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Poincaré-Chetaev 方程的统一对称性

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研究 Poincaré-Chetaev 方程的统一对称性及其导致的守恒量. 给出 Poincaré-Chetaev 方程统一对称性的定义和判据, 得到 Poincaré-Chetaev 方程统一对称性的三种守恒量:Noether 守恒量、Hojman 守恒量和 Mei 守恒量.

关键词:Poincaré-Chetaev 方程, 统一对称性, 守恒量

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1. 引 言

关于力学系统对称性与守恒量理论的研究一直是数学、力学、物理学等领域的重要课题. 寻求守恒量的近代方法主要有 Noether 对称性、Lie 对称性和 Mei 对称性, 得到的守恒量主要有 Noether 守恒量、Hojman 守恒量和 Mei 守恒量[1—12]. 最近,力学系统统一对称性的提出进一步发展了对称性理论, 由力学系统的统一对称性可以同时导致上述三种类型的守恒量[13—16].

Poincaré[17] 于 1901 年利用无限小的可迁群建立了完整力学系统的一类新型运动微分方程. Chetaev[18] 将其发展到变换群为非可迁、约束是非定常、变量是不独立的情况. 他们建立的方程称为 Poincaré-Chetaev 方程. Poincaré-Chetaev 方程对 Hamilton 系统近代理论的发展有重要意义. 此外, 多余坐标下的广义 Lagrange 方程、准坐标下的 Euler-Lagrange 方程等都是 Poincaré-Chetaev 方程的特例. 对 Poincaré-Chetaev 方程的研究已取得重要成果[19—22]. 本文将研究 Poincaré-Chetaev 方程的统一对称性及其导致的守恒量. 给出 Poincaré-Chetaev 方程统一对称性的定义和判据, 并得出 Poincaré-Chetaev 方程统一对称性的三种守恒量:Noether 守恒量、Hojman 守恒量和 Mei 守恒量.

2. Poincaré-Chetaev 方程

研究具有 n 个自由度的完整力学系统, 其在空间的位置由变量 x_i $(i=1,2,\cdots,m;m\geqslant n)$ 来确定. 如果 $m=n$, 则 x_i 为独立坐标; 如果 $m>n$, 则当 $i>n$ 时 x_i 为不独立坐标或多余坐标.

假设由某种方法引入加在微分不可积分约束组上的参数化, 广义速度有形式

$$\dot{x}_i = \zeta_i^s(t,x)\eta_s + \zeta_i(t,x),$$
$$\mathrm{rank}(\zeta_i^s) = n \quad (i=1,2,\cdots,m;s=1,2,\cdots,n). \tag{1}$$

存在无限小线性算子的封闭组

$$X_0 = \frac{\partial}{\partial t} + \zeta_i\frac{\partial}{\partial x_i},$$
$$X_s = \zeta_i^s\frac{\partial}{\partial x_i} \qquad (i,s=1,2,\cdots,n), \tag{2}$$

满足

$$\begin{aligned}[X_s,X_k]f &= X_sX_kf - X_kX_sf \\ &= C_{sk}^rX_rf \\ &= -C_{ks}^rX_rf \quad (s,k,r=0,1,2,\cdots,n),\end{aligned} \tag{3}$$

这里 $m=n$. 一般情况下结构系数 C_{sk}^r 可以是变量[20],

$$C_{sk}^r = C_{sk}^r(t,x).$$

由(2)和(3)式可得如下关系:

$$C_{sk}^r\zeta_i^r = \zeta_j^s\frac{\partial\zeta_i^k}{\partial x_j} - \zeta_j^k\frac{\partial\zeta_i^s}{\partial x_j},$$
$$C_{0k}^r\zeta_i^r = \frac{\partial\zeta_i^k}{\partial t} + \zeta_j\frac{\partial\zeta_i^k}{\partial x_j} - \zeta_j^k\frac{\partial\zeta_i}{\partial x_j} \tag{4}$$
$$(s,k,r=1,2,\cdots,n).$$

Chetaev 对 Poincaré 的结果进行推广, 得出了 Poincaré-Chetaev 方程[20]

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$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \eta_s} = C_{rs}^{k}\eta_r\frac{\partial L}{\partial \eta_k} + C_{0s}^{k}\frac{\partial L}{\partial \eta_k} + X_sL + Q_s$$
$$(s,k,r = 1,2,\cdots,n), \tag{5}$$

其中 $L = L(t,\boldsymbol{x},\boldsymbol{\eta})$ 是系统的 Lagrange 函数，Q_s 为非势广义力. 假设方程(5)非奇异，即

$$\det\left(\frac{\partial^2 L}{\partial \eta_s\partial \eta_k}\right) \neq 0, \tag{6}$$

则由方程(5)可解出所有 $\dot{\eta}_s$，简记为

$$\dot{\eta}_s = \alpha_s(t,\boldsymbol{x},\boldsymbol{\eta}) \quad (s = 1,2,\cdots,n). \tag{7}$$

3. Poincaré-Chetaev 方程统一对称性的定义和判据

引进时间和准坐标下的无限小变换

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{x},\boldsymbol{\eta}),$$
$$\pi_s^*(t^*) = \pi_s(t) + \varepsilon\xi_s(t,\boldsymbol{x},\boldsymbol{\eta}), \tag{8}$$

其中，ε 为无限小参数，ξ_0，ξ_s 为无限小群变换的生成元. 这里 π_s，π_s^* 只是一种记号，而 $\Delta\pi_s$ 有意义，表示 π_s 的非等时变分. 由(1)式有

$$\delta x_i = \zeta_i^s\delta\pi_s, \tag{9}$$

因此有

$$\begin{aligned}\Delta x_i &= \delta x_i + \dot{x}_i\Delta t\\ &= \zeta_i^s(\varepsilon\xi_s - \varepsilon\eta_s\xi_0) + \varepsilon(\zeta_i^s\eta_s + \zeta_i)\xi_0\\ &= \varepsilon(\zeta_i^s\xi_s + \zeta_i\xi_0),\end{aligned} \tag{10}$$

又有[20]

$$\begin{aligned}\Delta\eta_s &= \delta\eta_s + \dot{\eta}_s\Delta t\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\delta\pi_s + C_{rk}^{s}\eta_r\delta\pi_k + C_{0k}^{s}\delta\pi_k + \dot{\eta}_s\Delta t\\ &= \varepsilon[\dot{\xi}_s - \eta_s\dot{\xi}_0 + (C_{rk}^{s}\eta_r + C_{0k}^{s})(\xi_k - \eta_k\xi_0)].\end{aligned} \tag{11}$$

定义 在无限小群变换(8)式下，如果 Poincaré-Chetaev 方程的一种对称性既是 Noether 对称性，又是 Lie 对称性同时也是 Mei 对称性，则称这种对称性为 Poincaré-Chetaev 方程的统一对称性.

对 Poincaré-Chetaev 方程(5)，其 Noether 等式为[22]

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \eta_s\xi_0) + \dot{G}_{\mathrm{N}} = 0, \tag{12}$$

其中 $G_{\mathrm{N}} = G_{\mathrm{N}}(t,\boldsymbol{x},\boldsymbol{\eta})$ 为规范函数，且

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + (\zeta_i^s\xi_s + \zeta_i\xi_0)\frac{\partial}{\partial x_i} + [\dot{\xi}_s - \eta_s\dot{\xi}_0 + (C_{rk}^{s}\eta_r + C_{0k}^{s})(\xi_k - \eta_k\xi_0)]\frac{\partial}{\partial \eta_s}. \tag{13}$$

根据 Lie 对称性理论，Poincaré-Chetaev 方程(5)的 Lie 对称性确定方程为

$$X^{(2)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \eta_s}\right) - X^{(1)}\left(C_{rs}^{k}\eta_r\frac{\partial L}{\partial \eta_k}\right) - X^{(1)}\left(C_{0s}^{k}\frac{\partial L}{\partial \eta_k}\right) - X^{(1)}(X_sL) - X^{(1)}(Q_s) = 0, \tag{14}$$

其中

$$X^{(2)} = X^{(1)} + \left\{[\dot{\xi}_s - \eta_s\dot{\xi}_0 + (C_{rk}^{s}\eta_r + C_{0k}^{s})(\xi_k - \eta_k\xi_0)]^{\cdot} - \dot{\eta}_s\dot{\xi}_0\right\}\frac{\partial}{\partial \dot{\eta}_s}. \tag{15}$$

方程(14)的等价形式为

$$X^{(2)}(\dot{\eta}_s) - X^{(1)}(\alpha_s) = 0. \tag{16}$$

根据 Mei 对称性理论，Poincaré-Chetaev 方程(5) Mei 对称性的判据方程为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \eta_s}[X^{(1)}(L)] - C_{rs}^{k}\eta_r\frac{\partial}{\partial \eta_k}[X^{(1)}(L)] - C_{0s}^{k}\frac{\partial}{\partial \eta_k}[X^{(1)}(L)] - X_s[X^{(1)}(L)] - X^{(1)}(Q_s) = 0. \tag{17}$$

根据定义，我们有以下判据.

判据 如果存在规范函数 $G_{\mathrm{N}} = G_{\mathrm{N}}(t,\boldsymbol{x},\boldsymbol{\eta})$，使得无限小群变换的生成元 ξ_0，ξ_s 满足

$$\begin{aligned}&[L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \eta_s\xi_0) + \dot{G}_{\mathrm{N}}]^2\\ &+ \left[X^{(2)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \eta_s}\right) - X^{(1)}\left(C_{rs}^{k}\eta_r\frac{\partial L}{\partial \eta_k}\right) - X^{(1)}\left(C_{0s}^{k}\frac{\partial L}{\partial \eta_k}\right) - X^{(1)}(X_sL) - X^{(1)}(Q_s)\right]^2\\ &+ \left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \eta_s}[X^{(1)}(L)] - C_{rs}^{k}\eta_r\frac{\partial}{\partial \eta_k}[X^{(1)}(L)] - C_{0s}^{k}\frac{\partial}{\partial \eta_k}[X^{(1)}(L)] - X_s[X^{(1)}(L)] - X^{(1)}(Q_s)\right\}^2 = 0,\end{aligned} \tag{18}$$

则相应的对称性是 Poincaré-Chetaev 方程的统一对称性.

4. Poincaré-Chetaev 方程统一对称性导致的守恒量

Poincaré-Chetaev 方程的统一对称性在一定条件下能够同时导致 Noether 守恒量、Hojman 守恒量和 Mei 守恒量.

命题 1　在无限小群变换(8)式下，Poincaré-Chetaev 方程的统一对称性能够导致 Noether 守恒量

$$I_{\mathrm{N}} = L\xi_0 + \frac{\partial L}{\partial \eta_s}(\xi_s - \eta_s\xi_0) + G_{\mathrm{N}} = \mathrm{const}. \tag{19}$$

证明　因为 Poincaré-Chetaev 方程的统一对称性一定是 Noether 对称性，存在一个规范函数 $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{x}, \boldsymbol{\eta})$ 满足 Noether 等式. 因此，根据 Noether 定理 Poincaré-Chetaev 方程存在守恒量(19)式. 命题 1 得证.

命题 2　在时间不变的特殊无限小群变换下，如果无限小群变换的生成元满足(18)式以及

$$\frac{\partial C^s_{rt}}{\partial x_i}\zeta^s_i\eta_r\xi_t + \frac{\partial C^s_{0t}}{\partial x_i}\zeta^s_i\xi_t = 0, \tag{20}$$

并且存在函数 $\mu = \mu(t, \boldsymbol{x}, \boldsymbol{\eta})$ 满足

$$\frac{\partial \alpha_s}{\partial \eta_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{21}$$

则 Poincaré-Chetaev 方程的统一对称性导致 Hojman 守恒量

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial x_i}(\mu\xi_s)\zeta^s_i + \frac{1}{\mu}\frac{\partial}{\partial \eta_s}\Big[\mu\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t\Big)\Big] = \mathrm{const}. \tag{22}$$

这里

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\zeta^k_i\eta_k + \zeta_i)\frac{\partial}{\partial x_i} + \alpha_k\frac{\partial}{\partial \eta_k}. \tag{23}$$

证明　Poincaré-Chetaev 方程的统一对称性一定是 Lie 对称性，因此(16)式成立. 在时间不变的特殊无限小群变换下，展开(16)式得

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial x_i}\xi_k\zeta^k_i + \frac{\partial \alpha_s}{\partial \eta_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k + \frac{\partial \alpha_s}{\partial \eta_k}(C^k_{rt}\eta_r\xi_t + C^k_{0t}\xi_t) - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}(C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t). \tag{24}$$

将(22)式对时间求导数，有

$$\frac{\bar{\mathrm{d}}I_{\mathrm{H}}}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial x_i}\xi_s\zeta^s_i\Big) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial \xi_s}{\partial x_i}\zeta^s_i\Big) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big[\frac{1}{\mu}\frac{\partial \mu}{\partial \eta_s}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t\Big)\Big] + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big[\frac{\partial}{\partial \eta_s}(C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t)\Big]. \tag{25}$$

注意到

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big) = \frac{\partial}{\partial \eta_s}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big) - \frac{\partial}{\partial \eta_k}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big)\frac{\partial \alpha_k}{\partial \eta_s} - \frac{\partial}{\partial x_i}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big)\zeta^s_i, \tag{26}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big[\frac{\partial}{\partial \eta_s}(C^s_{rk}\eta_r\xi_k + C^s_{0k}\xi_k)\Big] = \frac{\partial}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t) - \frac{\partial}{\partial \eta_k}(C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t)\frac{\partial \alpha_k}{\partial \eta_s} - \frac{\partial}{\partial x_i}(C^s_{rt}\eta_r\xi_t + C^s_{0t}\xi_t)\zeta^s_i, \tag{27}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial \xi_s}{\partial x_i}\zeta^s_i\Big) = \Big(\frac{\partial}{\partial x_i}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \xi_s}{\partial \eta_l}\frac{\partial \alpha_l}{\partial x_i} - \frac{\partial \xi_s}{\partial x_j}\frac{\partial \zeta^l_j}{\partial x_i}\eta_l - \frac{\partial \xi_s}{\partial x_j}\frac{\partial \zeta_j}{\partial x_i}\Big)\zeta^s_i + \frac{\partial \xi_s}{\partial x_i}\frac{\partial \zeta^s_i}{\partial t} + \frac{\partial \xi_s}{\partial x_i}\frac{\partial \zeta^s_i}{\partial x_j}(\zeta^k_j\eta_k + \zeta_j), \tag{28}$$

利用(21)式做计算，有

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial x_i}\xi_s\zeta^s_i\Big) = \Big\{-\frac{\partial^2 \alpha_l}{\partial \eta_l\partial x_i}\zeta^s_i - \frac{1}{\mu}\frac{\partial \mu}{\partial \eta_k}\frac{\partial \alpha_k}{\partial x_i}\zeta^s_i - \frac{1}{\mu}\frac{\partial \mu}{\partial x_j}\frac{\partial \zeta^l_j}{\partial x_i}\eta_l\zeta^s_i - \frac{1}{\mu}\frac{\partial \mu}{\partial x_j}\frac{\partial \zeta_j}{\partial x_i}\zeta^s_i + \frac{1}{\mu}\frac{\partial \mu}{\partial x_j}\frac{\partial \zeta^s_j}{\partial t} + \frac{1}{\mu}\frac{\partial \mu}{\partial x_j}\frac{\partial \zeta^s_j}{\partial x_i}(\zeta^l_i\eta_l + \zeta_i)\Big\}\xi_s + \frac{1}{\mu}\frac{\partial \mu}{\partial x_i}\zeta^s_i\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s, \tag{29}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial \eta_s}\Big) = -\frac{\partial^2 \alpha_l}{\partial \eta_l\partial \eta_s} - \frac{1}{\mu}\frac{\partial \mu}{\partial \eta_l}\frac{\partial \alpha_l}{\partial \eta_s} - \frac{1}{\mu}\frac{\partial \mu}{\partial x_i}\zeta^s_i. \tag{30}$$

将(24)式两端对 η_s 求偏导数，并对 s 求和，得

$$\frac{\partial}{\partial \eta_s}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big) = \frac{\partial^2 \alpha_s}{\partial \eta_s\partial x_i}\xi_k\zeta^k_i + \frac{\partial a_s}{\partial x_i}\frac{\partial \xi_k}{\partial \eta_s}\zeta^k_i + \frac{\partial \alpha_s}{\partial \eta_k}\frac{\partial}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k + \frac{\partial^2 \alpha_s}{\partial \eta_s\partial \eta_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k + \frac{\partial^2 \alpha_s}{\partial \eta_s\partial \eta_k}(C^k_{rt}\eta_r\xi_t + C^k_{0t}\xi_t)$$

$$+\frac{\partial \alpha_s}{\partial \eta_k}\frac{\partial}{\partial \eta_s}(C_{rt}^k\eta_r\xi_t + C_{0t}^k\xi_t)$$
$$-\frac{\partial}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(C_{rt}^s\eta_r\xi_t + C_{0t}^s\xi_t). \tag{31}$$

将(26)—(31)式代入(25)式，并利用(4),(20),(24)式可得

$$\frac{\bar{\mathrm{d}} I_{\mathrm{H}}}{\mathrm{d}t} = -\frac{\partial C_{rt}^s}{\partial x_i}\zeta_i^s\eta_r\xi_t - \frac{\partial C_{0t}^s}{\partial x_i}\zeta_i^s\xi_t = 0. \tag{32}$$

命题2得证.

当 $\zeta_i = 0, \zeta_i^s = \zeta_i^s(\boldsymbol{x})$ 时，$C_{0t}^s = 0$，(22)式变为文献[2]中提到的 Poincaré 方程的 Hojman 守恒量.

命题3 在无限小群变换(8)式下，如果存在函数 $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{x}, \boldsymbol{\eta})$ 满足

$$\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] + \tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0$$
$$+\tilde{X}^{(1)}(Q_s)(\xi_s - \eta_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{M}} = 0, \tag{33}$$

则 Poincaré-Chetaev 方程的统一对称性导致 Mei 守恒量

$$I_{\mathrm{M}} = \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_s}(\xi_0 - \eta_s\xi_0) + G_{\mathrm{M}}$$
$$= \mathrm{const}. \tag{34}$$

这里

$$\tilde{X}^{(1)} = \xi_0\frac{\partial}{\partial t} + (\zeta_i^s\xi_s + \zeta_i\xi_0)\frac{\partial}{\partial x_i}$$
$$+\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \eta_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0$$
$$+(C_{rk}^s\eta_r + C_{0k}^s)(\xi_k - \eta_k\xi_0)\Big]\frac{\partial}{\partial \eta_s}. \tag{35}$$

证明 Poincaré-Chetaev 方程的统一对称性一定是 Mei 对称性，因此(17)式成立. 将(34)式对时间求导数，有

$$\frac{\bar{\mathrm{d}} I_{\mathrm{M}}}{\mathrm{d}t} = \xi_0\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[\tilde{X}^{(1)}(L)] + \tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0$$
$$+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big[\frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_s}\Big](\xi_s - \eta_s\xi_0)$$
$$+\frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\xi_s - \eta_s\xi_0)$$
$$-\xi_0\frac{\partial}{\partial t}[\tilde{X}^{(1)}(L)]$$
$$-(\zeta_i^s\xi_s + \zeta_i\xi_0)\frac{\partial}{\partial x_i}[\tilde{X}^{(1)}(L)]$$
$$-\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \eta_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0$$
$$+(C_{rk}^s\eta_r + C_{0k}^s)(\xi_k - \eta_k\xi_0)\Big]\frac{\partial}{\partial \eta_s}[\tilde{X}^{(1)}(L)]$$
$$-\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \tilde{X}^{(1)}(Q_s)(\xi_s - \eta_s\xi_0)$$
$$=\Big\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_s} - C_{rs}^k\eta_r\frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_k}$$
$$-C_{0s}^k\frac{\partial \tilde{X}^{(1)}(L)}{\partial \eta_k} - X_s[\tilde{X}^{(1)}(L)]$$
$$-\tilde{X}^{(1)}(Q_s)\Big\}(\xi_s - \eta_s\xi_0)$$
$$=0. \tag{36}$$

命题3得证.

5. 算 例

系统在广义坐标下的 Lagrange 函数为

$$\tilde{L} = \frac{1}{2}(q_1^2\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_1^2 - q_2, \tag{37}$$

试研究 Poincaré-Chetaev 方程的统一对称性及其守恒量.

首先建立系统(37)的 Poincaré-Chetaev 方程.

令

$$x_1 = q_1,\quad x_2 = q_2,\quad \dot{x}_1 = \frac{\eta_1}{x_1},\quad \dot{x}_2 = \eta_2, \tag{38}$$

则有

$$\zeta_1 = \zeta_2 = 0,\quad \zeta_1^1 = \frac{1}{x_1},\quad \zeta_1^2 = \zeta_2^1 = 0,\quad \zeta_2^2 = 1,$$
$$X_1 = \frac{1}{x_1}\frac{\partial}{\partial x_1},\quad X_2 = \frac{\partial}{\partial x_2},$$
$$C_{rs}^k = C_{0s}^k = 0 \quad (r, s, k = 1, 2),$$
$$L = \frac{1}{2}(\eta_1^2 + \eta_2^2) + \frac{1}{2}x_1^2 - x_2. \tag{39}$$

由(5)式可得 Poincaré-Chetaev 方程为

$$\dot{\eta}_1 = 1,\quad \dot{\eta}_2 = -1. \tag{40}$$

取生成元

$$\xi_0 = 0,\quad \xi_1 = \eta_1,\quad \xi_2 = \eta_2 + t, \tag{41}$$

可以验证该组生成元满足(18)式，因此该组生成元是统一对称性的.

由(18)式可得规范函数

$$G_N = -\eta_1^2 - \frac{1}{2}\eta_2^2 + \frac{1}{2}t^2, \tag{42}$$

根据命题 1，方程存在 Noether 守恒量

$$I_N = \frac{1}{2}\eta_2^2 + t\eta_2 + \frac{1}{2}t^2 = \text{const}. \tag{43}$$

由(21)式可得

$$\frac{\bar{d}}{dt}\ln\mu = 0. \tag{44}$$

(44)式有解

$$\mu = t\eta_2 - x_2 + \frac{1}{2}t^2. \tag{45}$$

根据命题 2，方程存在 Hojman 守恒量

$$I_H = -\left(t\eta_2 - x_2 + \frac{1}{2}t^2\right)^{-1}(\eta_2 + t) = \text{const}. \tag{46}$$

由(33)式可得

$$G_M = -2t. \tag{47}$$

根据命题 3，方程存在 Mei 守恒量

$$I_M = 2\eta_1 - \eta_2 - 3t = \text{const}. \tag{48}$$

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Unified symmetry of Poincaré-Chetaev equations

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Abstract

The unified symmetry and conserved quantities of Poincaré-Chetaev equations are studied. The definition and criterion of the unified symmetry are given. The Noether conserved quantity, the Hojman conserved quantity, and the Mei conserved quantity induced from the symmetry are obtained.

Keywords: Poincaré-Chetaev equations, unified symmetry, conserved quantity

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