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Volterra 差分微分方程和 KdV 差分微分方程新的精确解*

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辅助方程法和试探函数法为基础,给出函数变换与辅助方程相结合的一种方法,借助符号计算系统 Mathematica 构造了 Volterra 差分微分方程和 KdV 差分微分方程新的精确孤立波解和三角函数解.该方法也适合求解其他非线性差分微分方程的精确解.

关键词:辅助方程,函数变换,非线性差分微分方程,孤立波解

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1. 引 言

随着非线性科学技术的不断发展,非线性科学理论的研究问题在自然科学和社会科学领域当中正在蓬勃发展.构造非线性发展方程精确解是孤立子理论的重要研究课题之一.对于在连续非线性系统当中人们建立了构造精确解的许多有效直接方法[1—6].如双曲正切函数展开法,Jacobi 椭圆函数展开法,辅助方程法,齐次平衡法,试探函数法等.许多经济问题、人口问题、生物问题、化学问题的模型是离散的.近年来,人们为了得到非线性离散系统的精确解,提出一些方法,并获得了许多研究成果[7—13].文献[7—11]分别用双曲函数型试探函数法、三角函数型试探函数法和 Jacobi 椭圆函数型试探函数法,得到了(2+1)维 Hybrid-Lattice 系统等非线性离散系统的精确解.文献[12,13]利用耦合的 Riccati 方程组得到了 Volterra 差分微分方程等,非线性离散方程新的精确解.本文在文献[6—13]的基础上,给出函数变换与辅助方程相结合的一种方法,并用符号计算系统 Mathematica 构造了 Volterra 差分微分方程和 KdV 差分微分方程新的精确孤立波解和三角函数解.

2. 方法的介绍

设给定的非线性差分微分方程为

$$\begin{aligned}&H(u_{n+p_1}(X),u_{n+p_2}(X),\cdots,u_{n+p_k}(X),\\&u'_{n+p_1}(X),u'_{n+p_2}(X),\cdots,u'_{n+p_k}(X),\cdots,\\&u^{(r)}_{n+p_1}(X),u^{(r)}_{n+p_2}(X),\cdots,u^{(r)}_{n+p_k}(X))=0,\end{aligned}\qquad(1)$$

其中

$$\begin{aligned}u_{n+p_l}(X)&=(u_{n+p_l(1)}(X),u_{n+p_l(2)}(X),\cdots,\\&\qquad u_{n+p_l(M)}(X)),\\X&=(x_1,x_2,\cdots,x_N),\ n=(n_1,n_2,\cdots,n_h),\\p_l&=(p_{l_1},p_{l_2},\cdots,p_{l_h}),\\u^{(r)}_{n+p_l}(X)&=\frac{\partial^{a_1+a_2+\cdots+a_N}u_{n+p_l}(X)}{\partial x_1^{a_1}\partial x_2^{a_2}\cdots\partial x_N^{a_N}},\end{aligned}$$

$$(a_1+a_2+\cdots+a_N=r,l=1,2,\cdots,k).$$

对方程(1)进行下列行波变换 $u_{n+p_l}(X)=u_{n+p_l}(\xi)$,$\xi=\sum_{i=1}^{h}d_in_i+\sum_{j=1}^{N}\lambda_jx_j$,其中 d_i,λ_j 为待定常数.这样得到如下方程:

$$\begin{aligned}&H(u_{n+p_1}(\xi),u_{n+p_2}(\xi),\cdots,u_{n+p_k}(\xi),\\&u'_{n+p_1}(\xi),u'_{n+p_2}(\xi),\cdots,u'_{n+p_k}(\xi),\cdots,\end{aligned}$$

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$$u_{n+p_1}^{(r)}(\xi), u_{n+p_2}^{(r)}(\xi), \cdots, u_{n+p_k}^{(r)}(\xi)) = 0. \tag{2}$$

方程(2)的解取为下列形式：

$$u_n(\xi) = g_0 + \frac{g_1 z(\xi)}{g_2 + g_3 z(\xi)}, \tag{3}$$

$$u_{n+p_l}(\xi) = g_0 + \frac{g_1 z(\xi + \eta_l)}{g_2 + g_3 z(\xi + \eta_l)}, \tag{4}$$

其中 $\eta_l = p_{l_1} d_1 + p_{l_2} d_2 + \cdots + p_{l_h} d_h$；$d_1, d_2, \cdots, d_h$，$g_s (s = 0,1,2,3)$ 是常数. $u_{n+p_k}^{(r)}(\xi)$ 表示 $u_{n+p_k}(\xi)$ 关于 ξ 的 r 阶导数. 另外，$z(\xi)$，$z(\xi + \eta_l)$ 以下列辅助方程[6]来确定：

$$(z'(\xi))^2 = \left(\frac{dz(\xi)}{d\xi}\right)^2 = az^2(\xi) + bz(\xi) + c. \tag{5}$$

我们得到方程(5)的如下解.

2.1. 双曲函数解

$$z(\xi) = -\frac{1}{a}\left(b - 2\sqrt{ac}\tanh\left(\frac{1}{2}\sqrt{a}\xi\right)\right) \times \cosh^2\left(\frac{1}{2}\sqrt{a}\xi\right), (a > 0, c > 0); \tag{6}$$

$$z(\xi) = \frac{1}{a}\left(b - 2\sqrt{ac}\coth\left(\frac{1}{2}\sqrt{a}\xi\right)\right) \times \sinh^2\left(\frac{1}{2}\sqrt{a}\xi\right), (a > 0, c > 0); \tag{7}$$

$$z(\xi) = \frac{1}{4a}\Big(-2b - (1 + b^2 - 4ac)\cosh(\sqrt{a}\xi) + (1 - b^2 + 4ac)\sinh(\sqrt{a}\xi)\Big), (a > 0); \tag{8}$$

$$z(\xi) = \frac{b - 2\sqrt{ac}(\coth(\sqrt{a}\xi) \pm \operatorname{csch}(\sqrt{a}\xi))}{-a + a(\coth(\sqrt{a}\xi) \pm \operatorname{csch}(\sqrt{a}\xi))^2}, (c > 0, a > 0); \tag{9}$$

$$z(\xi) = -\frac{1}{2a}(b \pm b\cosh(\sqrt{a}\xi) \mp 2\sqrt{ac}\sinh(\sqrt{a}\xi)), (c > 0, a > 0). \tag{10}$$

2.2. 三角函数解

$$z(\xi) = -\frac{1}{a}\left(b + 2\sqrt{-ac}\tan\left(\frac{1}{2}\sqrt{-a}\xi\right)\right) \times \cos^2\left(\frac{1}{2}\sqrt{-a}\xi\right), (a < 0, c > 0); \tag{11}$$

$$z(\xi) = -\frac{1}{a}\left(b - 2\sqrt{-ac}\cot\left(\frac{1}{2}\sqrt{-a}\xi\right)\right) \times \sin^2\left(\frac{1}{2}\sqrt{-a}\xi\right), (a < 0, c > 0); \tag{12}$$

$$z(\xi) = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\sin(\sqrt{-a}\xi), (a < 0, b^2 - 4ac > 0); \tag{13}$$

$$z(\xi) = -\frac{b + 2\sqrt{-ac}(\tan(\sqrt{-a}\xi) \pm \sec(\sqrt{-a}\xi))}{a(1 + (\tan(\sqrt{-a}\xi) \pm \sec(\sqrt{-a}\xi))^2)}, (c > 0, a < 0); \tag{14}$$

$$z(\xi) = \frac{-b + 2\sqrt{-ac}(\cot(\sqrt{-a}\xi) \pm \csc(\sqrt{-a}\xi))}{a(1 + (\cot(\sqrt{-a}\xi) \pm \csc(\sqrt{-a}\xi))^2)}, (c > 0, a < 0). \tag{15}$$

我们根据下列(16)—(18)和(6)—(10)式即可用 a，b，c，$z(\xi)$，$z'(\xi)$，$\sinh(\Omega)$，$\cosh(\Omega)$（其中 $\Omega = \frac{1}{2}\sqrt{ad}$ 或 $\Omega = \sqrt{ad}$）来表示 $z(\xi + \eta_l)$.

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y), \tag{16}$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y), \tag{17}$$

$$-\sinh^2(x) + \cosh^2(x) = 1. \tag{18}$$

同理，根据如下(19)—(21)和(11)—(15)式可用 a，b，c，$\sin(\gamma)$，$\cos(\gamma)$，$z(\xi)$，$z'(\xi)$（其中 $\gamma = \frac{1}{2}\sqrt{-ad}$ 或 $\gamma = \sqrt{-ad}$）来表示 $z(\xi + \eta_l)$.

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y), \tag{19}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y), \tag{20}$$

$$\sin^2(x) + \cos^2(x) = 1. \tag{21}$$

下面说明用辅助方程(5)的解(6)，(8)，(11)和函数变换(3)相结合的情况下，如何构造非线性差分微分方程(1)的精确解(其余情况这里不讨论).

(6)式和(18) $\left(x = \frac{1}{2}\sqrt{a}\xi\right)$ 式联立即可得到下列表达式：

$$\sinh\left(\frac{1}{2}\sqrt{a}\xi\right) = \frac{\mp(\sqrt{c} + z'(\xi))}{\sqrt{a}z(\xi)} \times \sqrt{\frac{a(2c + bz(\xi) - 2\sqrt{c}z'(\xi))}{-b^2 + 4ac}}, \tag{22}$$

$$\cosh\left(\frac{1}{2}\sqrt{a}\xi\right) = \pm\sqrt{\frac{a(2c + bz(\xi) - 2\sqrt{c}z'(\xi))}{-b^2 + 4ac}}, \tag{23}$$

$$\sinh\left(\frac{1}{2}\sqrt{a}\xi\right) = \frac{\pm(\sqrt{c} - z'(\xi))}{\sqrt{a}z(\xi)} \times \sqrt{\frac{a(2c + bz(\xi) + 2\sqrt{c}z'(\xi))}{-b^2 + 4ac}}, \tag{24}$$

$$\cosh\left(\frac{1}{2}\sqrt{a}\xi\right)=\mp\sqrt{\frac{a\left(2c+bz(\xi)+2\sqrt{c}z'(\xi)\right)}{-b^2+4ac}}.\tag{25}$$

同理，将(8)式和(18)($x=\sqrt{a}\xi$)式；(11)式和(21)$\left(x=\frac{1}{2}\sqrt{a}\xi\right)$式联立后得到下列表达式：

$$\cosh(\sqrt{a}\xi)=\frac{1}{2(b^2-4ac)}(-b^3+b(1+4ac)+2a(1-b^2+4ac)z(\xi)$$
$$\pm 2\sqrt{a}(b^2+1-4ac)z'(\xi)),\tag{26}$$

$$\sinh(\sqrt{a}\xi)=\frac{1}{2(b^2-4ac)}(b^3-b(1+4ac)+2a(-1+b^2-4ac)z(\xi)$$
$$\mp 2\sqrt{a}(b^2-(1+4ac))z'(\xi)),\tag{27}$$

$$\sin\left(\frac{1}{2}\sqrt{a}\xi\right)=\frac{\pm(\sqrt{c}+z'(\xi))}{\sqrt{-a}z(\xi)}\sqrt{\frac{a\left(2c+bz(\xi)-2\sqrt{c}z'(\xi)\right)}{-b^2+4ac}},\tag{28}$$

$$\cos\left(\frac{1}{2}\sqrt{a}\xi\right)=\pm\sqrt{\frac{a\left(2c+bz(\xi)-2\sqrt{c}z'(\xi)\right)}{-b^2+4ac}},\tag{29}$$

$$\sin\left(\frac{1}{2}\sqrt{a}\xi\right)=\frac{\mp(\sqrt{c}-z'(\xi))}{\sqrt{-a}z(\xi)}\sqrt{\frac{a\left(2c+bz(\xi)+2\sqrt{c}z'(\xi)\right)}{-b^2+4ac}},\tag{30}$$

$$\cos\left(\frac{1}{2}\sqrt{a}\xi\right)=\mp\sqrt{\frac{a\left(2c+bz(\xi)+2\sqrt{c}z'(\xi)\right)}{-b^2+4ac}}.\tag{31}$$

这样对应与解(6)，(8)，(11)的 $u_{n+p_l}(\xi)$ 为如下形式：

$$u_{n+p_l}(\xi)=g_0-\frac{g_1\left(b-2\sqrt{ac}\tanh\left(\frac{1}{2}\sqrt{a}(\xi+\eta_l)\right)\right)\cosh^2\left(\frac{1}{2}\sqrt{a}(\xi+\eta_l)\right)}{ag_2-g_3\left(b-2\sqrt{ac}\tanh\left(\frac{1}{2}\sqrt{a}(\xi+\eta_l)\right)\right)\cosh^2\left(\frac{1}{2}\sqrt{a}(\xi+\eta_l)\right)},\tag{32}$$

$$u_{n+p_l}(\xi)=g_0+\frac{g_1(-2b-(1+b^2-4ac)\cosh(\sqrt{a}(\xi+\eta_l))+(1-b^2+4ac)\sinh(\sqrt{a}(\xi+\eta_l)))}{4ag_2+g_3(-2b-(1+b^2-4ac)\cosh(\sqrt{a}(\xi+\eta_l))+(1-b^2+4ac)\sinh(\sqrt{a}(\xi+\eta_l)))},\tag{33}$$

$$u_{n+p_l}(\xi)=g_0-\frac{g_1\left(b+2\sqrt{-ac}\tan\left(\frac{1}{2}\sqrt{-a}(\xi+\eta_l)\right)\right)\cos^2\left(\frac{1}{2}\sqrt{-a}(\xi+\eta_l)\right)}{ag_2-g_3\left(b+2\sqrt{-ac}\tan\left(\frac{1}{2}\sqrt{-a}(\xi+\eta_l)\right)\right)\cos^2\left(\frac{1}{2}\sqrt{-a}(\xi+\eta_l)\right)}.\tag{34}$$

将(3)，(5)，(22)，(23)，(32)式一起代入方程(2)，并令 $z'(\xi)z^q(\xi)$，($q=1,2,3,\cdots$) 的(或(3)，(5)，(24)，(25)，(32)式；(3)，(5)，(26)，(27)，(33)式；(3)，(5)，(28)，(29)，(34)式；(3)，(5)，(30)，(31)，(34)式)系数为零后得到一个 $a,b,c,d_i(i=1,2,\cdots,h)$，$\lambda_j(j=1,2,\cdots,N)$，$g_s(s=0,1,2,3)$ 为未知量的非线性代数方程组，用符号计算系统 Mathematica 求出该方程组的解，再将求出的每一组解分别与(6)式(或(8)式，(11)式)一起代入(3)式后得到非线性差分微分方程(1)的精确解.

3. Volterra 差分微分方程和 KdV 差分微分方程的精确解

例 1　Volterra 差分微分方程[12]

$$\dot{u}_n(t)=u_n(u_{n+1}-u_{n-1}),\tag{35}$$

将 $u_n(n,t)=u_n(\xi)$，$u_{n+1}(n,t)=u_{n+1}(\xi)$，$u_{n-1}(n,t)=u_{n-1}(\xi)$，$\xi=dn+\omega t$ 代入方程(35)后得到下列方程：

$$\omega u'_n(\xi)=u_n(\xi)(u_{n+1}(\xi)-u_{n-1}(\xi)),\tag{36}$$

方程(36)的解取为(3)式,其中 $z(\xi)$ 是由(6)式来确定.这样根据(22),(23),(32)式即可得到

$$u_{n+1}(\xi)=\frac{\sqrt{c}\left(2ag_0g_2-b\left(g_1+g_0g_3\right)\right)+\left(b+2az(\xi)\right)\left(g_1+g_0g_3\right)\cosh(\sqrt{ad})+\Phi(\xi)}{\left(2ag_2-bg_3\right)\sqrt{c}+\left(b+2az(\xi)\right)\sqrt{c}g_3\cosh(\sqrt{ad})-2\sqrt{ac}g_3\sinh(\sqrt{ad})z'(\xi)},\tag{37}$$

$$u_{n-1}(\xi)=\frac{\sqrt{c}\left(2ag_0g_2-b\left(g_1+g_0g_3\right)\right)+\left(b+2az(\xi)\right)\left(g_1+g_0g_3\right)\cosh(\sqrt{ad})-\Phi(\xi)}{\left(2ag_2-bg_3\right)\sqrt{c}+\left(b+2az(\xi)\right)\sqrt{c}g_3\cosh(\sqrt{ad})+2\sqrt{ac}g_3\sinh(\sqrt{ad})z'(\xi)}.\tag{38}$$

其中

$$\Phi(\xi)=-2\left(g_1+g_0g_3\right)\sqrt{ac}g_1\sinh(\sqrt{ad})z'(\xi),$$

$$a>0,c>0.$$

将(3),(5),(37),(38)式一起代入方程(36),并令 $z^q(\xi)z'(\xi)(q=0,1,2)$ 的系数为零后得到一组非线性代数方程组(限于篇幅未列出).用符号计算系统 Mathematica 求出该方程组的如下解:

$$\begin{aligned}
g_0&=-\frac{\left(MN+2b\left(b\mp\sqrt{2MP}\sinh(\sqrt{ad})\right)\right)\sqrt{ac}\omega\operatorname{csch}\left(\frac{\sqrt{ad}}{2}\right)\operatorname{sech}\left(\frac{\sqrt{ad}}{2}\right)}{4\sqrt{c}\left(b^2+4ac+M\cosh(\sqrt{ad})\right)},\\
g_1&=\frac{\sqrt{ac}\omega g_3\left(\mp\sqrt{2MP}b+M\sinh(\sqrt{ad})\right)}{\sqrt{c}\left(b^2+4ac+M\cosh(\sqrt{ad})\right)},\\
g_2&=\frac{g_3}{4a}\left(2b\mp\sqrt{2MP}\coth\left(\frac{\sqrt{ad}}{2}\right)\right).
\end{aligned}\tag{39}$$

其中

$$M=-b^2+4ac>0,$$

$$N=\cosh(\sqrt{ad})+\cosh(2\sqrt{ad}),$$

$$P=1-\cosh(\sqrt{ad}),a>0,c>0.$$

将(6),(39)式一起代入(3)式后得到方程(35)的如下形式的精确孤波解:

$$u_{n(1,2)}(n,t)=g_0-\frac{2g_1\left(b-2\sqrt{ac}\tanh\left(\frac{\sqrt{a}}{2}(dn+\omega t)\right)\right)\cosh^2\left(\frac{\sqrt{a}}{2}(dn+\omega t)\right)}{2ag_2-bg_3\left(1+\cosh\left(\sqrt{a}(dn+\omega t)\right)\right)+2g_3\sqrt{ac}\sinh\left(\sqrt{a}(dn+\omega t)\right)};$$

其中 g_0,g_1,g_2 由(39)式来确定.

方程(36)的解取为(3)式,其中 $z(\xi)$ 是由(11)式来确定.这样根据(28),(29),(34)式即可得到

$$u_{n+1}(\xi)=\frac{\sqrt{c}\left(2ag_0g_2-b\left(g_1+g_0g_3\right)\right)+\left(b+2az(\xi)\right)\left(g_1+g_0g_3\right)\cos(\sqrt{-ad})+\Psi(\xi)}{\left(2ag_2-bg_3\right)\sqrt{c}+\left(b+2az(\xi)\right)\sqrt{c}g_3\cos(\sqrt{-ad})-2\sqrt{-ac}g_3\sin(\sqrt{-ad})z'(\xi)},\tag{40}$$

$$u_{n-1}(\xi)=\frac{\sqrt{c}\left(2ag_0g_2-b\left(g_1+g_0g_3\right)\right)+\left(b+2az(\xi)\right)\left(g_1+g_0g_3\right)\cos(\sqrt{-ad})-\Psi(\xi)}{\left(2ag_2-bg_3\right)\sqrt{c}+\left(b+2az(\xi)\right)\sqrt{c}g_3\cos(\sqrt{-ad})+2\sqrt{-ac}g_3\sin(\sqrt{-ad})z'(\xi)}.\tag{41}$$

其中

$$\Psi(\xi)=-2\left(g_1+g_0g_3\right)\sqrt{-ac}g_1\sin(\sqrt{-ad})z'(\xi),$$

$$a<0,c>0.$$

把(3),(5),(40),(41)式一起代入方程(36),并令 $z^q(\xi)z'(\xi)(q=0,1,2)$ 的系数为零后得到一组非线性代数方程组(限于篇幅未列出).用符号计算系统 Mathematica 求出该方程组的如下解:

$$g_0=\frac{\sqrt{c}\omega\left(aMg_3Q+2b\left(abg_3\mp 4\left|a\cos\left(\frac{\sqrt{-ad}}{2}\right)g_3\right|\sin^2\left(\frac{\sqrt{-ad}}{2}\right)\sqrt{-M}\right)\right)\csc\left(\frac{\sqrt{-ad}}{2}\right)\sec\left(\frac{\sqrt{-ad}}{2}\right)}{4\sqrt{-ac}\left(b^2+4ac+M\cos(\sqrt{-ad})\right)g_3},$$

$$g_1=\frac{\sqrt{c}\omega\left((ab^2-4a^2c)g_3+aMg_3\cos(2\sqrt{-ad})\mp 8b\left|ag_3\cos\left(\frac{\sqrt{-ad}}{2}\right)\right|\sin^2\left(\frac{\sqrt{-ad}}{2}\right)\right)\tan\left(\frac{\sqrt{-ad}}{2}\right)}{2\sqrt{-ac}\left(b^2\left(1-2\cos(\sqrt{-ad})+\cos^2(\sqrt{-ad})\right)+4ac\sin^2(\sqrt{-ad})\right)},$$

$$g_2=\frac{1}{4a^2}\left((1-\cos(\sqrt{-ad}))abg_3\pm 2\left|a\cos\left(\frac{\sqrt{-ad}}{2}\right)g_3\right|\sin^2\left(\frac{\sqrt{-ad}}{2}\right)\sqrt{-M}\right)\csc^2\left(\frac{\sqrt{-ad}}{2}\right).\tag{42}$$

其中

$$Q=\cos(\sqrt{-ad})+\cos(2\sqrt{-ad}),$$
$$M=-b^2+4ac<0,\ a<0,c>0.$$

将(11),(42)式一起代入(3)式后得到方程(35)下列形式的精确三角函数解：

$$u_{n(3,4)}(n,t)=g_0+\frac{2g_1\left(b+2\sqrt{ac}\tan\left(\frac{\sqrt{-a}}{2}(dn+\omega t)\right)\right)\cos^2\left(\frac{\sqrt{-a}}{2}(dn+\omega t)\right)}{-2ag_2+bg_3\left(1+\cos(\sqrt{-a}(dn+\omega t))\right)+2g_3\sqrt{-ac}\sin(\sqrt{-a}(dn+\omega t))}.$$

其中 g_0,g_1,g_2 由(42)式来确定.

例 2　KdV 差分微分方程[12]

$$\dot{u}_n(t)=u_n^2(u_{n+1}-u_{n-1}),\tag{43}$$

将

$$u_n(n,t)=u_n(\xi),u_{n+1}(n,t)=u_{n+1}(\xi),$$
$$u_{n-1}(n,t)=u_{n-1}(\xi),\xi=dn+\omega t$$

代入方程(43)后得到下列方程：

$$\omega u'_n(\xi)=u_n^2(\xi)(u_{n+1}(\xi)-u_{n-1}(\xi)),\tag{44}$$

方程(44)的解取为(3)式,其中 $z(\xi)$ 是由(8)式来确定.这样根据(26),(27),(33)式即可得到

$$u_{n+1}(\xi)=g_0+\frac{g_1\left(-b+(b+2az(\xi))\cosh(\sqrt{ad})+2\sqrt{a}\sinh(\sqrt{ad})z'(\xi)\right)}{2ag_2-bg_3+(b+2az(\xi))g_3\cosh(\sqrt{ad})+2\sqrt{a}g_3\sinh(\sqrt{ad})z'(\xi)},\tag{45}$$

$$u_{n-1}(\xi)=g_0+\frac{g_1\left(-b+(b+2az(\xi))\cosh(\sqrt{ad})-2\sqrt{a}\sinh(\sqrt{ad})z'(\xi)\right)}{2ag_2-bg_3+(b+2az(\xi))g_3\cosh(\sqrt{ad})-2\sqrt{a}g_3\sinh(\sqrt{ad})z'(\xi)}.\tag{46}$$

将(3),(5),(45),(46)式一起代入方程(44),并令 $z^q(\xi)z'(\xi)(q=0,1,2)$ 的系数为零后得到一组非线性代数方程组(限于篇幅未列出).用符号计算系统 Mathematica 求出该方程组的如下解：

$$g_0=\mp\frac{\sqrt{\Theta}}{4|acg_3|\sqrt{2g_3\sinh(\sqrt{ad})}},$$

$$g_1=\pm\frac{\left(aMg_3\cosh(\sqrt{ad})-Bb\right)\left|\sinh\left(\frac{\sqrt{ad}}{2}\right)\right|\sqrt{\Theta}}{a\left(b^2+4ac+M\cosh(2\sqrt{ad})\right)|acg_3|\sqrt{2g_3\sinh(\sqrt{ad})}},\ g_2=\frac{abg_3-B}{2a^2};\tag{47}$$

$$g_0=\pm\frac{\sqrt{\gamma}}{4|acg_3|\sqrt{2g_3\sinh(\sqrt{ad})}},$$

$$g_1=\mp\frac{\left(aMg_3\cosh(\sqrt{ad})+Bb\right)\left|\sinh\left(\frac{\sqrt{ad}}{2}\right)\right|\sqrt{\gamma}}{a\left(b^2+4ac+M\cosh(2\sqrt{ad})\right)|acg_3|\sqrt{2g_3\sinh(\sqrt{ad})}},\ g_2=\frac{abg_3+B}{2a^2}.\tag{48}$$

其中

$$\Theta=\frac{\omega g_3^2}{\sqrt{a}}\Big(A\big((ab^2-2a^2c)g_3+bB\big)-4b\cosh(\sqrt{a}d)\big((ab^3-4a^2bc)g_3+(b^2-2ac)B\big)\Big)>0,$$

$$A=3b^2-4ac-M\cosh(2\sqrt{a}d),$$

$$B=|ag_3|\sqrt{-M},M=-b^2+4ac<0,$$

$$\gamma=\frac{\omega g_3^2}{\sqrt{a}}\Big(A\big((ab^2-2a^2c)g_3-bB\big)-4b\cosh(\sqrt{a}d)\big((ab^3-4a^2bc)g_3-(b^2-2ac)B\big)\Big)>0,$$

$$g_3\sinh(\sqrt{a}d)>0,a>0.$$

将(47),(48)式分别与(8)式一起代入(3)式后得到方程(43)如下形式的精确孤波解:

$$u_{n(5,6)}(n,t)=g_0+\frac{g_1(2b+(1+b^2-4ac)\cosh(\sqrt{a}(dn+\omega t))+(-1+b^2-4ac)\sinh(\sqrt{a}(dn+\omega t)))}{-4ag_2+2bg_3+(1+b^2-4ac)\cosh(\sqrt{a}(dn+\omega t))+(-1+b^2-4ac)\sinh(\sqrt{a}(dn+\omega t))}.$$

其中 g_0,g_1,g_2 由(47),(48)式来确定.

4. 结 论

文献[12,14]用耦合的 Riccati 方程组(49),(50)得到了 Benjamin Ono 方程、Volterra 差分微分方程和 KdV 差分微分方程的精确解.

$$g(\xi_n)=\frac{f'(\xi_n)}{pf(\xi_n)},\tag{49}$$

$$g'(\xi_n)=q+pg^2(\xi_n)-rf(\xi_n).\tag{50}$$

将(49)式代入方程(50)经整理后得到下列方程:

$$f''(\xi_n)f(\xi_n)-2(f'(\xi_n))^2-pqf^2(\xi_n)+prf^3(\xi_n)=0,\tag{51}$$

对方程(51)进行变换 $f(\xi_n)w(\xi_n)=1$ 得到下列方程 (52)和它的三个解:

$$w''(\xi_n)+pqw(\xi_n)-pr=0.\tag{52}$$

$$w(\xi_n)=\frac{1}{q}\big(r+s\cosh(\sqrt{-pq}\xi_n)+h\sinh(\sqrt{-pq}\xi_n)\big),(qp<0);\tag{53}$$

$$w(\xi_n)=\frac{1}{q}\big(r+s\cos(\sqrt{pq}\xi_n)+h\sin(\sqrt{pq}\xi_n)\big),(qp>0);\tag{54}$$

$$w(\xi_n)=\frac{1}{2}(pr\xi_n^2+s\xi_n+h),(q=0).\tag{55}$$

其中 s,h 是任意常数.

当方程(5)的系数 a,b,c 之间满足一定关系时得到下列形式的解:

当 $c=\frac{b^2}{4a}-a(m_1^2-m_2^2)$ 时,

$$z(\xi)=-\frac{1}{2a}\big(b-2am_1\cosh(\sqrt{a}\xi)+2am_2\sinh(\sqrt{a}\xi)\big),(a>0);\tag{56}$$

当 $c=\frac{b^2}{4a}-a(m_1^2+m_2^2)$ 时,

$$z(\xi)=\frac{1}{2a}\big(-b+2am_1\cos(\sqrt{-a}\xi)+2am_2\sin(\sqrt{-a}\xi)\big),(a<0);\tag{57}$$

当 $c=-bm_1^2+m_2^2$ 时,

$$z(\xi)=\frac{b}{4}\xi^2+m_1\xi+m_2,(a=0).\tag{58}$$

其中 m_1,m_2 是任意常数.

在方程(5)的解(56)—(58)中取 $a=-pq,b=2qr$ 后即可得到方程(52)的型如(53—(55)的解.文献[12]在 $p=-1,q=1$ 和 $p=1,q=1$ 时,获得了 Volterra 差分微分方程和 KdV 差分微分方程的下列形式的精确解:

$$u_{n(1)}(n,t)=A_0+\frac{B_0}{r+p\cosh(\xi)+q\sinh(\xi)};$$

$$u_{n(2)}(n,t)=A_0+\frac{C_0}{r+p\cos(\xi)+q\sin(\xi)};$$

$$u_{n(3)}(n,t)=A_0+\frac{E_0}{(r+p\cosh(\xi)+q\sinh(\xi))^2};$$

$$u_{n(4)}(n,t)=A_0+\frac{H_0}{(p\cos(\xi)+q\sin(\xi))^2};$$

$$u_{n(5)}(n,t)=A_0+\frac{F_0}{r+p\cosh(\xi)+q\sinh(\xi)}+\frac{G_0}{(r+p\cosh(\xi)+q\sinh(\xi))^2};$$

$$u_{n(6)}(n,t)=A_0+\frac{K_0}{r+p\cos(\xi)\pm iq\sin(\xi)}+\frac{L_0}{(r+p\cos(\xi)\pm iq\sin(\xi))^2}.$$

这里 $A_0, B_0, C_0, D_0, E_0, F_0, H_0, K_0, L_0$ 是常数.

本文利用辅助方程(5)的三个解(6),(8),(11)和函数变换(3),构造了 Volterra 差分微分方程和 KdV 差分微分方程的型如 $u_{n(1,2)}(n,t)$, $u_{n(3,4)}(n,t)$, $u_{n(5,6)}(n,t)$ 的新精确解.辅助方程(5)的其余解与函数变换(3)相结合也可以得到差分微分方程(35),(43)的其他形式新的精确解.该方法也可以得到一般格子方程,(2 + 1)维 Hybrid-Lattice 系统和 mKdV 差分微分方程等[6—11,13]其他非线性差分微分方程新的精确解.

$$\dot{u}_n(t) = (\alpha + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}), \tag{59}$$

$$\dot{u}_n(t) = (1 + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}), \tag{60}$$

$$\dot{u}_n(t) = (\alpha - u_n^2)(u_{n+1} - u_{n-1}). \tag{61}$$

由于求出的非线性代数方程组解的形式比较复杂,这里未讨论.

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The new exact solutions of Volterra difference-differential equation and KdV difference-differential equation*

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Abstract

Based on the auxiliary equation method and the trial function method, a method for combining function transformation with auxiliary equation is proposed. And the method is applied to construct new exact solitary wave solutions and triangle function solutions of Volterra and KdV difference-differential equations with the help of symbolic computation system Mathematica. Our method can also be applied to other nonlinear difference-differential equations.

Keywords: auxiliary equation, function transformation, nonlinear difference-differential equation, solitary wave solution

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