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量子 ħ-deformed W_N 代数及其屏蔽流代数

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构造了 ħ-deformed W_N 代数的量子理论和与之相对应的量子 ħ-deformed Miura 变换. 还研究了 ħ-deformed W_N 代数的屏蔽流代数.

PACC: 0220; 0240; 0350; 1100

1 引 言

近来, 对无穷维 Lie 代数的 q-deformation——q-deformed 仿射代数[1—3], q-deformed Virasoro[4,5]代数和 q-deformed W_N 代数受到数学物理学界的广泛关注. 这些 q-deformed 代数被证明是格点可积模型的对称代数. 另一方面, 对于无穷维 Lie 代数的另一种 deformation——ħ-deformed 仿射代数(即 Yangian Double $DY_{\hbar}(\hat{g})_c$[6,7])、ħ-deformed Virasoro 代数、ħ-deformed W_N 代数同样也引起了广泛的关注. ħ-deformed 代数是许多由共形场扰动得到的可积有质量场论模型的对称代数. Smirnov 证明 $DY_{\hbar}(\widehat{sl}_2)_c$ 代数是 $SU(2)$-Invariant Thirring 模型的动力学对称代数[8]; 量子 ħ-deformed Virasoro 代数是量子 Restricted sine-Gordon 模型的对称代数[9]. 因此, 对于 ħ-deformed W_N 代数的经典及量子理论的研究具有极其重要的意义.

在文献[10]中, 我们构造了经典 ħ-deformed W_N 代数及其相应的 ħ-deformed Miura 变换, 同时实现了经典 ħ-deformed W_N 代数的玻色化. 本文在此基础上构造量子 ħ-deformed W_N 代数, 并研究它的屏蔽流代数.

2 经典 ħ-deformed W_N 代数及其玻色化

下面首先简要介绍文献[10]中有关经典 ħ-deformed W_N 代数及其玻色化的工作, 经典 ħ-deformed W_N 代数是由$\{t_m(\beta), (m=1, \cdots, N)\}$生成的 Poisson 代数, 它的生成元$\{t_m(\beta), (m=1, \cdots, N)\}$又可以通过经典 ħ-deformed Miura 变换与$\{\Lambda_m(\beta), (m=1, \cdots, N)\}$建立一一对应的关系,

$$[e^{i\hbar\partial_\beta}-\Lambda_1(\beta)][e^{i\hbar\partial_\beta}-\Lambda_2(\beta-i\hbar)]\cdots[e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)]$$

$$= \sum_{m=0}^{N}(-1)^m t_m\left(\beta - \mathrm{i}\frac{m-1}{2}\hbar\right)\mathrm{e}^{\mathrm{i}(N-m)\hbar\partial_\beta}, \tag{1}$$

且 $\Lambda_m(\beta)$ 满足 $\Lambda_1(\beta)\Lambda_2(\beta-\mathrm{i}\hbar)\cdots\Lambda_N(\beta-\mathrm{i}(N-1)\hbar)=1$.

由(1)式, 可以得到

$$t_m(\beta) = \sum_{1\leqslant j_1<j_2\cdots j_m\leqslant N}\Lambda_{j_1}\left(\beta+\mathrm{i}\frac{m-1}{2}\hbar\right)\Lambda_{j_2}\left(\beta+\mathrm{i}\frac{m-3}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta-\mathrm{i}\frac{m-1}{2}\hbar\right). \tag{2}$$

实际上, $\{t_m(\beta),(m=1,\cdots,N)\}$ 的 Poisson 代数完全可由 $\{\Lambda_m(\beta),(m=1,\cdots,N)\}$ 具有以下的 Poisson 括号得到:

$$\{\Lambda_m(\beta_1),\ \Lambda_m(\beta_2)\} = \hbar\phi_{m=m}(\beta_2-\beta_1)\Lambda_m(\beta_1)\Lambda_m(\beta_2), \tag{3}$$

其中 $\phi_{m=m}(\beta)=f_{1N}(\beta)$;

$$\{\Lambda_n(\beta_1),\ \Lambda_m(\beta_2)\} = \hbar\phi_{n<m}(\beta_2-\beta_1)\Lambda_n(\beta_1)\Lambda_m(\beta_2), \qquad n<m, \tag{4}$$

其中 $\phi_{n<m}(\beta)=-f_{1N-1}\left(\beta+\mathrm{i}\frac{N}{2}\hbar\right)$;

$$\{\Lambda_n(\beta_1),\ \Lambda_m(\beta_2)\} = \hbar\phi_{n>m}(\beta_2-\beta_1)\Lambda_n(\beta_1)\Lambda_m(\beta_2), \qquad n>m, \tag{5}$$

其中 $\phi_{n>m}(\beta)=-f_{1N-1}\left(\beta-\mathrm{i}\frac{N}{2}\hbar\right)$.

$$f_{nm}(\beta) = \frac{\partial}{\partial(\mathrm{i}\beta)} \times \ln\left(\frac{\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+\frac{n+m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1-\frac{n+m}{2N}\right)\Gamma\left(-\frac{\mathrm{i}\beta}{N\hbar}-\frac{n-m}{2N}\right)\Gamma\left(-\frac{\mathrm{i}\beta}{N\hbar}+1+\frac{n-m}{2N}\right)}{\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}-\frac{n-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1+\frac{n-m}{2N}\right)\Gamma\left(-\frac{\mathrm{i}\beta}{N\hbar}+\frac{n+m}{2N}\right)\Gamma\left(-\frac{\mathrm{i}\beta}{N\hbar}+1-\frac{n+m}{2N}\right)}\right).$$

定义经典玻色子 $\lambda_m^{(0)}(t)(m=1,\cdots,N;\ t\in R-\{0\})$, 它们满足以下 Heisenberg-Poisson 代数:

$$\{\lambda_m^{(0)}(t),\ \lambda_m^{(0)}(t')\} = \hbar\frac{2\,\mathrm{sh}\left(\frac{(N-1)\hbar}{2}t\right)\mathrm{sh}\left(\frac{\hbar}{2}t\right)}{\mathrm{sh}\left(\frac{N\hbar}{2}t\right)}\delta(t+t'), \tag{6}$$

$$\{\lambda_n^{(0)}(t),\ \lambda_m^{(0)}(t')\} = -\hbar\frac{2\,\mathrm{sh}\left(\frac{\hbar}{2}t\right)\mathrm{sh}\left(\frac{\hbar}{2}t\right)}{\mathrm{sh}\left(\frac{N\hbar}{2}t\right)}\mathrm{e}^{\mathrm{sgn}(m-n)\frac{N\hbar t}{2}}\delta(t+t'), \qquad n\neq m. \tag{7}$$

且 $\lambda_m^{(0)}(t)$ 满足约束条件

$$\sum_{m=1}^{N}\lambda_m^{(0)}(t)\mathrm{e}^{m\hbar t} = 0. \tag{8}$$

满足(3)—(5)式的 $\{\Lambda_m(\beta),(m=1,\cdots,N)\}$ 可以由 $\{\lambda_m^{(0)}(t)\}$ 通过以下方式实现:

$$\Lambda_m(\beta) = \exp\left(-\int_{-\infty}^{\infty}\lambda_m^{(0)}(t)\mathrm{e}^{\mathrm{i}\beta t}\mathrm{d}t\right). \tag{9}$$

实际上, (9)式给出了 $\{\Lambda_m(\beta)\}$ 的 Darboux 坐标. 利用(2)式, 我们就得到了 $\{t_m(\beta)\}$ 的玻色实现.

3　量子 ħ-deformed W_N 代数

通过量子化的过程, 经典玻色子$\{\lambda_m^{(0)}(t), (m=1, \ldots, N)\}$将量子化为 ħ-deformed 量子玻色子$\{\lambda_m(t), (m=1, \ldots, N)\}$, 满足以下的 Heisenberg 代数:

$$[\lambda_m(t), \lambda_m(t')] = \frac{4\,\mathrm{sh}\left(\frac{(N-1)\hbar}{2}t\right)\mathrm{sh}\left(\hbar\frac{\xi}{2}t\right)\mathrm{sh}\left(\frac{h(\xi+1)}{2}t\right)}{t\,\mathrm{sh}\left(\frac{N\hbar t}{2}\right)}\delta(t+t'), \quad (10)$$

$$[\lambda_n(t), \lambda_m(t')] = \frac{4\,\mathrm{sh}\left(\frac{\hbar}{2}t\right)\mathrm{sh}\left(\hbar\frac{\xi}{2}t\right)\mathrm{sh}\left(\frac{\hbar(\xi+1)}{2}t\right)}{t\,\mathrm{sh}\left(\frac{N\hbar t}{2}\right)}\mathrm{e}^{\mathrm{sgn}(m-n)\frac{N\hbar t}{2}}\delta(t+t'), \quad n\neq m. \quad (11)$$

且$\{\lambda_n(t), (n=1, \ldots, N)\}$满足约束条件

$$\sum_{m=1}^{N}\lambda_m(t)\mathrm{e}^{m\hbar t} = 0, \quad (12)$$

其中 ξ 为量子化参数(与通常量子化中的普朗克常数 ħ 具有相同的意义). 可以看出, 当 $\xi\to 0$ 时, 有

$$\{\lambda_m^{(0)}(t), \lambda_m^{(0)}(t')\} = \lim_{\xi\to 0}\frac{[\lambda_m(t), \lambda_m(t')]}{\xi} = \hbar\frac{2\,\mathrm{sh}\left(\frac{(N-1)\hbar}{2}t\right)\mathrm{sh}\left(\frac{\hbar}{2}t\right)}{\mathrm{sh}\left(\frac{N\hbar}{2}t\right)}\delta(t+t'),$$

$$\{\lambda_n^{(0)}(t), \lambda_m^{(0)}(t')\} = \lim_{\xi\to 0}\frac{[\lambda_m(t), \lambda_m(t')]}{\xi} = -\hbar\frac{2\,\mathrm{sh}\left(\frac{\hbar}{2}t\right)\mathrm{sh}\left(\frac{\hbar}{2}t\right)}{\mathrm{sh}\left(\frac{N\hbar}{2}t\right)}\mathrm{e}^{\mathrm{sgn}(m-n)\frac{N\hbar t}{2}}\delta(t+t'), \quad n\neq m.$$

因此, 量子 Heisenberg 代数(10), (11)式当 $\xi\to 0$ 时, 退化为经典 Heisenberg-Poisson 代数(6), (7)式, 即(10), (11)式是(6), (7)式的量子形式.

定义量子 $\Lambda_m(\beta), (m=1, \ldots, N)$,

$$\Lambda_m(\beta) = :\exp\left(-\int_{-\infty}^{\infty}\lambda_m(t)\mathrm{e}^{\mathrm{i}\beta t}\mathrm{d}t\right):, \quad (13)$$

其中符号: O: 代表玻色算子 O 的正规形式(normal order). 同时, 我们定义量子 ħ-deformed W_N 代数的生成元$\{T_m(\beta), (m=1, \ldots, N)\}$,

$$T_m(\beta) = \sum_{1\leqslant j_1<j_2\cdots j_m\leqslant N}:\Lambda_{j_1}\left(\beta+\mathrm{i}\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta-\mathrm{i}\frac{m-1}{2}\hbar\right):. \quad (14)$$

实现生成元$\{T_m(\beta), (m=1, \ldots, N)\}$与$\{\Lambda_m(\beta), (m=1, \ldots, N)\}$一一对应关系的(14)式实际上是由以下的量子 ħ-deformed Miura 变换所得到的:

$$:[e^{i\hbar\partial_\beta}-\Lambda_1(\beta)][e^{i\hbar\partial_\beta}-\Lambda_2(\beta-i\hbar)]\cdots[e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)]$$
$$=\sum_{m=1}^{N}(-1)^m T_m\left(\beta-i\frac{m-1}{2}\hbar\right)e^{i(N-m)\hbar\partial_\beta}. \tag{15}$$

约束条件(12)式等价于

$$T_N(\beta)=:\Lambda_1\left(\beta+i\frac{N-1}{t}\hbar\right)\cdots\Lambda_N\left(\beta-i\frac{N-1}{2}\hbar\right):=1.$$

为了得到$\{T_m(\beta),(m=1,\dots,N)\}$的生成关系,首先计算$\{\Lambda_m(\beta),(m=1,\dots,N)\}$的正规关系.

$$\Lambda_m(\beta_1)\Lambda_m(\beta_2)=\Phi_{m=m}(\beta_2-\beta_1):\Lambda_m(\beta_1)\Lambda_m(\beta_2):, \tag{16}$$

其中

$$\Phi_{m=m}(\beta)=\frac{\Gamma\left(\frac{i\beta}{N\hbar}-\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1-\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+\frac{1+\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1\right)}{\Gamma\left(\frac{i\beta}{N\hbar}\right)\Gamma\left(\frac{i\beta}{N\hbar}-\frac{1+\xi}{N}+1\right)\Gamma\left(\frac{i\beta}{N\hbar}+\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N}\right)};$$

$$\Lambda_n(\beta_1)\Lambda_m(\beta_2)=\Phi_{n<m}(\beta_2-\beta_1):\Lambda_n(\beta_1)\Lambda_m(\beta_2):,\qquad n<m, \tag{17}$$

其中

$$\Phi_{n<m}(\beta)=\frac{\Gamma\left(\frac{i\beta}{N\hbar}-\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}-\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+\frac{1+\xi}{N}\right)}{\Gamma\left(\frac{i\beta}{N\hbar}-\frac{1+\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+\frac{1}{N}\right)};$$

$$\Lambda_n(\beta_1)\Lambda_m(\beta_2)=\Phi_{n>m}(\beta_2-\beta_1):\Lambda_n(\beta_1)\Lambda_m(\beta_2):,\qquad n>m, \tag{18}$$

其中

$$\Phi_{n>m}(\beta)=\frac{\Gamma\left(\frac{i\beta}{N\hbar}+1-\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1-\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1+\frac{1+\xi}{N}\right)}{\Gamma\left(\frac{i\beta}{N\hbar}+1-\frac{1+\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\hbar}+1+\frac{1}{N}\right)}.$$

在上述计算过程中,我们利用了 Γ 函数的积分表达式,

$$\Gamma(z)=\exp\left\{\int_0^\infty\left(\frac{e^{-zt}-e^{-t}}{1-e^{-t}}+(z-1)e^{-t}\right)\frac{dt}{t}\right\},\qquad \mathrm{Re}(z)>0.$$

与经典 $\hbar$-deformed W_N 代数类似,关系式(16)—(18)式以及量子 $\hbar$-deformed Miura 变换(14)式已经完全定义了$\{T_m(\beta),(m=1,\dots,N)\}$之间的关系式. 因此,这里我们仅具体计算 $T_1(\beta)$与 $T_m(\beta)$之间的关系式,其它关系式的计算是类似的.

利用(16)—(18)式,我们有:

1) 当 $n=j_k,(k\in\{1,2,\dots,m\})$,且 $\mathrm{Im}(\beta_2)<\mathrm{Im}(\beta_1)$时,

$$\Lambda_n(\beta_1):\Lambda_{j_1}\left(\beta_2+i\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-i\frac{m-1}{2}\hbar\right):$$
$$=F_{1m}(\beta_2-\beta_1):\Lambda_n(\beta_1)\Lambda_{j_1}\left(\beta_2+i\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-i\frac{m-1}{2}\hbar\right):. \tag{19}$$

2) 当 $j_k<n<j_{k+1}$(当 $n<j_1$,定义 $k=0$;当 $n>j_m$,定义 $k=m$),且 $\mathrm{Im}(\beta_2)<\mathrm{Im}(\beta_1)$时,

$$\Lambda_n(\beta_1):\Lambda_{j_1}\left(\beta_2+i\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-i\frac{m-1}{2}\hbar\right):$$

$$= F_{1m}(\beta_2-\beta_1)\frac{\Gamma\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\hbar}-\frac{\xi}{N}-\frac{1}{2N}+\frac{2k-m}{2N}\right)\Gamma\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\hbar}+\frac{\xi}{N}+\frac{1}{2N}+\frac{2k-m}{2N}\right)}{\Gamma\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\hbar}-\frac{1}{2N}+\frac{2k-m}{2N}\right)\Gamma\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\hbar}+\frac{1}{2N}+\frac{2k-m}{2N}\right)}$$

$$\times:\Lambda_n(\beta_1)\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right):. \tag{20}$$

3) 当 $n=j_k,(k\in\{1,\ldots,m\})$, 且 $\mathrm{Im}(\beta_1)<\mathrm{Im}(\beta_2)$ 时,

$$:\Lambda_{j_1}\left(\beta_2+\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right):\Lambda_n(\beta_1)$$

$$= F_{1m}(\beta_1-\beta_2):\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right)\Lambda_n(\beta_1):. \tag{21}$$

4) 当 $j_k<n<j_{k+1}$(当 $n<j_1$, 定义 $k=0$; 当 $n>j_m$, 定义 $k=m$), 且 $\mathrm{Im}(\beta_1)<\mathrm{Im}(\beta_2)$,

$$:\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right):\Lambda_n(\beta_1)$$

$$= F_{1m}(\beta_1-\beta_2)\frac{\Gamma\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}-\frac{\xi}{N}-\frac{1}{2N}-\frac{2k-m}{2N}\right)\Gamma\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}+\frac{1}{2N}+\frac{\xi}{N}-\frac{2k-m}{2N}\right)}{\Gamma\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}-\frac{1}{2N}-\frac{2k-m}{2N}\right)\Gamma\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}+\frac{1}{2N}-\frac{2k-m}{2N}\right)}$$

$$\times:\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\hbar\right)\cdots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right)\Lambda_n(\beta_1):. \tag{22}$$

(19)—(22)式中,

$$F_{1m}(\beta)$$

$$=\frac{\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1-\frac{1+m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1+\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}-\frac{\xi}{N}-\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+\frac{\xi}{N}+\frac{1+m}{2N}\right)}{\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1-\frac{\xi}{N}-\frac{1+m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+1+\frac{\xi}{N}+\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}-\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\hbar}+\frac{1+m}{2N}\right)}.$$

根据(14)式, 同时利用(19)—(22)式和

$$\lim_{\varepsilon\to 0+}\left(\frac{1}{x+\mathrm{i}\varepsilon}-\frac{1}{x-\mathrm{i}\varepsilon}\right)=-2\mathrm{i}\pi\delta(x),$$

可以得到 $T_1(\beta)$ 与 $T_m(\beta)$ 的关系式

$$F_{1m}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_m(\beta_2)-F_{1m}^{-1}(\beta_1-\beta_2)T_m(\beta_2)T_1(\beta_1)$$

$$=2\pi\hbar\xi(\xi+1)\left\{T_{m+1}\left(\beta_2+\frac{\mathrm{i}\hbar}{2}\right)\delta\left(\beta_1-\beta_2-\mathrm{i}\frac{m+1}{2}\hbar\right)\right.$$

$$\left.-T_{m+1}\left(\beta_2-\mathrm{i}\frac{\hbar}{2}\right)\delta\left(\beta_1-\beta_2+\mathrm{i}\frac{m+1}{h}\right)\right\}. \tag{23}$$

利用同样的方法, 可以得到 $T_n(\beta)$ 与 $T_m(\beta)$ 的关系式.

当 $N=2$ 时, 量子 ħ-deformed W_2 代数就是量子 ħ-deformed Virasoro 代数, 它是量子 Restricted sine-Gordon 模型的对称代数. 作为一个例子, 我们给出量子 ħ-deformed W_3 代数的所有关系. 量子 ħ-deformed W_3 代数是由$\{T_1(\beta), T_2(\beta), T_3(\beta)=1\}$生成的结合代数, 满足以下关系:

$$T_1(\beta)=\Lambda_1(\beta)+\Lambda_2(\beta)+\Lambda_3(\beta), \tag{24}$$

$$T_2(\beta)=:\Lambda_1\left(\beta+\mathrm{i}\frac{\hbar}{2}\right)\Lambda_2\left(\beta-\mathrm{i}\frac{\hbar}{2}\right):+:\Lambda_1\left(\beta+\mathrm{i}\frac{\hbar}{2}\right)\Lambda_3\left(\beta-\mathrm{i}\frac{\hbar}{2}\right):$$
$$+:\Lambda_2\left(\beta+\mathrm{i}\frac{\hbar}{2}\right)\Lambda_3\left(\beta-\mathrm{i}\frac{\hbar}{2}\right):,\tag{25}$$

$$F_{11}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_1(\beta_2)-F_{11}^{-1}(\beta_1-\beta_2)T_1(\beta_2)T_1(\beta_1)$$
$$=2\pi\hbar\xi(\xi+1)\left\{T_2\left(\beta_2+\mathrm{i}\frac{\hbar}{2}\right)\delta(\beta_1-\beta_2-\mathrm{i}\hbar)-T_2\left(\beta_2-\frac{\mathrm{i}\hbar}{2}\right)\delta(\beta_1-\beta_2+\mathrm{i}\hbar)\right\},\tag{26}$$

$$F_{12}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_2(\beta_2)-F_{12}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_1(\beta_1)$$
$$=2\pi\hbar\xi(\xi+1)\left\{\delta\left(\beta_1-\beta_2-\mathrm{i}\frac{3}{2}\hbar\right)-\delta\left(\beta_1-\beta_2+\mathrm{i}\frac{3}{2}\hbar\right)\right\}.\tag{27}$$

$$F_{22}^{-1}(\beta_2-\beta_1)T_2(\beta_1)T_2(\beta_2)-F_{22}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_2(\beta_1)$$
$$=2\pi\hbar\xi(\xi+1)\left\{T_1\left(\beta_2+\mathrm{i}\frac{\hbar}{2}\right)\delta(\beta_1-\beta_2-\mathrm{i}\hbar)-T_1\left(\beta_2-\mathrm{i}\frac{\hbar}{2}\right)\delta(\beta_1-\beta_2+\mathrm{i}\hbar)\right\},\tag{28}$$

其中结构函数

$$F_{11}(\beta)=\frac{\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{2}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+1\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}-\frac{\xi}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{\xi}{3}+\frac{1}{3}\right)}{\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{2}{3}-\frac{\xi}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+1+\frac{\xi}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{3}\right)},$$

$$F_{12}(\beta)=\frac{\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{2}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{5}{6}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}-\frac{\xi}{3}+\frac{1}{6}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{2}+\frac{\xi}{3}\right)}{\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{2}-\frac{\xi}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{5}{6}+\frac{\xi}{3}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{6}\right)\Gamma\left(\frac{\mathrm{i}\beta}{3\hbar}+\frac{1}{2}\right)},$$

$$F_{22}(\beta)=F_{11}(\beta).$$

4 屏蔽流及流代数

在讨论量子 $\hbar$-deformed W_N 代数的屏蔽流之前,我们先引入 $A_{N-1}^{(1)}$型面权. 令 $\varepsilon_\mu(1\leqslant\mu\leqslant N)$是 R^N 空间一组正交归一在$\langle\varepsilon_\mu,\varepsilon_\upsilon\rangle=\delta_{\mu\upsilon}$, 由$\{\varepsilon_\mu\}$可以构造

$$\bar{\varepsilon}_\mu=\varepsilon_\mu-\varepsilon,\quad \varepsilon=\frac{1}{N_m}\sum_{m=1}^{N}\varepsilon_m.$$

定义 $A_{N-1}^{(1)}$型权格 $P=\left\{\sum_{\mu=1}^{N}Z\bar{\varepsilon}_\mu,(Z\text{ 为整数})\right\}$ 和 $A_{N-1}^{(1)}$型 Lie 代数的素根 $\alpha_\mu(1\leqslant\mu\leqslant N-1)$, $\alpha_\mu=\bar{\varepsilon}_\mu-\bar{\varepsilon}_{\mu+1}$.

对于任意一个矢量 $\alpha\in P$, 可以引入一对线性依赖于 α 的零模算子 P_α 和 Q_α, 它们满足以下的关系:

$$[\mathrm{i}P_\alpha,Q_\beta]=\langle\alpha,\beta\rangle,\qquad \alpha,\beta\in P.\tag{29}$$

定义 Fock 空间 $F_{l,k}(l,k\in P)$,

$$F_{l,k}=C[\lambda_{m_1}(-t_1),\lambda_{m_2}(-t_2)\cdots]\mid l,k\rangle,\qquad t_i>0,$$

$$\lambda_m(t)\,|\,l,k\rangle = 0,\qquad t>0,$$

且

$$P_\beta\,|\,l,k\rangle = <\beta, \alpha_+ l + a_- k>\,|\,l,k\rangle,$$
$$|\,l,k\rangle = e^{i\alpha_+ Q_l + i\alpha_- Q_k}\,|\,0,0\rangle,$$

其中

$$\alpha_+ = -\sqrt{\frac{1+\xi}{\xi}},\quad \alpha_- = \sqrt{\frac{\xi}{\xi+1}}. \tag{30}$$

由连续玻色子$\{\lambda_m(t)\}$, 可以引入两套玻色子$\{S_j^{\pm}(t)\}$,

$$S_j^+(t) = \frac{e^{j\frac{\hbar}{2}t}}{2\,\mathrm{sh}\frac{\hbar\xi}{2}t}(\lambda_j(t)-\lambda_{j+1}(t)), \tag{31}$$

$$S_j^-(t) = \frac{e^{j\frac{\hbar}{2}t}}{2\,\mathrm{sh}\frac{\hbar(\xi+1)}{2}t}(\lambda_j(t)-\lambda_{j+1}(t)). \tag{32}$$

定义屏蔽流

$$S_j^+(\beta) = :\exp\left[-\int_{-\infty}^{\infty} S_j^+(t)e^{i\beta t}dt\right]:e^{-i\alpha_+ Q_{\alpha_j}}, \tag{33}$$

$$S_j^-(\beta) = :\exp\left[\int_{-\infty}^{\infty} S_j^-(t)e^{i\beta t}dt\right]:e^{i\alpha_- Q_{\alpha_j}}. \tag{34}$$

通常, 量子 W_N 代数与它们屏蔽流的对易括号等于一个全微商[11]. 而对于 ħ-deformed W_N 代数, 与它的屏蔽流(33), (34)式的对易括号等于一个全差分(这与 q-deformed W_N 代数情形是类似的[4-5]). 当 $\hbar\to 0$ 时, (33), (34)式将给出通常 W_N 代数的屏蔽流.

$\{S_j^{\pm}(\beta)\}$与$\{\Lambda_m(\beta)\}$具有以下的正规关系, 这里我们仅给出$\{S_j^+(\beta)\}$与$\{\Lambda_m(\beta)\}$的关系:

$$\Lambda_j(\beta_1)S_j^+(\beta_2) = f_{jj}^+(\beta_2-\beta_1):\Lambda_j(\beta_1)S_j^+(\beta_2):,$$
$$S_j^+(\beta_1)\Lambda_j(\beta_2) = f_{jj}^+(\beta_2-\beta_1):S_j^+(\beta_1)\Lambda_j(\beta_1):,$$
$$\Lambda_{j+1}(\beta_1)S_j^+(\beta_2) = f_{j+1,j}^+(\beta_2-\beta_1):\Lambda_{j+1}(\beta_1)S_j^+(\beta_2):,$$
$$S_j^+(\beta_1)\Lambda_{j+1}(\beta_2) = f_{j+1,j}^+(\beta_2-\beta_1):S_j^+(\beta_1)\Lambda_{j+1}(\beta_2):,$$
$$S_{j+1}^+(\beta_1)\Lambda_j(\beta_2) = \Lambda_j(\beta_2)S_{j+1}^+(\beta_1) = :\Lambda_j(\beta_2)S_{j+1}^+(\beta_1):,$$
$$S_j^+(\beta_1)\Lambda_m(\beta_2) = S_j^+(\beta_1)\Lambda_m(\beta_2) = :\Lambda_m(\beta_2)S_j^+(\beta_1):,\qquad |j-m|>1,$$

其中

$$f_{ij}^+(\beta) = \frac{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}-\frac{1}{N}+\frac{j}{2N}}{\frac{i\beta}{N\hbar}+\frac{\xi}{2N}+\frac{j}{2N}},\quad f_{j+1,j}^+(\beta) = \frac{\frac{i\beta}{N\hbar}+\frac{\xi}{2N}+\frac{1}{N}+\frac{j}{2N}}{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}+\frac{j}{2N}}.$$

利用以上关系, 我们可以得到

$$\{:(e^{i\hbar\partial_\beta}-\Lambda_1(\beta_1))(e^{i\hbar\partial_\beta}-\Lambda_2(\beta-i\hbar))\cdots[e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)]:,\ S_j^+(\sigma)\}$$

$$= \mathrm{i}\hbar(1+\xi) D_{\sigma,\mathrm{i}\hbar\xi}\left\{2\mathrm{i}\pi\delta\left(\sigma-\beta-\mathrm{i}\frac{j+\xi}{2}\hbar\right):(\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_1(\beta))\cdots\right.$$
$$\times[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_{j-1}(\beta-\mathrm{i}(j-2)\hbar)]A_j^+(\sigma)\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_{j+2}(\beta-\mathrm{i}(j+1)\hbar)]\cdots$$
$$\left.\times[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_N(\beta-\mathrm{i}(N-1)\hbar)]:\right\}, \tag{35}$$

其中

$$A_j^+(\sigma) = :\Lambda_j\left(\sigma-\mathrm{i}\frac{j+\xi}{2}\hbar\right)S_j^+(\sigma):,$$
$$D_{\sigma,\eta}f(\sigma,\beta) = f(\sigma,\beta)-f(\sigma+\eta,\beta).$$

同理, 我们可以得到

$$\{:(\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_1(\beta_1))(\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_2(\beta-\mathrm{i}\hbar))\cdots[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_N(\beta-\mathrm{i}(N-1)\hbar)]:,S_j^-(\sigma)\}$$
$$= \mathrm{i}\hbar\xi D_{\sigma,-\mathrm{i}\hbar(1+\xi)}\left\{2\mathrm{i}\pi\delta\left(\sigma-\beta+\mathrm{i}\frac{-j+\xi+1}{2}\hbar\right):(\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_1(\beta))\cdots\right.$$
$$\times[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_{j-1}(\beta-\mathrm{i}(j-2)\hbar)]A_j^-(\sigma)\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_{j+2}(\beta-\mathrm{i}(j+1)\hbar)]\cdots$$
$$\left.\times[\mathrm{e}^{\mathrm{i}\hbar\partial_\beta}-\Lambda_N(\beta-\mathrm{i}(N-1)\hbar)]:\right\}, \tag{36}$$

其中

$$A_j^-(\sigma) = :\Lambda_j\left(\sigma+\mathrm{i}\frac{-j+1+\xi}{2}\hbar\right)S_j^-(\sigma):.$$

由(35), (36)式可以看出, 量子 $\hbar$-deformed W_N 代数与$\{S_j^\pm(\beta)\}$的对易括号等于一个全差分, 这说明(33), (34)式定义的$\{S_j^\pm(\beta)\}$是 $\hbar$-deformed W_N 代数的屏蔽流.

利用 normal order 方法, 我们可以得到量子 $\hbar$-deformed W_N 代数的屏蔽流的流代数,

$$S_i^+(\beta_1)S_j^+(\beta_2) = (-1)^{A_{ij}}\frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi}+\frac{A_{ij}}{2\xi}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi}-\frac{A_{ij}}{2\xi}\right)}S_j^+(\beta_2)S_i^+(\beta_1),$$

$$S_i^-(\beta_1)S_j^-(\beta_2) = (-1)^{A_{ij}}\frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)}-\frac{A_{ij}}{2(\xi+1)}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)}+\frac{A_{ij}}{2(\xi+1)}\right)}S_j^-(\beta_2)S_i^-(\beta_1),$$

$$[S_j^+(\beta_1),S_j^-(\beta_2)] = \delta\left(\beta_1-\beta_2-\frac{\sqrt{-1}\hbar}{2}\right)H_j^+\left(\beta_1-\frac{\sqrt{-1}\hbar}{4}\right)$$
$$-\delta\left(\beta_1-\beta_2+\frac{\sqrt{-1}\hbar}{2}\right)H_j^-\left(\beta_1+\frac{\sqrt{-1}\hbar}{4}\right),$$

$$S_j^+(\beta_1)S_{j+1}^-(\beta_2) = -S_{j+1}^-(\beta_2)S_j^+(\beta_1),$$
$$S_j^-(\beta_1)S_{j+1}^+(\beta_2) = -S_{j+1}^+(\beta_2)S_j^-(\beta_1),$$
$$[S_j^+(\beta_1),S_l^-(\beta_2)] = 0,\qquad |j-l|>1;$$

$$H_j^+(\beta) \equiv :S_j^+\left(\beta+\frac{\sqrt{-1}\hbar}{4}\right)S_j^-\left(\beta-\frac{\sqrt{-1}\hbar}{4}\right):,$$

$$H_j^-(\beta) \equiv :S_j^+\left(\beta - \frac{\sqrt{-1}\hbar}{4}\right) S_j^-\left(\beta + \frac{\sqrt{-1}\hbar}{4}\right):,$$

$$H_i^{\pm}(\beta_1) S_j^+(\beta_2) = \frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} + \frac{A_{ij}}{2\xi} \pm \frac{1}{4\xi}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} - \frac{A_{ij}}{2\xi} \pm \frac{1}{4\xi}\right)} S_j^+(\beta_2) H_j^{\pm}(\beta_1),$$

$$H_i^{\pm}(\beta_1) S_j^-(\beta_2) = \frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} - \frac{A_{ij}}{2(\xi+1)} \mp \frac{1}{4(\xi+1)}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} + \frac{A_{ij}}{2(\xi+1)} \mp \frac{1}{4(\xi+1)}\right)} S_j^-(\beta_2) H_j^{\pm}(\beta_1),$$

$$H_i^{\pm}(\beta_1) H_j^{\pm}(\beta_2)$$
$$= \frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} + \frac{A_{ij}}{2\xi}\right)\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} - \frac{A_{ij}}{2(\xi+1)}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} - \frac{A_{ij}}{2\xi}\right)\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} + \frac{A_{ij}}{2(\xi+1)}\right)} H_j^{\pm}(\beta_2) H_j^{\pm}(\beta_1),$$

$$H_i^{\pm}(\beta_1) H_j^{\mp}(\beta_2)$$
$$= \frac{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} + \frac{A_{ij}}{2\xi} \pm \frac{1}{2\xi}\right)\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} - \frac{A_{ij}}{2(\xi+1)} \mp \frac{1}{2(\xi+1)}\right)}{\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar\xi} - \frac{A_{ij}}{2\xi} \pm \frac{1}{2\xi}\right)\sin\pi\left(\frac{\sqrt{-1}(\beta_1-\beta_2)}{\hbar(\xi+1)} + \frac{A_{ij}}{2(\xi+1)} \mp \frac{1}{2(\xi+1)}\right)}$$
$$\times H_j^{\mp}(\beta_2) H_i^{\pm}(\beta_1),$$

$$A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}.$$

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QUANTUM ℏ-DEFORMED W_N ALGEBRA AND ITS SCREENING CURRENT ALGEBRA

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ABSTRACT

The quantum theory of ℏ-deformed W_N algebra is constructed. The corresponding quantum ℏ-deformed Miura transformation is given. We also study the algebra of the screening currents of quantum ℏ-deformed W_N algebra.

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