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自旋为 5/2 的 Bargmann-Wigner 方程的严格解*

黄时中[1)2)] 阮图南[1)2)†] 吴 宁[1)3)] 郑志鹏[1)3)]

1)(中国高等科学技术中心(世界实验室),北京 100080)

2)(中国科学技术大学近代物理系,合肥 230027)

3)(中国科学院高能物理研究所,北京 100039)

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在坐标表象中严格求解自旋为 5/2 的 Bargmann-Wigner 方程,导出了自旋为 5/2 的场($m \neq 0$)的相对论性方程和动量表象波函数.

关键词:自旋为 5/2 的场($m \neq 0$), Bargmann-Wigner 方程, 坐标表象, 严格解

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1 引 言

自旋为 5/2 的场的相对论性波动方程及其解在理论和应用上都具有相当重要的地位. 在理论上,自旋为 5/2 的场的相对论理论是自旋为任意半整数场的相对论性理论的基础;在应用上,许多高能物理过程的振幅分析依赖于这类场的动量表象波函数[1],另一方面,自旋为 5/2 的粒子的相对论性波函数对于在原子结构计算中深入考虑核与电子之间的相互作用有重要应用.

尽管在 Dirac 和 Fierz 等[2—4]工作的基础上, Bargmann 和 Wigner[5]早就系统地给出了任意自旋的相对论性波动方程,但是可以说,只是对自旋为½、1、3⁄2 的体系有严格系统的求解过程(例如,见 Salam 的论文[6]). 最近,我们提供了一种在动量表象中构造自旋为整数的相对论性波函数的方案[7,8]. 通过进一步的研究,本文将严格求解自旋为 5/2 的 Bargmann-Wigner 方程,在坐标表象中,给出自旋为 5/2 的场的相对论性波动方程和动量表象波函数.

2 Bargmann-Wigner 方程之解

自旋为 5/2 的 Bargmann-Wigner 方程为[5]

$$(\partial\!\!\!/ + m)_{\alpha\alpha'}\Psi_{\alpha'\beta\delta\tau\rho}(x) = 0, \tag{1a}$$

$$(\partial\!\!\!/ + m)_{\beta\beta'}\Psi_{\alpha\beta'\delta\tau\rho}(x) = 0, \tag{1b}$$

$$(\partial\!\!\!/ + m)_{\delta\delta'}\Psi_{\alpha\beta\delta'\tau\rho}(x) = 0, \tag{1c}$$

$$(\partial\!\!\!/ + m)_{\tau\tau'}\Psi_{\alpha\beta\delta\tau'\rho}(x) = 0, \tag{1d}$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\Psi_{\alpha\beta\delta\tau\rho'}(x) = 0, \tag{1e}$$

其中 $\Psi_{\alpha\beta\delta\tau\rho}(x)$ 为 5 阶全对称多重旋量波函数. 为了便于在坐标表象中进行求解,我们借鉴求解自旋为 2 的 Bargmann-Wigner 方程的方法[8]. 例如,自旋为 2 的 Bargmann-Wigner 方程为[5]

$$(\partial\!\!\!/ + m)_{\alpha\alpha'}\Psi_{\alpha'\beta\delta\tau}(x) = 0, \tag{2a}$$

$$(\partial\!\!\!/ + m)_{\beta\beta'}\Psi_{\alpha\beta'\delta\tau}(x) = 0, \tag{2b}$$

$$(\partial\!\!\!/ + m)_{\delta\delta'}\Psi_{\alpha\beta\delta'\tau}(x) = 0, \tag{2c}$$

$$(\partial\!\!\!/ + m)_{\tau\tau'}\Psi_{\alpha\beta\delta\tau'}(x) = 0, \tag{2d}$$

其中 $\Psi_{\alpha\beta\delta\tau}(x)$ 为 4 阶全对称多重旋量波函数,按文献[8],它的解为

$$\Psi_{\alpha\beta\delta\tau}(x) = (\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta} \cdot (\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}A^{\nu_1\nu_2}(x), \tag{3}$$

其中 $C = \gamma_2\gamma_4$ 为电荷共轭矩阵;$\gamma_\mu C$ 和 $\Sigma_{\mu\nu}C$ 为对称矩阵;$A^{\nu_1\nu_2}(x)$ 为二阶张量场,满足下列方程:

$$(\Box - m^2)A^{\nu_1\nu_2}(x) = 0, \tag{4a}$$

$$\partial_{\nu_1}A^{\nu_1\nu_2}(x) = 0, \tag{4b}$$

$$\partial_{\nu_2}A^{\nu_1\nu_2}(x) = 0, \tag{4c}$$

$$A^{\nu\nu}(x) = 0, \tag{4d}$$

$$A^{\nu_1\nu_2}(x) = A^{\nu_2\nu_1}(x). \tag{4e}$$

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† 通讯联系人:阮图南,合肥 4 信箱 4 系,合肥 230027.

由于方程(1a)—(1d)与方程(2a)—(2d)关于旋量指标 $\alpha\beta\delta\tau$ 所满足的 Dirac 方程相同,并且 $\Psi_{\alpha\beta\delta\tau\rho}(x)$ 关于旋量指标 $\alpha\beta\delta\tau$ 是对称的,所以采用与推导(3)和(4)式完全相同的步骤,可得到

$$\Psi_{\alpha\beta\delta\tau\rho}(x) = \left(\mathrm{i}m\gamma_{\nu_1} C + \Sigma_{\mu_1\nu_1} C\partial_{\mu_1}\right)_{\alpha\beta} \cdot \left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau} \Psi_{\rho}^{\nu_1\nu_2}(x), \quad (5)$$

其中 $\Psi_{\rho}^{\nu_1\nu_2}(x)$ 为二阶张量-旋量,满足下列方程:

$$(\Box - m^2)\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \quad (6a)$$

$$\partial_{\nu_1}\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \quad (6b)$$

$$\partial_{\nu_2}\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \quad (6c)$$

$$\Psi_{\rho}^{\nu\nu}(x) = 0, \quad (6d)$$

$$\Psi_{\rho}^{\nu_1\nu_2}(x) = \Psi_{\rho}^{\nu_2\nu_1}(x). \quad (6e)$$

由于 $\Psi_{\rho}^{\nu_1\nu_2}(x)$ 关于指标 ρ 满足 Dirac 方程,将(5)式代入(1e)式得到

$$\left(\mathrm{i}m\gamma_{\nu_1} C + \Sigma_{\mu_1\nu_1} C\partial_{\mu_1}\right)_{\alpha\beta}\left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau} \cdot (\partial\!\!\!/ + m)_{\rho\rho'}\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0,$$

利用 γ 矩阵的独立性,上式给出

$$(\partial\!\!\!/ + m)_{\rho\rho'}\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0. \quad (6f)$$

另一方面,(5)式等号右边关于旋量指标 $\alpha\beta\delta\tau$ 是对称的. 为保证 $\Psi_{\alpha\beta\delta\tau\rho}(x)$ 关于旋量指标 $\alpha\beta\delta\tau\rho$ 是全对称的,我们进一步要求(5)式等号右边关于旋量指标 $\delta\tau\rho$ 也是对称的. 为此,我们要求(5)式中的因子 $\left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau}\Psi_{\rho}^{\nu_1\nu_2}(x)$ 与所有反对称 Dirac 矩阵,即 C^{-1}, $C^{-1}\gamma_5$, $C^{-1}\gamma_5\gamma_\lambda$,关于指标 $\tau\rho$ 的收缩为零,亦即

$$\left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau}\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1})_{\tau\rho} = 0, \quad (7a)$$

$$\left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau}\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1}\gamma_5)_{\tau\rho} = 0, \quad (7b)$$

$$\left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau}\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\tau\rho} = 0. \quad (7c)$$

展开(7)式,利用 γ 矩阵的乘法公式将每一等式左边化为 γ 矩阵的线性组合,并注意利用(6b)—(6e)式,可分别得到(以下按惯例在场方程中隐去旋量指标 ρ)

$$\gamma_{\nu_2}(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2}(x) = 0, \quad (8a)$$

$$-2m\gamma_\nu\Psi^{\nu\nu_2}(x) + \gamma_{\nu_2}(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2}(x) = 0, \quad (8b)$$

$$(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2}(x) = 0. \quad (8c)$$

由(6f)式可知,(8a)和(8c)式为恒等式,而(8b)式等号左端后一项为零,但前一项给出

$$\gamma_\nu\Psi^{\nu\nu_2}(x) = 0, \quad (6g)$$

综合以上结果,我们得到自旋为 5/2 的 Bargmann-Wigner 方程的解为

$$\Psi_{\alpha\beta\delta\tau\rho}(x) = \left(\mathrm{i}m\gamma_{\nu_1} C + \Sigma_{\mu_1\nu_1} C\partial_{\mu_1}\right)_{\alpha\beta} \cdot \left(\mathrm{i}m\gamma_{\nu_2} C + \Sigma_{\mu_2\nu_2} C\partial_{\mu_2}\right)_{\delta\tau} \Psi_{\rho}^{\nu_1\nu_2}(x), \quad (9)$$

其中 $\Psi_{\rho}^{\nu_1\nu_2}(x)$ 满足下列方程(隐去旋量指标):

$$(\Box - m^2)\Psi^{\nu_1\nu_2}(x) = 0, \quad (10a)$$

$$\partial_{\nu_1}\Psi^{\nu_1\nu_2}(x) = 0, \quad (10b)$$

$$\partial_{\nu_2}\Psi^{\nu_1\nu_2}(x) = 0, \quad (10c)$$

$$\Psi^{\nu\nu}(x) = 0, \quad (10d)$$

$$\Psi^{\nu_1\nu_2}(x) = \Psi^{\nu_2\nu_1}(x), \quad (10e)$$

$$(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2}(x) = 0, \quad (10f)$$

$$\gamma_\nu\Psi^{\nu\nu_2}(x) = 0. \quad (10g)$$

3 自旋为 5/2 的场的平面波展开

下面根据场方程(10),将 $\Psi^{\nu_1\nu_2}(x)$ 作平面波展开

$$\Psi^{\nu_1\nu_2}(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}px}\Psi^{\nu_1\nu_2}(p), \quad (11)$$

代入方程(10a)式得到

$$(p^2 + m^2)\Psi^{\nu_1\nu_2}(p) = 0. \quad (12)$$

按照博格留波夫方法[9],利用 $x\delta(x) = 0$,可将满足上式的 $\Psi^{\nu_1\nu_2}(p)$ 写作

$$\Psi^{\nu_1\nu_2}(p) = \delta(p^2 + m^2)B^{\nu_1\nu_2}(p) = \frac{1}{2E}[\delta(E - p_0) + \delta(E + p_0)] \cdot B^{\nu_1\nu_2}(p), \quad (13)$$

其中 $E = \sqrt{\boldsymbol{p}^2 + m^2}$. 将(13)式代入(11)式并对 p_0 进行积分可得到

$$\Psi^{\nu_1\nu_2}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^4}\frac{1}{2E}\left[\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r} - \mathrm{i}Et}B^{\nu_1\nu_2}(\boldsymbol{p}, E) + \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r} + \mathrm{i}Et}B^{\nu_1\nu_2}(-\boldsymbol{p}, -E)\right]. \quad (14)$$

过渡到分立形式,并作符号简化,得到

$$\Psi^{\nu_1\nu_2}(x) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}}\left[a^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right], \quad (15)$$

其中 $a^{\nu_1\nu_2}(\boldsymbol{p})$ 和 $b^{\nu_1\nu_2}(\boldsymbol{p})$ 分别对应于正能解和负能解,V 是归一化体积.

将(15)式代入(10b)—(10e)式可得到

$$\begin{cases} p_{\nu_1} a^{\nu_1\nu_2}(\boldsymbol{p}) = 0, \\ p_{\nu_1} b^{\nu_1\nu_2}(\boldsymbol{p}) = 0, \end{cases} \quad \begin{cases} p_{\nu_2} a^{\nu_1\nu_2}(\boldsymbol{p}) = 0, \\ p_{\nu_2} b^{\nu_1\nu_2}(\boldsymbol{p}) = 0, \end{cases} \tag{16a}$$

$$a^{\nu\nu}(\boldsymbol{p}) = 0, \quad b^{\nu\nu}(\boldsymbol{p}) = 0, \tag{16b}$$

$$a^{\nu_1\nu_2}(\boldsymbol{p}) = a^{\nu_2\nu_1}(\boldsymbol{p}), \quad b^{\nu_1\nu_2}(\boldsymbol{p}) = b^{\nu_2\nu_1}(\boldsymbol{p}). \tag{16c}$$

因为 $p_\mu e^\mu_\lambda(\boldsymbol{p})=0$,方程(16a)之解为

$$a^{\nu_1\nu_2}(\boldsymbol{p}) = e^{\nu_1}_{\lambda_1}(\boldsymbol{p}) e^{\nu_2}_{\lambda_2}(\boldsymbol{p}) a_{\lambda_1\lambda_2}(\boldsymbol{p}) \quad (\lambda_1,\lambda_2 = 1,0,-1), \tag{17a}$$

$$b^{\nu_1\nu_2}(\boldsymbol{p}) = \bar{e}^{\nu_1}_{\lambda_1}(\bar{\boldsymbol{p}}) \bar{e}^{\nu_2}_{\lambda_2}(\boldsymbol{p}) b^+_{\lambda_1\lambda_2}(\boldsymbol{p}), \tag{17b}$$

其中 $a_{\lambda_1\lambda_2}(\boldsymbol{p})$和 $b^+_{\lambda_1\lambda_2}(\boldsymbol{p})$待定,$e^\nu_\lambda(\boldsymbol{p})$为自旋为 1 的螺旋度算符 $S\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}$的本征态,本征值为 $\lambda=1,0,-1$,即

$$e^\nu_{+1}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\cos\phi + i\sin\phi \\ -\cos\theta\sin\phi - i\cos\phi \\ \sin\theta \\ 0 \end{pmatrix},$$

$$e^\nu_0(\boldsymbol{p}) = \begin{pmatrix} (E/m)\sin\theta\cos\phi \\ (E/m)\sin\theta\sin\phi \\ (E/m)\cos\theta \\ i|\boldsymbol{p}|/m \end{pmatrix}, \tag{18a}$$

$$e^\nu_{-1}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \\ 0 \end{pmatrix}. \tag{18b}$$

这里 θ,ϕ 是$\boldsymbol{p}$ 的方位角,而

$$\bar{e}^\nu_\lambda(\boldsymbol{p}) = g_{\nu\mu}(e^\mu_\lambda(\boldsymbol{p}))^*, \quad g_{\nu\mu} = \mathrm{diag}\{1,1,1,-1\}. \tag{19}$$

在静止系中,$\boldsymbol{p}=0, E=m$,上式简化为

$$e^\nu_{+1}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \\ 0 \end{pmatrix}, \quad e^\nu_0(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$e^\nu_{-1}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

它们是 S_z 的本征态,本征值依次为 1,0,-1. $e^\nu_\lambda(\boldsymbol{p})$和 $e^\nu_\lambda(0)$之间存在如下洛沦兹变换关系:

$$e^\nu_\lambda(\boldsymbol{p}) = L^{\nu\nu'} e^{\nu'}_\lambda(0) \quad (\lambda = 1,0,-1). \tag{21}$$

此处,

$$L = e^{-iS_3\phi} e^{-iS_2\theta} e^{\varepsilon K_3} \quad (\mathrm{ch}\varepsilon = E/m, \mathrm{sh}\varepsilon = |\boldsymbol{p}|/m) \tag{22}$$

是先沿 z 轴推动(Boost)再转动到 $\boldsymbol{p}$ 方向的洛沦兹变换,其显式为

$$L = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & (E/m)\sin\theta\cos\phi & -ip_1/m \\ \cos\theta\sin\phi & \cos\phi & (E/m)\sin\theta\sin\phi & -ip_2/m \\ -\sin\theta & 0 & (E/m)\cos\theta & -ip_3/m \\ 0 & 0 & i|\boldsymbol{p}|/m & E/m \end{pmatrix}. \tag{23}$$

将(17)式代回方程(16b)得到

$$e^\nu_{\lambda_1}(\boldsymbol{p}) e^\nu_{\lambda_2}(\boldsymbol{p}) a_{\lambda_1\lambda_2}(\boldsymbol{p}) = 0, \tag{24a}$$

$$\bar{e}^\nu_{\lambda_1}(\boldsymbol{p}) \bar{e}^\nu_{\lambda_2}(\boldsymbol{p}) b^+_{\lambda_1\lambda_2}(\boldsymbol{p}) = 0. \tag{24b}$$

利用(21)式,并注意到 $L^{\nu\nu_1}L^{\nu\nu_2}=\delta_{\nu_1\nu_2}$,可将(24)式改写作

$$e^\nu_{\lambda_1}(0) e^\nu_{\lambda_2}(0) a_{\lambda_1\lambda_2}(\boldsymbol{p}) = 0, \tag{25a}$$

$$\bar{e}^\nu_{\lambda_1}(0) \bar{e}^\nu_{\lambda_2}(0) b^+_{\lambda_1\lambda_2}(\boldsymbol{p}) = 0. \tag{25b}$$

先讨论方程(25a)式. $a_{\lambda_1\lambda_2}(\boldsymbol{p})$只与两个磁量子数($\lambda_1,\lambda_2=1,0,-1$)有关,联系到两个自旋为 1 的角动量耦合的 CG 系数,可将 $a_{\lambda_1\lambda_2}(\boldsymbol{p})$一般地表示为

$$\begin{aligned} a_{\lambda_1\lambda_2}(\boldsymbol{p}) = & \sum_m \langle 1,\lambda_1;1,\lambda_2 | 1,1,2,m\rangle a_{2m}(\boldsymbol{p}) \\ & + \sum_{m'} \langle 1,\lambda_1;1,\lambda_2 | 1,1,1,m'\rangle a_{1m'}(\boldsymbol{p}) \\ & + \langle 1,\lambda_1;1,\lambda_2 | 1,1,0,0\rangle a_{00}(\boldsymbol{p}) \\ & (m = 2,1,0,-1,-2; m' = 1,0,-1), \end{aligned} \tag{26}$$

其中$\langle S_1,\lambda_1;S_2,\lambda_2 | S_1,S_2,S,m\rangle$是两个自旋为 1 的角动量 S_1 与 S_2 耦合为总角动量 $S=S_1+S_2$($S=2,1,0$)的 CG 系数. 令

$$e^{\nu_1\nu_2}_{2m}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0) e^{\nu_2}_{\lambda_2}(0) \cdot \langle 1,\lambda_1;1,\lambda_2 | 1,1,2,m\rangle, \tag{27a}$$

$$e^{\nu_1\nu_2}_{1m'}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0) e^{\nu_2}_{\lambda_2}(0) \cdot \langle 1,\lambda_1;1,\lambda_2 | 1,1,1,m'\rangle, \tag{27b}$$

$$e^{\nu_1\nu_2}_{00}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0) e^{\nu_2}_{\lambda_2}(0) \cdot \langle 1,\lambda_1;1,\lambda_2 | 1,1,0,0\rangle. \tag{27c}$$

将(26)和(27)式代入(25a)式得到

$$e^{\nu\nu}_{2m}(0) a_{2m}(\boldsymbol{p}) + e^{\nu\nu}_{1m'}(0) a_{1m'}(\boldsymbol{p}) + e^{\nu\nu}_{00}(0) a_{00}(\boldsymbol{p}) = 0. \tag{28}$$

计算出各个 CG 系数之值,可得到(27)式的显式为

$$e^{\nu_1\nu_2}_{22}(0) = e^{\nu_1}_{+1}(0) e^{\nu_2}_{+1}(0), \tag{29a}$$

$$e_{21}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{2}}\left[e_{+1}^{\nu_1}(0)e_0^{\nu_2}(0) + e_0^{\nu_1}(0)e_{+1}^{\nu_2}(0)\right], \tag{29b}$$

$$e_{20}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{6}}\left[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) + 2e_0^{\nu_1}(0)e_0^{\nu_2}(0) + e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)\right], \tag{29c}$$

$$e_{2-1}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{2}}\left[e_0^{\nu_1}(0)e_{-1}^{\nu_2}(0) + e_{-1}^{\nu_1}(0)e_0^{\nu_2}(0)\right], \tag{29d}$$

$$e_{2-2}^{\nu_1\nu_2}(0) = e_{-1}^{\nu_1}(0)e_{-1}^{\nu_2}(0), \tag{29e}$$

$$e_{11}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{2}}\left[e_{+1}^{\nu_1}(0)e_0^{\nu_2}(0) - e_0^{\nu_1}(0)e_{+1}^{\nu_2}(0)\right], \tag{30a}$$

$$e_{10}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{2}}\left[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)\right], \tag{30b}$$

$$e_{1-1}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{2}}\left[e_0^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_{-1}^{\nu_1}(0)e_0^{\nu_2}(0)\right], \tag{30c}$$

$$e_{00}^{\nu_1\nu_2}(0) = \frac{1}{\sqrt{3}}\left[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_0^{\nu_1}(0)e_0^{\nu_2}(0) + e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)\right]. \tag{31}$$

利用(20)式,不难算出

$$e_{2m}^{\nu\nu}(0) = 0(m = 2,1,0,-1,-2),\ e_{1m'}^{\nu\nu}(0) = 0(m' = 1,0,-1),\ e_{00}^{\nu\nu}(0) = -\sqrt{3}. \tag{32}$$

将(32)式代入(28)式得到

$$a_{00}(\boldsymbol{p}) = 0. \tag{33}$$

于是(26)式简化为

$$a_{\lambda_1\lambda_2}(\boldsymbol{p}) = \sum_m \langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle a_{2m}(\boldsymbol{p}) + \sum_{m'}\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,1,m'\rangle \cdot a_{1m'}(\boldsymbol{p}). \tag{34a}$$

同理,由方程(25b)可得到

$$b^+_{\lambda_1\lambda_2}(\boldsymbol{p}) = \sum_m \langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle b^+_{2m}(\boldsymbol{p}) + \sum_{m'}\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,1,m'\rangle \cdot b^+_{1m'}(\boldsymbol{p}). \tag{34b}$$

另一方面,对称性条件(16c)式要求

$$a_{\lambda_1\lambda_2}(\boldsymbol{p}) = a_{\lambda_2\lambda_1}(\boldsymbol{p}),\quad b^+_{\lambda_1\lambda_2}(\boldsymbol{p}) = b^+_{\lambda_2\lambda_1}(\boldsymbol{p}). \tag{35}$$

但 CG 系数的对称性给出

$$\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle = \langle 1,\lambda_2;1,\lambda_1 \mid 1,1,2,m\rangle, \tag{36a}$$

$$\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,1,m'\rangle = -\langle 1,\lambda_2;1,\lambda_1 \mid 1,1,1,m'\rangle. \tag{36b}$$

将(34)式代入(35)式,并利用(36)式得到

$$a_{1m'}(\boldsymbol{p}) = 0,\ b^+_{1m'}(\boldsymbol{p}) = 0. \tag{37}$$

于是,(34)式简化为

$$a_{\lambda_1\lambda_2}(\boldsymbol{p}) = \sum_m \langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle a_{2m}(\boldsymbol{p}), \tag{38a}$$

$$b^+_{\lambda_1\lambda_2}(\boldsymbol{p}) = \sum_m \langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle b^+_{2m}(\boldsymbol{p}). \tag{38b}$$

将(38)式代回(17)式得到

$$a^{\nu_1\nu_2}(\boldsymbol{p}) = e_{2m}^{\nu_1\nu_2}(\boldsymbol{p})a_{2m}(\boldsymbol{p}),\quad b^{\nu_1\nu_2}(\boldsymbol{p}) = \bar{e}_{2m}^{\nu_1\nu_2}(\boldsymbol{p})b^+_{2m}(\boldsymbol{p}), \tag{39}$$

其中 $e_{2m}^{\nu_1\nu_2}(\boldsymbol{k}) = \sum_{\lambda_1\lambda_2} e_{\lambda_1}^{\nu_1}(\boldsymbol{k})e_{\lambda_2}^{\nu_2}(\boldsymbol{k})\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,m\rangle\quad (m = 2,1,0,-1,-2),$ (40)

$$\bar{e}_{2m}^{\nu_1\nu_2}(\boldsymbol{k}) = g_{\nu_1\mu_1}g_{\nu_1\mu_2}(e_{2m}^{\mu_1\mu_2}(\boldsymbol{k}))^*. \tag{41}$$

将(39)式代回(15)式,略去下标 2,同时将量子数 m 换成λ_{12}(便于后面的论述)我们得到

$$\Psi^{\nu_1\nu_2}(x) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}}\left[a_{\lambda_{12}}(\boldsymbol{p})e_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b^+_{\lambda_{12}}(\boldsymbol{p})\bar{e}_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right], \tag{42}$$

其中

$$e_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{k}) = \sum_{\lambda_1\lambda_2} e_{\lambda_1}^{\nu_1}(\boldsymbol{k})e_{\lambda_2}^{\nu_2}(\boldsymbol{k})\langle 1,\lambda_1;1,\lambda_2 \mid 1,1,2,\lambda_{12}\rangle\quad (\lambda_{12} = 0,\pm 1,\pm 2), \tag{43a}$$

$$\bar{e}_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{k}) = g_{\nu_1\mu_1}g_{\nu_1\mu_2}(e_{\lambda_{12}}^{\mu_1\mu_2}(\boldsymbol{k}))^*. \tag{43b}$$

将(42)式代入(10f)式得到

$$(\mathrm{i}\not{p} + m)a_{\lambda_{12}}(\boldsymbol{p}) = 0, \tag{44a}$$

$$(-\mathrm{i}\not{p} + m)b^+_{\lambda_{12}}(\boldsymbol{p}) = 0. \tag{44b}$$

这是正能旋量和负能旋量的 Dirac 方程,其解为

$$a_{\lambda_{12}}(\boldsymbol{p}) = u_r(\boldsymbol{p})a_{\lambda_{12},r}(\boldsymbol{p})\quad \left(r = \frac{1}{2},-\frac{1}{2}\right), \tag{45a}$$

$$b^+_{\lambda_{12}}(\boldsymbol{p}) = v_r(\boldsymbol{p})b^+_{\lambda_{12},r}(\boldsymbol{p}),\ v_r(\boldsymbol{p}) = \gamma_2 u_r(\boldsymbol{p})^*, \tag{45b}$$

其中 $u_r(\boldsymbol{p})$和 $v_r(\boldsymbol{p})$分别为正能和负能 Dirac 旋量,可表示为

$$u_r(\boldsymbol{p}) = \Lambda u_r(0)\sqrt{\frac{m}{E}},\ v_r(\boldsymbol{p}) = \Lambda v_r(0)\sqrt{\frac{m}{E}}. \tag{46}$$

此处
$$\Lambda = e^{-i\Sigma_3\phi/2}e^{-i\Sigma_2\theta/2}e^{\alpha_3\varepsilon/2}, \tag{47}$$

而
$$u_{1/2}(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ u_{-1/2}(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$
$$v_{1/2}(0)=\begin{pmatrix}0\\0\\0\\-1\end{pmatrix},\ v_{-1/2}(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{48}$$

是静止系中的正能和负能 Dirac 旋量.

将(45)式代入(42)式得到
$$\Psi^{\nu_1\nu_2}(x)=\frac{1}{\sqrt{V}}\sum_{p}\Big[e^{\nu_1\nu_2}_{\lambda_{12}}(p)u_r(p)a_{\lambda_{12},r}(p)e^{ipx} + \bar e^{\nu_1\nu_2}_{\lambda_{12}}(p)v_r(p)b^+_{\lambda_{12},r}(p)e^{-ipx}. \tag{49}$$

将(49)式代入(10g)式得到
$$\gamma_\nu e^{\nu\nu_2}_{\lambda_{12}}(p)u_r(p)a_{\lambda_{12},r}(p)=0, \tag{50a}$$
$$\gamma_\nu \bar e^{\nu\nu_2}_{\lambda_{12}}(p)v_r(p)b^+_{\lambda_{12},r}(p)=0. \tag{50b}$$

利用(43b)式以及 $\gamma_2\gamma_\nu^*\gamma_2=g_{\nu\mu}\gamma_\mu$ 和 $v_r(p)=\gamma_2u_r(p)^*$,不难证明(50b)与(50a)式是等价的,因此只需讨论方程(50a). 借助(21)和(46)式,并注意到
$$L^{\nu\nu_1}L^{\nu\nu_2}=\delta_{\nu_1\nu_2},\ \Lambda^{-1}\gamma_\nu\Lambda=L^{\nu\nu'}\gamma_{\nu'}. \tag{51}$$

可将(50a)式改写作
$$\gamma_\nu e^{\nu\nu_2}_{\lambda_{12}}(0)u_r(0)a_{\lambda_{12},r}(p)=0, \tag{52}$$

$a_{\lambda_{12},r}(p)$只与两个磁量子数($\lambda_{12}=2,1,0,-1,-2$;$r=1/2,-1/2$)有关,联系到自旋为 2 和自旋为 1/2 的两个角动量耦合的 CG 系数,可将 $a_{\lambda_{12},r}(p)$一般地表示为
$$a_{\lambda_{12},r}(p)=\sum_m\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac52,m\Big\rangle\cdot a_{\frac52,m}(p)+\sum_{m'}\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac32,m'\Big\rangle a_{\frac32,m'}(p)$$
$$\Big(m=\pm\frac12,\pm\frac32,\pm\frac52;\ m'=\pm\frac12,\pm\frac32\Big). \tag{53}$$

令
$$U^{\nu_1\nu_2}_{\frac52,m}(p)=\sum_{\lambda_{12}r}e^{\nu_1\nu_2}_{\lambda_{12}}(p)u_r(p)\cdot\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac52,m\Big\rangle, \tag{54a}$$
$$U^{\nu_1\nu_2}_{\frac32,m'}(p)=\sum_{\lambda_{12}r}e^{\nu_1\nu_2}_{\lambda_{12}}(p)u_r(p)\cdot\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac32,m'\Big\rangle, \tag{54b}$$

则(52)式可写作
$$\gamma_\nu U^{\nu\nu_2}_{\frac52,m}(0)a_{\frac52,m}(p)+\gamma_\nu U^{\nu\nu_2}_{\frac32,m'}(0)a_{\frac32,m'}(p)=0. \tag{55}$$

计算出(54)式中的 CG 系数,并利用由(20)和(48)式所导出的下列关系:
$$\gamma_\nu e^\nu_{+1}(0)u_{\frac12}(0)=0,$$
$$\gamma_\nu e^\nu_{+1}(0)u_{-\frac12}(0)=i\sqrt2\gamma_5u_{\frac12}(0),$$
$$\gamma_\nu e^\nu_0(0)u_{\frac12}(0)=-i\gamma_5u_{\frac12}(0),$$
$$\gamma_\nu e^\nu_0(0)u_{-\frac12}(0)=i\gamma_5u_{-\frac12}(0),$$
$$\gamma_\nu e^\nu_{-1}(0)u_{\frac12}(0)=-i\sqrt2\gamma_5u_{-\frac12}(0),$$
$$\gamma_\nu e^\nu_{-1}(0)u_{-\frac12}(0)=0,$$

可以计算出
$$\gamma_\nu U^{\nu\nu_2}_{\frac52,m}(0)=0\qquad\Big(m=\pm\frac12,\pm\frac32,\pm\frac52\Big), \tag{56a}$$
$$\gamma_\nu U^{\nu\nu_2}_{\frac32,m'}(0)=i\sqrt{\frac52}\sum_{\lambda_2r}e^{\nu_2}_{\lambda_2}(0)\gamma_5u_r(0)\cdot\Big\langle 1,\lambda_2;\frac12,r\,\Big|\,1,\frac12,\frac32,m'\Big\rangle\neq0$$
$$\Big(m'=\pm\frac12,\pm\frac32\Big). \tag{56b}$$

将(56)式代入(55)式得到
$$a_{\frac32,m'}(p)=0. \tag{57}$$

于是(53)式简化为(略去 $a_{\frac52,m'}(p)$中的下标$\frac52$)
$$a_{\lambda_{12},r}(p)=\sum_m\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac52,m\Big\rangle a_m(p). \tag{58a}$$

同理,由(50b)式可得到
$$b^+_{\lambda_{12},r}(p)=\sum_m\Big\langle 2,\lambda_{12};\frac12,r\,\Big|\,2,\frac12,\frac52,m\Big\rangle b^+_m(p). \tag{58b}$$

将(58)式代入(49)式,最终得到
$$\Psi^{\nu_1\nu_2}(x)=\frac{1}{\sqrt V}\sum_{p,m}\Big[a_m(p)U^{\nu_1\nu_2}_m(p)e^{ipx}+b^+_m(p)V^{\nu_1\nu_2}_m(p)e^{-ipx}\Big], \tag{59}$$

其中

$$U_m^{\nu_1\nu_2}(p) = \sum_{\lambda_{12},r} e_{\lambda_{12}}^{\nu_1\nu_2}(p)\,u_r(p)\left\langle 2,\lambda_{12};\frac{1}{2},r\,\middle|\,2,\frac{1}{2},\frac{5}{2},m\right\rangle$$

$$\equiv \sum_{\lambda_1,\lambda_2=-1}^{1}\sum_{r=-\frac{1}{2}}^{\frac{1}{2}} e_{\lambda_1}^{\nu_1}(p)\,e_{\lambda_2}^{\nu_2}(p)\,u_r(p)\,\delta(\lambda_1+\lambda_2+r,m)$$

$$\cdot\sqrt{\frac{\left(\frac{5}{2}+m\right)!\left(\frac{5}{2}-m\right)!}{30(1+\lambda_1)!(1-\lambda_1)!(1+\lambda_2)!(1-\lambda_2)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}, \tag{60a}$$

$$V_m^{\nu_1\nu_2}(p) = \bar{e}_{\lambda_{12}}^{\nu_1\nu_2}(p)\,v_r(p)\left\langle 2,\lambda_{12};\frac{1}{2},r\,\middle|\,2,\frac{1}{2},\frac{5}{2},m\right\rangle$$

$$\equiv \sum_{\lambda_1,\lambda_2=-1}^{1}\sum_{r=-\frac{1}{2}}^{\frac{1}{2}} \bar{e}_{\lambda_1}^{\nu_1}(p)\,\bar{e}_{\lambda_2}^{\nu_2}(p)\,v_r(p)\,\delta(\lambda_1+\lambda_2+r,m)$$

$$\cdot\sqrt{\frac{\left(\frac{5}{2}+m\right)!\left(\frac{5}{2}-m\right)!}{30(1+\lambda_1)!(1-\lambda_1)!(1+\lambda_2)!(1-\lambda_2)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}. \tag{60b}$$

上式最后一步利用了 CG.系数的 Wigner 公式. 这就是自旋为 5/2 的场的平面波展开. 利用 $e_\lambda^\nu(p)$,$u_r(p)$和 $v_r(p)$的正交归一性以及 CG 系数的幺正性,不难证明正能解的集合和负能解的集合都是正交归一的.

4　拉氏密度

最后,我们给出自旋为 5/2 的场的拉氏密度

$$L(x) = \bar{\Psi}^{\mu\nu'}(x)(\partial\!\!\!/ + m)\Psi^{\mu\nu'}(x)$$

$$-\frac{1}{3}\bar{\Psi}^{\mu\nu'}(x)(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\Psi^{\nu\nu'}(x)$$

$$+\frac{1}{3}\bar{\Psi}^{\mu\nu'}(x)\gamma_\mu(\partial\!\!\!/ - m)\gamma_\nu\Psi^{\nu\nu'}(x), \tag{61}$$

其中 $\bar{\Psi}^{\nu\nu'}(x) = g_{\nu\mu}g_{\nu'\mu'}\gamma_4(\Psi^{\mu\mu'}(x))^+$,$\Psi^{\nu\nu'}(x)$为对称张量-旋量,且 $\Psi^{\nu\nu}(x)=0$.

由拉氏函数给出的运动方程为

$$(\Box - m^2)\Psi^{\nu\nu'}(x) = 0,\ \partial_\nu\Psi^{\nu\nu'}(x) = 0, \tag{62a}$$

$$(\partial\!\!\!/ + m)\Psi^{\nu\nu'}(x) = 0,\ \gamma_\nu\Psi^{\nu\nu'}(x) = 0. \tag{62b}$$

它们与方程(10)是一致的.

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A RIGOROUS SOLUTION TO BARGMANN-WIGNER EQUATION FOR SPIN 5/2*

HUANG SHI-ZHONG[1)2)] RUAN TU-NAN[1)2)] WU NING[1)3)] ZHENG ZHI-PENG[1)3)]

[1)] (*China Center of Advanced Science and Technology* (*CCAST*) (*World Laboratory*), *Beijing* 100080, *China*)

[2)] (*Department of Modern Physics*, *University of Science and Technology of China*, *Hefei* 230027, *China*)

[3)] (*Institute of High Energy Physics*, *Chinese Academy of Sciences*, *Beijing* 100039, *China*)

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ABSTRACT

A rigorous solution of Bargmann-Wigner equation for spin-5/2 in coordinate representation is given in this paper. A relativistic equation and the representative function of momentum for the spin-5/2 are deduced.

Keywords: spin-5/2 fields, Bargmann-Wigner equation, coordinate representation, rigorous solution

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