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广义相对论中双荷场源的电磁场张量

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提出了一种求解双荷场源电磁场张量的方法. 作为实例, 讨论了含磁荷情况下 Kerr-Newman 时空中的电磁场张量, 所得结果与直接进行复延拓 $e \to e + \mathrm{i}q$ 所得相同.

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1 引 言

对应于场源荷电又荷磁且密度之比 ρ_e/ρ_g 为常量的情况, 文献[1]引入了一个广义的电磁场张量 $\tilde{F}_{\mu\nu}$, 从而使 Einstein 场方程的求解变得类似于场源只含电荷的情况. 本文进一步讨论在这种定义下如何求真实的电磁场张量 $F_{\mu\nu}$.

2 文献[1]中的定义

文献[1]中定义广义电磁场张量 $$\tilde{F}^{\mu\nu} = F^{\mu\nu}\cos\alpha_0 + F^{+\mu\nu}\sin\alpha_0, \tag{1}$$
式中 α_0 为常实角度, $F_{\mu\nu}$为真实电磁场张量, 其定义式为
$$F_{\mu\nu} = \Delta_\nu A_\mu - \Delta_\mu A_\nu + G^+_{\mu\nu}, \tag{2}$$
式中 $G^+_{\mu\nu}$为 Dirac 弦项.

用 $\tilde{F}^{\mu\nu}$表示的 Maxwell 方程为 $$\Delta_\nu \tilde{F}^{\mu\nu} = 4\pi h u^\mu, \tag{3}$$
$$\Delta_\nu \tilde{F}^{+\mu\nu} = 0. \tag{4}$$
h, α_0 满足如下方程: $$\rho_e\cos\alpha_0 + \rho_g\sin\alpha_0 = h, \tag{5}$$
$$-\rho_e\sin\alpha_0 + \rho_g\cos\alpha_0 = 0. \tag{6}$$
(3)和(4)两式的形式与只含电荷的情况相同, 求解 Einstein-Maxwell 方程时只需将 ρ_e 换成 h, q^2 换成 $e^2 + q^2$ 就可得到荷电又荷磁的 Einstein 场方程严格解.

3 求解真实的电磁场张量

3.1 广义 Maxwell 方程

引入零标架, 并借用 Newman-Penrose 所使用的符号, 类似于只含电荷的情况可得出 $\tilde{F}_{\mu\nu}$满足的方程为 $$D\tilde{\Phi}_1 - \bar{\delta}\tilde{\Phi}_0 = (\pi - 2\alpha)\tilde{\Phi}_0 + 2\rho\tilde{\Phi}_1 - \kappa\tilde{\Phi}_2, \tag{7}$$
$$D\tilde{\Phi}_2 - \bar{\delta}\tilde{\Phi}_1 = -\lambda\tilde{\Phi}_0 + 2\pi\tilde{\Phi}_1 + (\rho - 2\varepsilon)\tilde{\Phi}_2, \tag{8}$$

$$\delta\tilde{\Phi}_1 - \Delta\tilde{\Phi}_0 = (\mu - 2\gamma)\tilde{\Phi}_0 + 2\tau\tilde{\Phi}_1 - \sigma\tilde{\Phi}_2, \tag{9}$$

$$\delta\tilde{\Phi}_2 - \Delta\tilde{\Phi}_1 = -\nu\tilde{\Phi}_0 + 2\mu\tilde{\Phi}_1 + (\tau - 2\beta)\tilde{\Phi}_2. \tag{10}$$

其中
$$\tilde{\Phi}_0 = \tilde{F}_{\mu\nu}(l^\mu m^\nu), \tag{11}$$

$$\tilde{\Phi}_1 = \frac{1}{2}\tilde{F}_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu), \tag{12}$$

$$\tilde{\Phi}_2 = \tilde{F}_{\mu\nu}(\overline{m}^\mu n^\nu). \tag{13}$$

结合场方程, 解(7)—(10)式可求出 $\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2$, 而真实的 Φ_0, Φ_1, Φ_2 可用 $\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2$ 表示出. 下面讨论 Φ_0, Φ_1, Φ_2 与 $\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2$ 的具体关系.

3.2 电磁场张量的旋量形式

设 $F_{\mu\nu}$的旋量表示为$F_{AB'CD'}$, 即有
$$F_{AB'CD'} = \sigma^\mu_{AB'}\sigma^\nu_{CD'}F_{\mu\nu}, \tag{14}$$

对应地可得出
$$F_{AB'CD'} = \frac{1}{2}(\varepsilon_{AC}F_{GB'}{}^G{}_{D'} + \varepsilon_{B'D'}F_{CG'A}{}^{G'}). \tag{15}$$

令
$$\Phi_{AC} = \frac{1}{2}F_{CG'A}{}^{G'}, \tag{16}$$

则(15)式可写成
$$F_{AB'CD'} = \varepsilon_{AC}\overline{\Phi}_{B'D'} + \Phi_{AC}\varepsilon_{B'D'}, \tag{17}$$

式中 $\overline{\Phi}_{B'D'}$ 为 Φ_{BD} 的复共轭, ε_{AC}和 $\varepsilon_{B'D'}$ 为反对称的 Levi-Civita 符号. 同理, $F_{\mu\nu}$的对偶张量 $F^+_{\mu\nu}$对应的旋量表示为
$$F^+_{AB'CD'} = \mathrm{i}(\varepsilon_{AC}\overline{\Phi}_{B'D'} - \Phi_{AC}\varepsilon_{B'D'}). \tag{18}$$

对于 $\tilde{F}_{\mu\nu}$, 其旋量形式可写为
$$\tilde{F}_{AB'CD'} = \varepsilon_{AC}\bar{\tilde{\Phi}}_{B'D'} + \tilde{\Phi}_{AC}\varepsilon_{B'D'}. \tag{19}$$

由(17)和(18)式, 有

$$\varepsilon_{AC}\bar{\tilde{\Phi}}_{B'D'} + \tilde{\Phi}_{AC}\varepsilon_{B'D'} = \varepsilon_{AC}\overline{\Phi}_{B'D'}(\cos\alpha_0 + \mathrm{i}\sin\alpha_0) + \Phi_{AC}\varepsilon_{B'D'}(\cos\alpha_0 - \mathrm{i}\sin\alpha_0). \tag{20}$$

比较上式等号两边得
$$\tilde{\Phi}_{AC} = \Phi_{AC}(\cos\alpha_0 - \mathrm{i}\sin\alpha_0) = \Phi_{AC}\mathrm{e}^{-\mathrm{i}\alpha_0}. \tag{21}$$

我们称 Φ_{AC}为真实的 Maxwell 旋量.

在旋量空间中引入一组基 $\tilde{l}_A$ 和 $\tilde{n}_A$, 它们满足正规化条件 $\tilde{l}^A\tilde{n}_A = 1$, 这个基在 3 维 E_3 空间中引出另一组基
$$\xi_{0AC} = \tilde{n}_A\tilde{n}_C, \tag{22}$$

$$\xi_{1AC} = -(\tilde{l}_A\tilde{n}_C\tilde{l}_C\tilde{n}_A), \tag{23}$$

$$\xi_{2AC} = \tilde{l}_A\tilde{l}_C. \tag{24}$$

$\tilde{\Phi}_{AC}$及 Φ_{AC}可按这组基展开
$$\tilde{\Phi}_{AC} = \sum_{m=0}^{2}\tilde{\Phi}_m\xi_{mAC}, \tag{25}$$

$$\Phi_{AC} = \sum_{m=0}^{2}\Phi_m\xi_{mAC}, \tag{26}$$

其中 Φ_0, Φ_1, Φ_2 称为二旋量 Φ_{AC}的二重分量, $\tilde{\Phi}_m$ 与 Φ_m 的关系由(25)和(26)式可得

$$\Phi_m = \mathrm{e}^{\mathrm{i}\alpha_0}\tilde{\Phi}_m = (\cos\alpha_0 + \mathrm{i}\sin\alpha_0)\tilde{\Phi}_m. \tag{27}$$

由 $\tilde{l}_A$ 和 $\tilde{n}_C$ 构造出一组零标架
$$l^\mu = \sigma^\mu_{AC'}\tilde{l}^A\bar{\tilde{l}}^{C'}, \tag{28}$$

$$n^\mu = \sigma^\mu_{AC'}\tilde{n}^A\bar{\tilde{n}}^{C'}, \tag{29}$$

$$m^{\mu} = \sigma^{\mu}_{AC'} \bar{l}^{A} \bar{\tilde{n}}^{C'}, \tag{30}$$

$$\bar{m}^{\mu} = \sigma^{\mu}_{AC'} \tilde{n}^{A} \bar{\tilde{l}}^{C'}, \tag{31}$$

则 $\tilde{\Phi}_0$, $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ 可表达成(11)—(13)式, 如前文所说, 解场方程和(7)—(10)式, 可求出 $\tilde{\Phi}_0$, $\tilde{\Phi}_1$, $\tilde{\Phi}_2$, 而 Φ_0, Φ_1, Φ_2 可由(27)式求得. 对应电磁场张量及能动张量可写为

$$\begin{aligned} F_{\mu\nu} = & -4\mathrm{Re}(\Phi_1) l_{[\mu} n_{\nu]} + 4\mathrm{i}\,\mathrm{Im}(\Phi_1) m_{[\mu}\bar{m}_{\nu]} + 2\Phi_2 l_{[\mu} m_{\nu]} \\ & + 2\bar{\Phi}_2 l_{[\mu}\bar{m}_{\nu]} - 2\Phi_0 n_{[\mu} m_{\nu]} - 2\bar{\Phi}_0 n_{[\mu}\bar{m}_{\nu]}, \end{aligned} \tag{32}$$

$$\begin{aligned} 4\pi E_{\mu\nu} = & 2\left\{ |\Phi_2|^2 l_\mu l_\nu + |\Phi_0|^2 n_\mu n_\nu + \bar{\Phi}_0 \Phi_2 m_\mu m_\nu + \Phi_0 \bar{\Phi}_2 \bar{m}_\mu \bar{m}_\nu \right\} \\ & + 4|\Phi_1|^2 \left\{ l_{(\mu} n_{\nu)} + m_{(\mu}\bar{m}_{\nu)} \right\} - 4\Phi_1 \bar{\Phi}_2 l_{(\mu} m_{\nu)} \\ & - 4\Phi_1 \bar{\Phi}_2 l_{(\mu}\bar{m}_{\nu)} - 4\bar{\Phi}_0 \Phi_1 n_{(\mu} m_{\nu)} - 4\Phi_0 \bar{\Phi}_1 n_{(\mu}\bar{m}_{\nu)}. \end{aligned} \tag{33}$$

4 举 例

对于场源含磁荷的 Kerr-Newman 度规, 可类似于只含电荷的情况, 解场方程以及(7)—(10)式. 可得

$$l_\mu = \delta^0_\mu - a\sin^2\theta\,\delta^3_\mu, \tag{34}$$

$$n_\mu = \frac{1}{2}[1 + x(e^2 + q^2 - 2mr)]\,\delta^0_\mu + \delta^1_\mu + \frac{1}{2}a\sin^2\theta[1 + x(2mr - e^2 - q^2)]\,\delta^3_\mu, \tag{35}$$

$$m_\mu = -\frac{1}{\sqrt{2}(r + \mathrm{i}a\cos\theta)}\left[\frac{1}{x}\delta^2_\mu + \mathrm{i}\sin\theta(r^2 + a^2 - a^2\sin^2\theta)\,\delta^3_\mu\right], \tag{36}$$

$$\tilde{\Phi}_0 = 0, \tag{37}$$

$$\tilde{\Phi}_1 = h_0 / \sqrt{2}(r - \mathrm{i}a\cos\theta)^2, \tag{38}$$

$$\tilde{\Phi}_2 = \mathrm{i}h_0 a\sin\theta / (r - \mathrm{i}a\cos\theta)^3, \tag{39}$$

其中 $x = (r^2 + a^2\cos^2\theta)^{-1}$, e 和 q 分别为总电荷和总磁荷, 而 $h_0 = e\cos\alpha_0 + q\sin\alpha_0$.

由(27)式得

$$\Phi_0 = \mathrm{e}^{\mathrm{i}\alpha_0}\tilde{\Phi}_0 = 0, \tag{40}$$

$$\begin{aligned} \Phi_1 = \mathrm{e}^{\mathrm{i}\alpha_0}\tilde{\Phi}_1 & = (\cos\alpha_0 + \mathrm{i}\sin\alpha_0)\frac{h_0}{\sqrt{2}(r - \mathrm{i}a\cos\theta)^2} \\ & = (\cos\alpha_0 + \mathrm{i}\sin\alpha_0)\frac{e\cos\alpha_0 + q\sin\alpha_0}{\sqrt{2}(r - \mathrm{i}a\cos\theta)^2}. \end{aligned} \tag{41}$$

又由(6)式积分得 $$-e\sin\alpha_0 + q\cos\alpha_0 = 0. \tag{42}$$

把(42)式代入(41)式得 $$\Phi_1 = (e + \mathrm{i}q)/\sqrt{2}(r - \mathrm{i}a\cos\theta)^2. \tag{43}$$

同理 $$\Phi_2 = \mathrm{e}^{\mathrm{i}\alpha_0}\tilde{\Phi}_2. \tag{44}$$

利用(42)式得 $$\Phi_2 = \frac{\mathrm{i}(e + \mathrm{i}q)a\sin\theta}{(r - \mathrm{i}a\cos\theta)^3}. \tag{45}$$

得真实的 Maxwell 旋量的分量形式为

$$\Phi_0 = 0, \tag{46}$$

$$\Phi_1 = (e + iq)/\sqrt{2}(r - ia\cos\theta)^2, \tag{47}$$

$$\Phi_2 = i(e + iq)a\sin\theta/(r - ia\cos\theta)^3. \tag{48}$$

这与把带电 Kerr-Newman 时空中[2]的 Maxwell 旋量直接进行复延拓[3] $e \to e + iq$ 所得结果相同.

[1] 张靖仪, 物理学报, **46**(1997), 2294[J. Y. Zhang, *Acta Physica Sinica*, **46**(1997), 2294(in Chinese)].

[2] E. T. Newman *et al.*, *J. Math. Phys.*, **6**(1965), 918.

[3] M. Kasuya, *Phys. Rev.*, **D25**(1982), 995.

THE ELECTROMAGNETIC FIELD TENSOR FOR THE SOURCE OF FIELD POSSESSING BOTH ELECTRIC AND MAGNETIC CHARGES IN GENERAL RELATIVITY

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ABSTRACT

A method of obtaining electromagnetic field tensor for the source of the field possessing both electric and magnetic charges is given. As an example, in Kerr-Newman space-time with a large number of magnetic charges the method is used, and the results are the same as those obtained by making electric charge a complex extension directly to $e + iq$.

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