rangle - \mathrm{i}\langle N_{ds}|\ \hat{V}^+(t')\frac{\partial \hat{U}(t')}{\partial t}|N_{ds}\rangle\right]\mathrm{d}t'$$

$$= -\sum_m n_{mds}\int_0^t [\beta_s^{(d)}(p_m, t') + \beta_s^{(g)}(p_m, t')]\mathrm{d}t'.$$

这里 $\vartheta_{bs}(t)$, $\vartheta_{ds}(t)$是总的相位,包括动力学相和几何相.同样根据与不变量有关的幺正变换方法[2,8],可以得到与 $\hat{H}(t)$对应的含时 Schrödinger 方程(19)的特解

$$
\begin{aligned}
|\Psi_{N_{bs}N_{ds}}(t)\rangle_S &= \hat{V}(t)|N_{bs}, N_{ds}; t\rangle_{S_0} \\
&= \exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\right]\hat{V}(t)|N_{bs}, N_{ds}\rangle_{I_V} \\
&= \exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\right]|N_{bs}, N_{ds}; t\rangle_I.
\end{aligned}
\tag{35}
$$

其中$|N_{bs}, N_{ds}; t\rangle_I$ 为不变量 $\hat{I}(t)$的本征态.含时 Schrödinger 方程(19)的一般解是以上特解的叠加

$$
\begin{aligned}
|\Psi(t)\rangle_S &= \sum_{N_{bs}N_{ds}} C_{N_{bs}N_{ds}}\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\right]|N_{bs}, N_{ds}; t\rangle_I, \\
C_{N_{bs}N_{ds}} &= {}_I\langle N_{bs}, N_{ds}; 0|\Psi(0)\rangle_S,
\end{aligned}
\tag{36}
$$

其中初始态$|\Psi(0)\rangle$是可以任意选择的.

值得指出的是本文中的辅助方程远比含时标量场[7]中的辅助方程复杂,这些辅助方

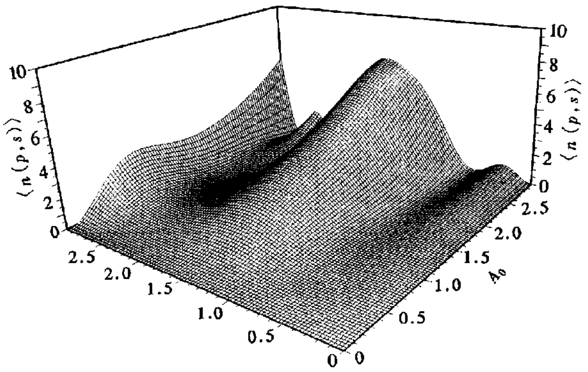

图1 粒子数算符 $\hat{n}(\mathbf{p}, s) \equiv \hat{n}_{b+s} + \hat{n}_{b-s} + \hat{n}_{d+s} + \hat{n}_{d-s}$关于态$|\Psi_{0,0}(t)\rangle$的期待值$\langle n(\mathbf{p}, s)\rangle$随时间和 A_0 变化 含时的电磁势的形式为 $\mathbf{A}(t) = \mathbf{A}_0\cos t$, 其中 $\mathbf{A}_0 = (A_0, 0, 0)$且 A_0 不含时;粒子质量 $m=1$, 粒子动量 $p_x = p_y = p_z = 1$

程只能运用数值方法进行求解. 作为一个示例, 我们选择了系统的初始态($t=0$)在基态以及作用在 Dirac 场上的含时电磁势的形式为 $\mathbf{A}(t)=\mathbf{A}_0\cos t$($\mathbf{A}_0=(A_0,0,0)$且 A_0 不含时)的情形, 计算了粒子数算符 $\hat{n}(\boldsymbol{p},s)\equiv\hat{n}_{b+s}+\hat{n}_{b-s}+\hat{n}_{d+s}+\hat{n}_{d-s}$关于态$|\Psi_{0,0}(t)\rangle_S$(当 $t=0$ 时该态为基态)的期待值随时间和 A_0 的变化, 这一计算结果显示在图 1 中.

4　讨　　论

(1) 可以看出, (15)式中的 $\hat{b}_s^+(\boldsymbol{p})$, $\hat{b}_s(\boldsymbol{p})$, $\hat{d}_s^+(\boldsymbol{p})$和 $\hat{d}_s(\boldsymbol{p})$与(17)式中的哈密顿量 $\hat{H}(t)$构成一个准代数[13]. 正是这一代数才得以找到(24)式中的幺正变换 $\hat{V}(t)$. 这意味着相表述和与不变量有关的幺正变换方法只能用来研究一些具有准代数的特殊的含时量子系统, 本文研究的 Dirac 场就属于这种特殊的系统. 尽管它很特殊, 但研究它还是有意义的, 因为据我们所知, 还没有人得到过具有含时哈密顿量的量子 Dirac 场的泛函 Schrödinger 方程的精确解. 此外, 这一精确解在相应的含时微扰量子场论中作为微扰展开的起点是有用的, 这方面的工作在标量场中已经作了研究[8].

(2) 值得注意的是, 甚至当选择 $\hat{H}(t=0)$为 $\hat{I}_V$ 时, "基态"解也仅仅是当 $t=0$ 时的 $\hat{H}(t)$的最低本征态, 因此, 一般来说这里"基态"一词的含义没有通常不含时系统中基态所具有的意义.

(3) 有趣的是系统的辅助方程是一个微分几何中的 pfaffian 系统. 运用微分几何的方法来研究这一系统及其解的局域性质和整体性质是一件有趣的工作.

(4) 使用同样的方法可以用来研究在其他含时背景下的 Dirac 场, 比如 Friedmann-Robertson-Walker 平直时空中的 Dirac 场. 这一方面的工作已经完成并将发表在其他地方.

附　　录

θ_m, ϕ_m, ($m=1,2,\cdots,6$)满足的辅助方程为

$$\begin{aligned}\dot{\theta}_1 = \frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)}\{&\sin2\theta_3\cos\theta_1\cos\theta_4[\sin(\phi_3+\phi_4)\lambda_{3r}+\cos(\phi_3+\phi_4)\lambda_{3i}]\\&-\sin2\theta_2\sin\theta_1\sin\theta_4[\sin(\phi_2+\phi_4)\lambda_{3r}-\cos(\phi_2+\phi_4)\lambda_{3i}]+\lambda_1\sin2\theta_2\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_2+\phi_4)\\&+\sin2\theta_3\sin\theta_1\sin\theta_4[\sin(\phi_3+\phi_4)\lambda_{3r}-\cos(\phi_3+\phi_4)\lambda_{3i}]-\lambda_1\sin2\theta_3\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_3+\phi_4)\\&-\sin2\theta_2\cos\theta_1\cos\theta_4[\sin(\phi_1+\phi_2)\lambda_{3r}+\cos(\phi_1+\phi_2)\lambda_{3i}]+\lambda_1\sin2\theta_3\cos\theta_4\sin\theta_1\sin\phi_3\\&-\lambda_1\sin2\theta_2\cos\theta_4\sin\theta_1\sin\phi_2\},\end{aligned}$$

$$\begin{aligned}\dot{\theta}_2 = &-\cos\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_2+\phi_4)+\lambda_{3i}\cos(\phi_2+\phi_4)]+\cos\theta_4\sin\theta_1[\lambda_{3r}\sin(\phi_1+\phi_2)+\lambda_{3i}\cos(\phi_1+\phi_2)]\\&-\lambda_1\cos\theta_4\cos\theta_1\sin\phi_2-\lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_2+\phi_4),\end{aligned}$$

$$\begin{aligned}\dot{\theta}_3 = &-\cos\theta_4\sin\theta_1[\lambda_{3r}\sin(\phi_1+\phi_3)+\lambda_{3i}\cos(\phi_1+\phi_3)]+\cos\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_3+\phi_4)-\lambda_{3i}\cos(\phi_3+\phi_4)]\\&+\lambda_1\cos\theta_4\cos\theta_1\sin\phi_3+\lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4),\end{aligned}$$

$$\dot{\theta}_4 = \frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)}\{\sin2\theta_2\cos\theta_1\cos\theta_4[\sin(\phi_2+\phi_4)\lambda_{3r}-\cos(\phi_2+\phi_4)\lambda_{3i}]$$

$$
\begin{aligned}
&+\sin 2\theta_2\sin\theta_1\sin\theta_4[\sin(\phi_1+\phi_2)\lambda_{3r}+\cos(\phi_1+\phi_2)\lambda_{3i}]+\lambda_1\sin 2\theta_2\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_2+{}_4)\\
&-\sin 2\theta_3\sin\theta_1\sin\theta_4[\sin(\phi_1+\phi_3)\lambda_{3r}+\cos(\phi_1+\phi_3)\lambda_{3i}]-\lambda_1\sin 2\theta_3\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4)\\
&-\sin 2\theta_3\cos\theta_1\cos\theta_4[\sin(\phi_3+\phi_4)\lambda_{3r}-\cos(\phi_3+\phi_4)\lambda_{3i}]+\lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\sin\phi_3\\
&-\lambda_1\sin 2\theta_2\sin\theta_4\cos\theta_1\sin\phi_2\},
\end{aligned}
$$

$$\dot\theta_5=\cos\phi_5[\chi_8^{(d)}(p,t)+\chi_8^{(g)}(p,t)]-\sin\phi_5[\chi_7^{(d)}(p,t)+\chi_7^{(g)}(p,t)],$$

$$\dot\theta_6=\cos\phi_6[\chi_6^{(d)}(p,t)+\chi_6^{(g)}(p,t)]-\sin\phi_6[\chi_5^{(d)}(p,t)+\chi_5^{(g)}(p,t)],$$

$$
\begin{aligned}
\dot\phi_1=&\frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)\sin 2\theta_1}\{\sin 2\theta_3\cos\theta_1\cos\theta_4[\cos(\phi_3+\phi_4)\lambda_{3r}-\sin(\phi_3+\phi_4)\lambda_{3i}]\\
&-\sin 2\theta_2\sin\theta_1\sin\theta_4[\cos(\phi_2+\phi_4)\lambda_{3r}+\sin(\phi_2+\phi_4)\lambda_{3i}]+\lambda_1\sin 2\theta_2\cos\theta_1\sin\theta_4\cos(\phi_1+\phi_2+{}_4)\\
&+\sin 2\theta_3\sin\theta_1\sin\theta_4[\cos(\phi_3+\phi_4)\lambda_{3r}+\sin(\phi_3+\phi_4)\lambda_{3i}]-\lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\cos(\phi_1+\phi_3+\phi_4)\\
&-\sin 2\theta_2\cos\theta_1\cos\theta_4[\cos(\phi_1+\phi_2)\lambda_{3r}-\sin(\phi_1+\phi_2)\lambda_{3i}]+\lambda_1\sin 2\theta_3\cos\theta_4\sin\theta_1\cos\phi_3\\
&-\lambda_1\sin 2\theta_2\cos\theta_4\sin\theta_1\cos\phi_2\},
\end{aligned}
$$

$$
\begin{aligned}
\dot\phi_2=&2E(t)-\sin^2\theta_1\dot\phi_1-\sin^2\theta_4\dot\phi_4-2\cot 2\theta_2\cos\theta_1\sin\theta_4[\cos(\phi_2+\phi_4)\lambda_{3r}+\sin(\phi_2+\phi_4)\lambda_{3i}]\\
&+2\cot 2\theta_2\sin\theta_1\cos\theta_4[\cos(\phi_1+\phi_2)\lambda_{3r}-\sin(\phi_1+\phi_2)\lambda_{3i}]-2\lambda_1\cot 2\theta_2\sin\theta_1\sin\theta_4\cos(\phi_1+\phi_2+\phi_4)\\
&-2\lambda_1\cot 2\theta_2\cos\theta_1\cos\theta_4\cos\phi_4,
\end{aligned}
$$

$$
\begin{aligned}
\dot\phi_3=&-2E(t)-\sin^2\theta_1\dot\phi_1-\sin^2\theta_4\dot\phi_4+2\cot 2\theta_3\cos\theta_1\sin\theta_4[\cos(\phi_3+\phi_4)\lambda_{3r}+\sin(\phi_3+\phi_4)\lambda_{3i}]\\
&-2\cot 2\theta_3\sin\theta_1\cos\theta_4[\cos(\phi_1+\phi_3)\lambda_{3r}-\sin(\phi_1+\phi_3)\lambda_{3i}]+2\lambda_1\cot 2\theta_3\sin\theta_1\sin\theta_4\cos(\phi_1+\phi_3+\phi_4)\\
&+2\lambda_1\cot 2\theta_3\cos\theta_1\cos\theta_4\cos\phi_3,
\end{aligned}
$$

$$
\begin{aligned}
\dot\phi_4=&\frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)\sin 2\theta_4}\{\sin 2\theta_2\cos\theta_1\cos\theta_4[\cos(\phi_2+\phi_4)\lambda_{3r}+\sin(\phi_2+\phi_4)\lambda_{3i}]\\
&+\sin 2\theta_2\sin\theta_1\sin\theta_4[\cos(\phi_1+\phi_2)\lambda_{3r}-\sin(\phi_1+\phi_2)\lambda_{3i}]+\lambda_1\sin 2\theta_2\cos\theta_4\sin\theta_1\cos(\phi_1+\phi_2+{}_4)\\
&-\sin 2\theta_3\sin\theta_1\sin\theta_4[\cos(\phi_1+\phi_3)\lambda_{3r}-\sin(\phi_1+\phi_3)\lambda_{3i}]-\lambda_1\sin 2\theta_3\cos\theta_4\sin\theta_1\cos(\phi_1+\phi_3+\phi_4)\\
&-\sin 2\theta_3\cos\theta_1\cos\theta_4[\cos(\phi_3+\phi_4)\lambda_{3r}+\sin(\phi_3+\phi_4)\lambda_{3i}]+\lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\cos\phi_3\\
&-\lambda_1\sin 2\theta_2\sin\theta_4\cos\theta_1\cos\phi_2\},
\end{aligned}
$$

$$
\begin{aligned}
\dot\phi_5=&[\chi_3^{(d)}(p,t)+\chi_3^{(g)}(p,t)]-[\chi_2^{(d)}(p,t)+\chi_2^{(g)}(p,t)]-2\cos\phi_5\cot 2\theta_5[\chi_7^{(d)}(p,t)+\chi_7^{(g)}(p,t)]\\
&-2\sin\phi_5\cot 2\theta_5[\chi_8^{(d)}(p,t)+\chi_8^{(g)}(p,t)],
\end{aligned}
$$

$$
\begin{aligned}
\dot\phi_6=&[\chi_4^{(d)}(p,t)+\chi_4^{(g)}(p,t)]-[\chi_1^{(d)}(p,t)+\chi_1^{(g)}(p,t)]-2\cos\phi_6\cot 2\theta_6[\chi_5^{(d)}(p,t)+\chi_5^{(g)}(p,t)]\\
&-2\sin\phi_6\cot 2\theta_6[\chi_6^{(d)}(p,t)+\chi_6^{(g)}(p,t)],
\end{aligned}
$$

其中

$$
\begin{aligned}
\chi_1^{(d)}(p,t)=&\cos 2\theta_2E(t)-\sin\theta_1\cos\theta_4\sin 2\theta_2[\lambda_{3r}\cos(\phi_1+\phi_2)-\lambda_{3i}\sin(\phi_1+\phi_3)]\\
&+\sin\theta_4\cos\theta_1\sin 2\theta_2[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)]\\
&+\lambda_1[\sin\theta_1\sin\theta_4\sin 2\theta_2\cos(\phi_1+\phi_2+\phi_4)+\cos\theta_1\cos\theta_4\sin 2\theta_2\cos(\phi_2)],
\end{aligned}
$$

$$\chi_1^{(g)}(p,t)=-\sin^2\theta_1\cos^2\theta_2\dot\phi_1+\sin^2\theta_2\dot\phi_2+\sin^2\theta_2\sin^2\theta_4\dot\phi_4,$$

$$
\begin{aligned}
\chi_2^{(d)}(p,t)=&\cos 2\theta_3E(t)+\sin\theta_4\cos\theta_1\sin 2\theta_3[\lambda_{3r}\cos(\phi_3+\phi_4)+\lambda_{3i}\sin(\phi_3+\phi_4)]\\
&-\sin\theta_1\cos\theta_4\sin 2\theta_3[\lambda_{3r}\cos(\phi_1+\phi_3)-\lambda_{3i}\sin(\phi_1+\phi_3)]\\
&+\lambda_1[\sin\theta_1\sin\theta_4\sin 2\theta_3\cos(\phi_1+\phi_3+\phi_4)+\cos\theta_1\cos\theta_4\sin 2\theta_3\cos(\phi_3)],
\end{aligned}
$$

$$\chi_2^{(g)}(p,t)=\sin^2\theta_1\cos^2\theta_3\dot\phi_1-\sin^2\theta_3\dot\phi_3-\sin^2\theta_3\sin^2\theta_4\dot\phi_4,$$

$$\chi_3^{(d)}(p,t)=-\cos 2\theta_2E(t)+\sin\theta_1\cos\theta_4\sin 2\theta_2[\lambda_{3r}\cos(\phi_1+\phi_2)-\lambda_{3i}\sin(\phi_1+\phi_2)]$$

$$
\begin{aligned}
&-\sin\theta_4\cos\theta_1\sin 2\theta_2[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)]\\
&-\lambda_1[\sin\theta_1\sin\theta_4\sin 2\theta_2\cos(\phi_1+\phi_2+\phi_4)+\cos\theta_1\cos\theta_4\sin 2\theta_2\cos(\phi_2)],
\end{aligned}
$$

$$
\chi_3^{(g)}(p,t)=\sin^2\theta_4\cos^2\theta_2\dot{\phi}_4-\sin^2\theta_2\dot{\phi}_2-\sin^2\theta_2\sin^2\theta_1\dot{\phi}_1,
$$

$$
\begin{aligned}
\chi_4^{(d)}(p,t)=&-\cos 2\theta_3 E(t)+\sin\theta_4\cos\theta_1\sin 2\theta_3[\lambda_{3r}\cos(\phi_3+\phi_4)-\lambda_{3i}\sin(\phi_3+\phi_4)]\\
&-\sin\theta_1\cos\theta_4\sin 2\theta_3[\lambda_{3r}\cos(\phi_1+\phi_3)+\lambda_{3i}\sin(\phi_1+\phi_3)]\\
&-\lambda_1[\sin\theta_1\sin\theta_4\sin 2\theta_3\cos(\phi_1+\phi_3+\phi_4)+\cos\theta_1\cos\theta_4\sin 2\theta_3\cos(\phi_3)],
\end{aligned}
$$

$$
\chi_4^{(g)}(p,t)=\sin^2\theta_1\sin^2\theta_3\dot{\phi}_1+\sin^2\theta_3\dot{\phi}_3-\cos^2\theta_3\sin^2\theta_4\dot{\phi}_4,
$$

$$
\begin{aligned}
\chi_5^{(d)}(p,t)=&\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4[\lambda_{3r}\cos(-\phi_1+\phi_2-\phi_3+\phi_4)+\lambda_{3i}\sin(-\phi_1+\phi_2-\phi_3+\phi_4)]\\
&+\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4[\lambda_{3r}\cos(\phi_1-\phi_4)-\lambda_{3i}\sin(\phi_1-\phi_4)]\\
&+\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4[\lambda_{3r}\cos(\phi_2-\phi_3)-\lambda_{3i}\sin(\phi_2-\phi_3)]+\lambda_{3r}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\\
&+\lambda_1[-\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos(\phi_2-\phi_3+\phi_4)+\sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\cos\phi_1\\
&+\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\cos(\phi_2-\phi_3-\phi_1)+\cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4\cos\phi_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_5^{(g)}(p,t)=&\dot{\theta}_1\sin(\phi_1+\phi_3)\cos\theta_2\sin\theta_3+\dot{\theta}_4\sin(\phi_2+\phi_4)\sin\theta_2\cos\theta_3\\
&+\frac{1}{2}[\dot{\phi}_1\cos(\phi_1+\phi_3)\cos\theta_2\sin\theta_3\sin 2\theta_1+\dot{\phi}_4\cos(\phi_2+\phi_4)\sin\theta_2\cos\theta_3\sin 2\theta_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_6^{(d)}(p,t)=&-\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4[\lambda_{3r}\sin(-\phi_1+\phi_2-\phi_3+\phi_4)-\lambda_{3i}\cos(-\phi_1+\phi_2-\phi_3+\phi_4)]\\
&+\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)]\\
&-\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4[\lambda_{3r}\sin(\phi_2-\phi_3)+\lambda_{3i}\cos(\phi_2-\phi_3)]-\lambda_{3r}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\\
&+\lambda_1[\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin(\phi_2-\phi_3+\phi_4)+\sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\sin\phi_1\\
&-\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\sin(\phi_2-\phi_3-\phi_1)+\cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4\sin\phi_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_6^{(g)}(p,t)=&-\dot{\theta}_1\cos(\phi_1+\phi_2)\cos\theta_2\sin\theta_3+\dot{\theta}_4\cos(\phi_2+\phi_4)\sin\theta_2\cos\theta_3\\
&+\frac{1}{2}[\dot{\phi}_1\sin(\phi_1+\phi_3)\cos\theta_2\sin\theta_3\sin 2\theta_1-\dot{\phi}_4\sin(\phi_2+\phi_4)\sin\theta_2\cos\theta_3\sin 2\theta_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_7^{(d)}(p,t)=&\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4[\lambda_{3r}\cos(\phi_1+\phi_2-\phi_3-\phi_4)-\lambda_{3i}\sin(\phi_1+\phi_2-\phi_3-\phi_4)]\\
&+\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4[\lambda_{3r}\cos(\phi_1-\phi_4)-\lambda_{3i}\sin(\phi_1-\phi_4)]\\
&+\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4[\lambda_{3r}\cos(\phi_2-\phi_3)+\lambda_{3i}\sin(\phi_2-\phi_3)]+\lambda_{3r}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\\
&+\lambda_1[-\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos(\phi_2-\phi_3-\phi_4)+\sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\cos\phi_1\\
&+\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\cos(\phi_1+\phi_2-\phi_3)-\cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4\cos\phi_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_7^{(g)}(p,t)=&-\dot{\theta}_1\sin(\phi_1+\phi_2)\sin\theta_2\cos\theta_3-\dot{\theta}_4\sin(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\\
&-\frac{1}{2}[\dot{\phi}_1\cos(\phi_1+\phi_2)\sin\theta_2\cos\theta_3\sin 2\theta_1+\dot{\phi}_4\cos(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\sin 2\theta_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_8^{(d)}(p,t)=&-\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4[\lambda_{3r}\sin(\phi_1+\phi_2-\phi_3-\phi_4)+\lambda_{3i}\cos(\phi_1+\phi_2-\phi_3-\phi_4)]\\
&-\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)]\\
&-\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4[\lambda_{3r}\sin(\phi_2-\phi_3)-\lambda_{3i}\cos(\phi_2-\phi_3)]+\lambda_{3i}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\\
&+\lambda_1[\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin(\phi_2-\phi_3-\phi_4)-\sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\sin\phi_1\\
&-\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\sin(\phi_1+\phi_2-\phi_3)-\cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4\sin\phi_4],
\end{aligned}
$$

$$
\begin{aligned}
\chi_8^{(g)}(p,t)=&-\dot{\theta}_1\cos(\phi_1+\phi_2)\sin\theta_2\cos\theta_3+\dot{\theta}_4\cos(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\\
&+\frac{1}{2}[\dot{\phi}_1\sin(\phi_1+\phi_2)\sin\theta_2\cos\theta_3\sin 2\theta_1-\dot{\phi}_4\sin(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\sin 2\theta_4].
\end{aligned}
$$

其中 $\lambda_{3r}(t)$, $\lambda_{3i}(t)$分别为 $\lambda_3(t)$的实部和虚部.

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INVARIANT-RELATED UNITARY TRANSFORMATION METHOD AND EXACT SOLUTIONS FOR THE QUANTUM DIRAC FIELD IN A TIME-DEPENDENT SPATIALLY HOMOGENEOUS ELECTRIC FIELD*

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ABSTRACT

On the basis of the generalized invariant formulation, the invariant-related unitary transformation method is used to study the evolution of the quantum Dirac field in a time-dependent spatially homogeneous electric field. We solve the functional Schrödinger equation for the Dirac field and obtain the exact solutions and corresponding total phase. The total phase includes both the dynamical phase and geometric phase (Aharonov-Anandan phase).

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