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非线性浅水长波近似方程组的显式精确解*

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利用两种不同的方法考虑非线性浅水长波近似方程组. 用 sine-cosine 方法并借助 Mathematica 软件和吴文俊消元法, 获得三组孤子解. 通过引入新的变换, 又获得若干有理形式的解. 这两种方法也适合于其他非线性方程(组).

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1 引 言

Whitham[1] 和 Broer[2] 在研究浅水波运动的过程中, 首先提出了非线性浅水长波近似方程组

$$u_t - uu_x - H_x + \frac{1}{2}u_{xx} = 0, \tag{1}$$

$$H_t - (uH)_x - \frac{1}{2}H_{xx} = 0. \tag{2}$$

在变量变换下方程(1)和(2)等价于浅水长波双向传播 Boussinesq 方程. Kupershmidt[3] 研究了方程(1)和(2)的对称和守恒律. 最近, Wang 等人[4] 利用齐次平衡方法, 获得了方程(1)和(2)的一组孤子解:

$$u(x,t) = \frac{a}{2}\left[1 + \tanh\frac{1}{2}\left(ax + \frac{1}{2}a^2t\right)\right], \quad H(x,t) = \frac{a^2}{4}\operatorname{sech}^2\frac{1}{2}\left(ax + \frac{1}{2}at^2\right). \tag{3}$$

文献[5]在齐次平衡方法的基础上, 通过引一个新的变换, 获得了方程(1)和(2)的多孤子解.

本文旨在研究两个方面问题. 第一, 最近 Yan[6] 直接从著名的 sine-Gordon 方程得出一个变换, 并将它应用于很多单个的非线性波动方程, 得到了更多的孤子解. 受其启发, 本文试图将此法改进并运用于方程(1)和(2)中, 获得了包括(3)式在内的更多孤子解. 第二, 基于文献[4, 5]的中间结论, 采用新的函数变换, 获得了若干有理形式的精确解.

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2 方程(1)和(2)的孤子解

对给定的方程(1)和(2), 设

$$u(x,t)=\theta(\alpha),\ H(x,t)=\phi(\alpha),\ \alpha=k(x-\lambda t+c), \tag{4}$$

其中 $\theta(\alpha)$, $\phi(\alpha)$ 为待定函数, k, λ 为待定常数, c 为任意常数. 将(4)式代入(1)和(2)式, 得

$$-\lambda\theta'-\theta\theta'-\phi'+\frac{1}{2}k\theta''=0, \tag{5}$$

$$\lambda\phi'+(\theta\phi)'+\frac{1}{2}k\phi''=0, \tag{6}$$

其中上撇表示 $\mathrm{d}/\mathrm{d}\alpha$.

为了求解方程(5)和(6), 设其有如下解:

$$\theta(\alpha)=a\cos\omega+b\sin\omega+a_0, \tag{7}$$

$$\phi(\alpha)=A_2\cos^2\omega+B_2\sin\omega\cos\omega+A_1\cos\omega+B_1\sin\omega+A_0, \tag{8}$$

且

$$\mathrm{d}\omega/\mathrm{d}\alpha=\sin\omega, \tag{9}$$

其中 $\omega=\omega(\alpha)$ 仅为 α 的函数. $a, b, a_0, A_2, B_2, A_1, B_1, A_0$ 为待定常数.

借助 Mathematica 软件, 从(7)—(9)式易得

$$\theta'=a\cos^2\omega+b\sin\omega\cos\omega-a, \tag{10}$$

$$\begin{aligned}\theta\theta'=&(a^2-b^2)\cos^3\omega+2ab\sin\omega\cos^2\omega+aa_0\cos^2\omega+a_0b\sin\omega\cos\omega\\&+(b^2-a^2)\cos\omega-ab\sin\omega-aa_0,\end{aligned} \tag{11}$$

$$\begin{aligned}\phi'=&2A_2\cos^3\omega+2B_2\sin\omega\cos^2\omega+A_1\cos^2\omega+B_1\sin\omega\cos\omega-2A_2\cos\omega-B_2\sin\omega\\&+(-A_1),\end{aligned} \tag{12}$$

$$\theta''=2a\cos^3\omega+2b\sin\omega\cos^2\omega-2a\cos\omega-b\sin\omega, \tag{13}$$

$$\begin{aligned}(\theta\phi)'=&(3aA_2-3bB_2)\cos^4\omega+(3bA_2+3aB_2)\sin\omega\cos^3\omega+(2aA_1+2a_0A_2\\&-2bB_1)\cos^3\omega+(4B_2b-3aA_2+a_0A_1+aA_0)\cos^2\omega+(2bA_1+2aB_1\\&+2a_0B_2)\sin\omega\cos^2\omega+(bA_0+a_0B_1-2bA_2-2aB_2)\sin\omega\cos\omega+(2bB_1\\&-2aA_1-2a_0A_2)\cos\omega+(-bA_1-aB_1-a_0B_2)\sin\omega+(-bB_2-a_0A_1\\&-aA_0),\end{aligned} \tag{14}$$

$$\begin{aligned}\phi''=&6A_2\cos^4\omega+6B_2\sin\omega\cos^3\omega+2A_1\cos^3\omega+2B_1\sin\omega\cos^2\omega-8A_2\cos^2\omega\\&-5B_2\sin\omega\cos\omega-2A_1\cos\omega-B_1\sin\omega+2A_2.\end{aligned} \tag{15}$$

将(10)—(15)式代入(5)和(6)式, 得

$$\begin{aligned}-\lambda\theta'-\theta\theta'-\phi'+\frac{1}{2}k\theta''=&(\lambda a+aa_0+A_1)+\left|ab+B_2-\frac{1}{2}kb\right|\sin\omega+(a^2-b^2+2A_2\\&-ka)\cos\omega-(\lambda b+a_0b+B_1)\sin\omega\cos\omega-(\lambda_a+aa_0+A_1)\\&\times\cos^2\omega+(-2ab-2B_2+kb)\sin\omega\cos^2\omega\\&+(b^2-a^2-2A_2+ka)\cos^3\omega=0,\end{aligned} \tag{16}$$

$$\lambda\phi' + (\theta\phi)' + \frac{1}{2}k\phi'' = (-\lambda A_1 - B_2 b - a_0 A_1 - aA_0 + kA_2)$$
$$+ \left[-\lambda B_2 - bA_1 - aB_1 - a_0 B_2 - \frac{1}{2}kB_1\right]\sin\omega$$
$$+ (-2\lambda A_2 + 2bB_1 - 2aA_1 - 2a_0 A_2 - kA_1)\cos\omega$$
$$+ \left(\lambda B_1 - 2bA_2 - 2aB_2 + bA_0 + a_0 B_1 - \frac{5}{2}kB_2\right)\sin\omega\cos\omega$$
$$+ (\lambda A_1 + 4B_2 b - 3aA_2 + a_0 A_1 + aA_0 - 4kA_2)$$
$$\times \cos^2\omega + (2\lambda B_2 + 2bA_1 + 2aB_1 + 2aB_2 + kB_1)\sin\omega\cos^2\omega$$
$$+ (2\lambda A_2 + 2aA_1 + 2a_0 A_2 - 2bB_1 + kA_1)\cos^3\omega + (3bA_2 + 3aB_2$$
$$+ 3kB_2)\sin\omega\cos^3\omega + (3aA_2 - 3B_2 b + 3kA_2)\cos^4\omega = 0. \quad (17)$$

分别令方程(16)中的常数项及 $\sin\omega$, $\cos\omega$, $\sin\omega\cos\omega$, $\cos^2\omega$, $\sin\omega\cos^2\omega$, $\cos^3\omega$ 的系数和(17)式中的常数项及 $\sin\omega$, $\cos\omega$, $\sin\omega\cos\omega$, $\cos^2\omega$, $\sin\omega\cos^2\omega$, $\cos^3\omega$, $\sin\omega\cos^3\omega$, $\cos^4\omega$ 的系数为零, 得到超定代数方程组

$$\lambda a + aa_0 + A_1 = 0, \quad (18)$$
$$ab + B_2 - \frac{1}{2}kb = 0, \quad (19)$$
$$a^2 - b^2 + 2A_2 - ka = 0, \quad (20)$$
$$-\lambda b - a_0 b - B_1 = 0, \quad (21)$$
$$-\lambda a - a_0 a - A_1 = 0, \quad (22)$$
$$-2ab - 2B_2 + kb = 0, \quad (23)$$
$$b^2 - a^2 - 2A_2 + ka = 0, \quad (24)$$
$$-\lambda A_1 - B_2 b - a_0 A_1 - aA_0 + kA_2 = 0, \quad (25)$$
$$-\lambda B_2 - bA_1 - aB_1 - a_0 B_2 - \frac{1}{2}kB_1 = 0, \quad (26)$$
$$-2\lambda A_2 + 2bB_1 - 2aA_1 - 2a_0 A_2 - kA_1 = 0, \quad (27)$$
$$\lambda B_1 - 2bA_2 - 2aB_2 + bA_0 + a_0 B_1 - \frac{5}{2}kB_2 = 0, \quad (28)$$
$$\lambda A_1 + 4B_2 b - 3aA_2 + a_0 A_1 + aA_0 - 4kA_2 = 0, \quad (29)$$
$$2\lambda B_2 + 2bA_1 + 2aB_1 + 2a_0 B_2 + kB_1 = 0, \quad (30)$$
$$2\lambda A_2 + 2aA_1 + 2a_0 A_2 - 2bB_1 + kA_1 = 0, \quad (31)$$
$$3bA_2 + 3aB_2 + 3kB_2 = 0, \quad (32)$$
$$3aA_2 - 3B_2 b + 3kA_2 = 0. \quad (33)$$

采用吴文俊消元法[7,8]约化求解方程(18)—(33), 得

情形 1　$b = A_1 = B_1 = B_2 = 0$, $a_0 = -\lambda$, $a = -k$, $A_2 = -k^2$, $A_0 = k^2$.

情形 2　$a = A_1 = B_1 = 0$, $a_0 = -\lambda$, $b = -\varepsilon k\mathrm{i}$, $A_2 = -\frac{k^2}{2}$, $A_0 = \frac{k^2}{4}$, $\varepsilon = \pm 1$, $\mathrm{i} = \sqrt{-1}$, $B_2 = -\frac{k^2}{2}\varepsilon\mathrm{i}$.

情形 3　$A_1 = B_2 = 0$, $a_0 = -\lambda$, $a = -\frac{1}{2}k$, $b = -\frac{k}{2}\mu i$, $A_0 = \frac{1}{2}k^2$, $A_2 = -\frac{1}{2}k^2$, $B_2 = -\frac{1}{2}k^2\mu i$, $\mu = \pm 1$, $i = \sqrt{-1}$.

下面考虑方程(9), 通过变量分离积分, 得

$$\sin\omega = \left.\frac{2N\exp(\pm\alpha)}{1+N^2\exp(\pm 2\alpha)}\right|_{N=1} = \operatorname{sech}\alpha, \tag{34}$$

或

$$\cos\omega = \left.\frac{1-N^2\exp(\pm 2\alpha)}{1+N^2\exp(\pm 2\alpha)}\right|_{N=1} = \mp\tanh\alpha, \tag{35}$$

因此由(4), (7), (8), (34), (35)式及情形 1, 2, 3, 得

ⅰ　方程(1)和(2)的一组(扭状、钟状)孤子解:

$$u(x,t) = \pm k\tanh k(x-\lambda t+c) - \lambda, \tag{36}$$

$$H(x,t) = k^2\operatorname{sech}^2[k(x-\lambda t+c)]. \tag{37}$$

ⅱ　方程(1)和(2)的一组(钟状、扭状)孤波解:

$$u(x,t) = -\varepsilon k i\operatorname{sech}[k(x-\lambda t+c)] - \lambda,$$

$$H(x,t) = -\frac{k^2}{2}\cos\omega e^{i\varepsilon\omega} + \frac{k^2}{4} = \pm\frac{k^2}{2}\tanh k(x-\lambda t+c)\exp\{i\varepsilon\arcsin[\operatorname{sech}k(x-\lambda t+c)]\} + \frac{k^2}{4},$$

这组解为钟状和扭状孤子解的非线性组合, 在复标量场中, 也是孤波解. 并且$|u(x,t)+\lambda| = |k|\operatorname{sech}\alpha$, $\left|H(x,t)-\frac{k^2}{4}\right| = \frac{k^2}{2}|\tanh\alpha|$. 当$|\alpha|\to\infty$时, $u(x,t)\to\lambda$, $H(x,t)\to\frac{3}{4}k^2$或$-\frac{k^2}{4}$. 这表明波运动时能量是集中的. 这也扩充了方程(1)和(2)解析解的范围.

ⅲ　方程(1)和(2)的一组孤波解:

$$u(x,t) = \pm\frac{1}{2}k\tanh\alpha - \frac{1}{2}k\mu i\operatorname{sech}\alpha - \lambda = -\frac{1}{2}k\exp(i\mu\omega) - \lambda$$
$$= -\frac{1}{2}k\exp\{i\mu\arcsin[\operatorname{sech}k(x-\lambda t+c)]\} - \lambda,$$

$$H(x,t) = -\frac{1}{2}k^2\tanh^2\alpha - \frac{1}{2}k^2\mu i\operatorname{sech}\alpha\tanh\alpha + \frac{k^2}{2} = -\frac{1}{2}k^2\cos\omega\exp(i\mu\omega) + \frac{k^2}{2}$$
$$= \pm\frac{1}{2}k^2\tanh[k(x-\lambda t+c)]\exp\{i\mu\arcsin[\operatorname{sech}k(x-\lambda t+c)]\} + \frac{k^2}{2}.$$

这组解类似于ⅱ, 都为ⅰ的扩展.

3　基于文献[4, 5]扩展方程(1)和(2)的精确解

文献[4, 5]($a=0$)采用齐次平衡方法获得

$$u(x,t) = \phi_x/\phi,\quad H(x,t) = -(\phi_x/\phi)^2 + \phi_{xx}/\phi, \tag{38}$$

其中 $\phi = \phi(x,t)$满足

$$\phi_t - \frac{1}{2}\phi_{xx} = 0. \tag{39}$$

事实上,(38)式为方程(1)和(2)和热传导方程(39)之间的一个 Bäcklund 变换. 设(39)式有如下形式解:

$$\phi(x,t) = \sum_{i=0}^{n} k_i(x)t^i = k_n(x)t^n + \cdots + k_1(x)t + k_0(x). \tag{40}$$

将(40)式代入(39)式, 并根据 1, t, …, t^n 线性无关, 得

$$k_n''(x) = 0, \tag{41}$$

$$nk_n(x) - \frac{1}{2}k_{n-1}''(x) = 0, \tag{42}$$

$$(n-1)k_{n-1}(x) - \frac{1}{2}k_{n-2}''(x) = 0, \tag{43}$$

………

$$k_1(x) - \frac{1}{2}k_0''(x) = 0. \tag{41'}$$

对(41)—(41)′式从上到下依次可求出 $k_i(x)(i = 1, 2, \cdots, n, 0)$

$$k_i(x) = 2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n+1-i)} a_j\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!},$$

其中 $a_j(j = 1, 2, \cdots, 2(n+1-i))$ 为任意常数. 因此, 得

$$\phi(x,t) = \sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n+1-i)} a_j\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}\right]t^i.$$

将上式代入(38)式, 得方程(1)和(2)的有理分式解:

$$u(x,t) = \frac{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n-i)+1} a_j\frac{x^{2(n-i)+1-j}}{(2(n-i)+1-j)!}\right]t^i}{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n+1-i)} a_j\frac{x^{2(n+1-i)-j}}{(2(n-i+1)-j)!}\right]t^i},$$

$$H(x,t) = \frac{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n-i)} a_j\frac{x^{2(n-i)-j}}{(2(n-i)-j)!}\right]t^j}{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n+1-i)} a_j\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}\right]t^i} - \frac{\left\{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n-i)+1} a_j\frac{x^{2(n-i)+1-j}}{(2(n-i)+1-j)!}\right]t^i\right\}^2}{\left\{\sum_{i=1}^{n}\left[2^{n-i}(n-i)!\,C_n^{n-i}\sum_{j=1}^{2(n+1-i)} a_j\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}\right]t^i\right\}^2}.$$

下面给出此解的特例(借助 Mathematica 软件).

i 当 $n = 1$ 时, 得方程(1)和(2)的有理分式解:

$$u(x,t) = \frac{3a_1x^2 + 6a_2x + 6a_3 + 3a_1t}{a_1x^3 + 3a_2x^2 + 6a_3x + 6a_4 + 3(a_1x + a_2)t},$$

$$H(x,t)=\frac{-3a_1^2x^4-12a_1a_2x^3-18a_2^2x^2+36(a_1a_4-a_2a_3)x-9a_1^2t^2}{(a_1x^3+3a_2x^2+6a_3x+6a_4+3a_1xt+3a_2t)^2}$$
$$+\frac{-36a_1a_3t+18a_2^2t+36a_2a_4-36a_3^2}{(a_1x^3+3a_2x^3+6a_3x+6a_4+3a_1xt+3a_2t)^2}.$$

ii　当 $n=2$ 时, 另得方程(1)和(2)的有理分式解:

$$u(x,t)=[3a_1t^2+6(a_1x^2+2a_2x+2a_3)t+a_1x^4+4a_2x^3+12a_3x^3+24(a_4x+a_5)]/3A(x,t),$$
$$H(x,t)=[12(a_1x+a_2)t+4(a_1x^3+a_2x^2+2a_3x+2a_4)]/3A(x,t)-\{[3a_1t^2+6(a_1x^2+2a_2x+2a_3)t+a_1x^4+4a_2x^3+12a_3x^3+24(a_4x+a_5)]^2\}/9A(x,t)^2,$$

其中

$$A(x,t)=\frac{a_1}{15}x^5+\frac{a_2}{3}x^4+\frac{4}{3}a_3x^3+4a_4x^2+8a_5x+8a_6+t\left(\frac{2}{3}a_1x^3+2a_2x^2+4a_3x+4a_4\right)+(a_1x+a_2)t^2.$$

对 $n\geqslant 3$ 由于表示式较长, 不易书写, 这里不再逐个展开.

本文仅以方程(1)和(2)来说明两种途径的有效性, 事实上, 它们同样适合于其他的非线性方程组, 其中包括高维的. 这将另文给出.

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EXPLICIT EXACT SOLUTIONS FOR NONLINEAR APPROXIMATE EQUATIONS WITH LONG WAVES IN SHALLOW WATER*

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ABSTRACT

In this paper, the nonlinear approximate equations for long wave in shallow water are studied via using two different methods. Firstly, with the aid of Mathematica and Wu-elimination method, by using sine-cosine method, three families of soliton solutions are obtained. Secondly, based on the known references, through using a new transformation, several rational formal solutions are found again. The two methods can be also applied to other nonlinear equations.

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