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广义 Pfaff-Birkhoff-d'Alembert 原理与广义 Birkhoff 系统的形式不变性*

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研究了广义 Pfaff-Birkhoff-d'Alembert 原理和广义 Birkhoff 系统在群的无限小变换下的形式不变性质. 给出了形式不变性的判据,并由形式不变性导出 Noether 守恒量和新型守恒量.

关键词: 广义 Pfaff-Birkhoff-d'Alembert 原理, 广义 Birkhoff 系统, Noether 守恒量, 形式不变性

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1. 引 言

对称性与守恒量的研究具有重要的数学和物理意义. 对称性理论可以用于常微分方程的降阶和偏微分方程的约化,也可用于分岔理论和控制理论,还可用于寻找守恒量. 对于约束力学系统的微分方程,如能找到某个守恒量,便可了解系统动力学的某些信息;如能找到足够多的守恒量,便可知道系统的解. 对称性的研究主要有 Noether 对称性[1—7]、Lie 对称性[6—10] 和形式不变性[11—16] 等. Noether 对称性是 Hamilton 作用量在无限小群变换下的不变性质. Lie 对称性是微分方程在无限小变换下的不变性质. 形式不变性是指动力学函数(Lagrange 函数、动能、势能、广义力、约束函数、广义约束力等)在无限小群变换后仍满足原来方程的一种不变性质. 本文提出广义 Pfaff-Birkhoff-d'Alembert 原理,由这个原理导出形式不变性的判据,并由形式不变性导出 Noether 守恒量和新型守恒量.

2. 广义 Pfaff-Birkhoff-d'Alembert 原理

广义 Pfaff-Birkhoff 原理可表示为[17]

$$\int_{t_1}^{t_2}\left\{\delta\left[\sum_{\nu=1}^{2n}\mathrm{R}_\nu(t,a)\dot{a}^\nu - B(t,a)\right] + \Lambda_\mu(t,a)\,\delta a^\mu\right\}\mathrm{d}t = 0. \tag{1}$$

(1)式也可表示为

$$\int_{t_1}^{t_2}\sum_{\mu=1}^{2n}\left\{\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu\right\}\delta a^\mu \mathrm{d}t = 0\,. \tag{2}$$

考虑到时间区间$[t_1,t_2]$的任意性,由(2)式得到

$$\sum_{\mu=1}^{2n}\left\{\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu\right\}\delta a^\mu = 0\,. \tag{3}$$

称(3)式为广义 Pfaff-Birkhoff-d'Alembert 原理.

由(3)式可导出广义 Birkhoff 方程

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu = 0 \quad (\mu,\nu=1,\cdots,2n)\,. \tag{4}$$

3. 形式不变性的判据

取时间 t 和变量 a^μ 的无限小变换为

$$t^* = t+\Delta t,$$
$$a^{\mu*}(t^*) = a^\mu(t)+\Delta a^\mu \qquad (\mu = 1,\cdots,2n), \tag{5}$$

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相应的展开式为

$$t^{*}=t+\varepsilon\xi_0(t,a),$$
$$a^{\mu *}(t^{*})=a^{\mu}(t)+\varepsilon\xi_{\mu}(t,a)\qquad(\mu=1,\cdots,2n).\tag{6}$$

由(5)和(6)式得到

$$\begin{aligned}\delta a^{\mu}&=\Delta a^{\mu}-\dot a^{\mu}\Delta t\\&=\varepsilon(\xi_{\mu}-\dot a^{\mu}\xi_0)\qquad(\mu=1,\cdots,2n).\end{aligned}\tag{7}$$

将(7)式代入(3)式得到

$$\varepsilon\sum_{\mu=1}^{2n}\left\{\sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot a^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\Lambda_{\mu}\right\}(\xi_{\mu}-\dot a^{\mu}\xi_0)=0.$$

形式不变性定义为

$$\varepsilon\sum_{\mu=1}^{2n}\left\{\sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}^{*}}{\partial a^{\mu}}-\frac{\partial R_{\mu}^{*}}{\partial a^{\nu}}\right)\dot a^{\nu}-\frac{\partial B^{*}}{\partial a^{\mu}}-\frac{\partial R_{\mu}^{*}}{\partial t}+\Lambda_{\mu}^{*}\right\}(\xi_{\mu}-\dot a^{\mu}\xi_0)=0.\tag{8}$$

注意到

$$\begin{aligned}R_{\mu}^{*}&=R_{\mu}(t^{*},a^{*})\\&=R_{\mu}(t,a)+\varepsilon X^{(0)}(R_{\mu})+o(\varepsilon^2),\\B^{*}&=B(t^{*},a^{*})\\&=B(t,a)+\varepsilon X^{(0)}(B)+o(\varepsilon^2),\\\Lambda_{\mu}^{*}&=\Lambda_{\mu}(t^{*},a^{*})\\&=\Lambda_{\mu}(t,a)+\varepsilon X^{(0)}(\Lambda_{\mu})+o(\varepsilon^2),\end{aligned}\tag{9}$$

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\sum_{\mu=1}^{2n}\xi_{\mu}\frac{\partial}{\partial a^{\mu}}.$$

将(9)式代入(8)式得

$$\begin{aligned}&\varepsilon\sum_{\mu=1}^{2n}\left\{\sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)a^{\nu}-\frac{\partial \dot B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\Lambda_{\mu}\right\}(\xi_{\mu}-\dot a^{\mu}\xi_0)\\&+\varepsilon^2\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\left(\frac{\partial X^{(0)}(R_{\nu})}{\partial a^{\mu}}-\frac{\partial X^{(0)}(R_{\mu})}{\partial a^{\nu}}\right)\dot a^{\nu}-\frac{\partial X^{(0)}(B)}{\partial a^{\mu}}-\frac{\partial X^{(0)}(R_{\mu})}{\partial t}+X^{(0)}(\Lambda_{\mu})\right]\\&\times(\xi_{\mu}-\dot a^{\mu}\xi_0)+o(\varepsilon^3)=0.\end{aligned}\tag{10}$$

(10)式中 ε 项为零[18—20]，略去 ε^3 项，则得到广义 Birkhoff 系统形式不变性的判据方程

$$\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\left(\frac{\partial X^{(0)}(R_{\nu})}{\partial a^{\mu}}-\frac{\partial X^{(0)}(R_{\mu})}{\partial a^{\nu}}\right)\dot a^{\nu}-\frac{\partial X^{(0)}(B)}{\partial a^{\mu}}-\frac{\partial X^{(0)}(R_{\mu})}{\partial t}+X^{(0)}(\Lambda_{\mu})\right]\times(\xi_{\mu}-\dot a^{\mu}\xi_0)=0.\tag{11}$$

4. 形式不变性导致的守恒量

由广义 Birkhoff 系统形式不变性可导出 Noether 守恒量.

命题 1　对于满足判据方程(11)的无限小生成元 ξ_0 和 ξ_{μ}，如果存在规范函数 $G_{\rm N}=G_{\rm N}(t,a)$ 满足结构方程

$$\sum_{\mu=1}^{2n}X^{(0)}(R_{\mu})\dot a^{\mu}-X^{(0)}(B)+\sum_{\mu=1}^{2n}R_{\mu}\dot\xi_{\mu}-B\dot\xi_0+\sum_{\mu=1}^{2n}\Lambda_{\mu}(\xi_{\mu}-\dot a^{\mu}\xi_0)+\dot G_{\rm N}=0,\tag{12}$$

则系统不变性导出 Noether 守恒量

$$I_{\rm N}=\sum_{\mu=1}^{2n}R_{\mu}\xi_{\mu}-B\xi_0+G_{\rm N}={\rm const}.\tag{13}$$

展开广义 Birkhoff 方程，记作

$$\dot a^{\mu}=\sigma^{\mu}(t,a).\tag{14}$$

由广义 Birkhoff 系统的形式不变性可导出一类新型守恒量.

命题 2　对于满足判据方程(11)的无限小生成元 ξ_0 和 ξ_{μ}，如果存在规范函数 $G_{\rm F}=G_{\rm F}(t,a)$ 满足结构方程

$$\begin{aligned}&\sum_{\mu=1}^{2n}X^{(0)}\{X^{(0)}(R_{\mu})\}\delta^{\mu}-X^{(0)}\{X^{(0)}(B)\}\\&+\sum_{\mu=1}^{2n}X^{(0)}(R_{\mu})\frac{\bar{\rm d}}{{\rm d}t}\xi_{\mu}-X^{(0)}(B)\frac{\bar{\rm d}}{{\rm d}t}\xi_0\\&+\sum_{\mu=1}^{2n}X^{(0)}(\Lambda_{\mu})(\xi_{\mu}-\delta^{\mu}\xi_0)+\frac{\bar{\rm d}}{{\rm d}t}G_{\rm F}=0,\end{aligned}\tag{15}$$

则系统形式不变性导致新型守恒量

$$\begin{aligned}I_{\rm F}&=\sum_{\mu=1}^{2n}X^{(0)}(R_{\mu})\xi_{\mu}-X^{(0)}(B)\xi_0+G_{\rm F}\\&={\rm const}.\end{aligned}\tag{16}$$

这里

$$\frac{\bar{\rm d}}{{\rm d}t}=\frac{\partial}{\partial t}+\delta^{\mu}\frac{\partial}{\partial a^{\mu}}.\tag{17}$$

5. 算　　例

4 阶广义 Birkhoff 系统为

$$
\begin{aligned}
&R_1 = a^3, \\
&R_2 = a^4, \\
&R_3 = R_4 = 0, \\
&B = -\frac{1}{2}(a^3)^2 - \frac{1}{2}(a^4)^2, \\
&\Lambda_1 = \Lambda_2 = 0, \\
&\Lambda_3 = -\frac{1}{b}\arctan(bt), \\
&\Lambda_4 = -\frac{1}{2b}\ln(1 + b^2t^2).
\end{aligned} \tag{18}
$$

相应的广义 Birkhoff 方程为

$$
\begin{aligned}
&-\dot{a}^3 = 0, \\
&-\dot{a}^4 = 0, \\
&\dot{a}^1 + a^3 = \frac{1}{b}\arctan(bt), \\
&\dot{a}^2 + a^4 = -\frac{1}{2b}\ln(1 + b^2t^2).
\end{aligned} \tag{19}
$$

取生成元为

$$
\begin{aligned}
&\xi_0 = \xi_2 = \xi_3 = \xi_4 = 0, \\
&\xi_1 = 1,
\end{aligned} \tag{20}
$$

则有

$$
\begin{aligned}
&X^{(0)}(R_1) = \xi_3 = 0, \\
&X^{(0)}(R_2) = \xi_4 = 0, \\
&X^{(0)}(R_3) = X^{(0)}(R_4) = 0, \\
&X^{(0)}(B) = -a^3\xi_3 - a^4\xi_4 = 0, \\
&\Lambda_1(\xi_1 - \dot{a}^1\xi_0) + \Lambda_2(\xi_2 - \dot{a}^2\xi_0) \\
&\quad + \Lambda_3(\xi_3 - \dot{a}^3\xi_0) + \Lambda_4(\xi_4 - \dot{a}^4\xi_0) = 0.
\end{aligned} \tag{21}
$$

由(12)式得

$$G_{\mathrm{N}} = 0.$$

因此,生成元(20)式是系统形式不变性的生成元.

命题 1 的守恒量(14)式给出 Noether 守恒量

$$I_{\mathrm{N}} = a^3 = \mathrm{const}. \tag{22}$$

由于

$$X^{(0)}(\Lambda_\mu) = 0 \qquad (\mu = 1, \cdots, 4), \tag{23}$$

将(21)和(23)式代入(15)式得

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} = 0. \tag{24}$$

由(17),(19)和(24)式得

$$G_{\mathrm{F}} = 0, \tag{25}$$

$$G_{\mathrm{F}} = a^3, \tag{26}$$

$$G_{\mathrm{F}} = a^4. \tag{27}$$

由新型守恒量(16)式分别给出

$$I_{\mathrm{F}} = 0, \tag{28}$$

$$I_{\mathrm{F}} = a^3, \tag{29}$$

$$I_{\mathrm{F}} = a^4, \tag{30}$$

其中(28)式为平凡守恒量,而(29)和(30)式给出非平凡守恒量.

6. 结 论

本文研究了广义 Pfaff-Birkhoff-d'Alembert 原理和广义 Birkhoff 系统在群的无限小变换下的形式不变性质.研究表明,广义 Pfaff-Birkhoff-d'Alembert 原理比普通的 Pfaff-Birkhoff-d'Alembert 原理更具一般性,广义 Birkhoff 系统在群的无限小变换下的形式不变性能够导出 Noether 守恒量和一类新型守恒量.

[1] Noether A E 1918 *Nachr Akad. Wiss Göttingen Math. Phys.* **1** 235

[2] Djukic D, Vujanovic B 1975 *Acta Mech.* **23** 17

[3] Li Z P 1981 *Acta Phys. Sin.* **30** 1659 (in Chinese)[李子平 1981 物理学报 **30** 1659]

[4] Liu D 1989 *Acta Mech. Sin.* **21** 75 (in Chinese)[刘 端 1989 力学学报 **21** 75]

[5] Li Z P 1993 *Classical and Quantal Dynamics of Constrained Systems and Their Symmetrical Properties* (Beijing: Beijing Polytechnic University Press)(in Chinese)[李子平 1993 经典和量子约束系统及其对称性质(北京:北京工业大学出版社)]

[6] Zhao Y Y, Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press)(in Chinese)[赵跃宇、梅凤翔 1999 力学系统的对称性与不变量(北京:科学出版社)]

[7] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)[梅凤翔 1999 李群和李代数对约束力学系统的应用(北京:科学出版社)]

[8] Lutzky M 1979 *J. Phys.* A **12** 973

[9] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380 (in Chinese)[赵跃宇 1994 力学学报 **26** 380]

[10] Mei F X 2000 *Acta Mech.* **141** 135

[11] Mei F X 2000 *J. Beijing Inst. Techn.* **9** 120

[12] Wang S Y, Mei F X 2002 *Chin. Phys.* **11** 5

[13] Mei F X, Xie J F, Gang T Q 2008 *Acta Mech. Sin.* **24** 583 (in Chinese)[梅凤翔、解加芳、江铁强 2008 物理学报 **24** 583]

[14] Mei F X, Gang T Q, Xie J F 2006 *Chin. Phys.* **15** 1678

[15] Ge W K, Mei F X 2009 *Acta Phys. Sin.* **58** 699 (in Chinese) [葛伟宽、梅凤翔 2009 物理学报 **58** 699]

[16] Mei F X, Cai J L 2008 *Acta Phys. Sin.* **57** 4657 (in Chinese) [梅凤翔、蔡建乐 2008 物理学报 **57** 4657]

[17] Mei F X, Zhang Y F, He G, Gang T Q, Xie J F 2007 *J. Beijing Inst. Techn.* **27** 1035 (in Chinese) [梅凤翔、张永发、何　光、江铁强、解加芳 2007 北京理工大学学报 **27** 1035]

[18] Mei F X 1993 *Sci. China* **36** 1456

[19] Mei F X, Shang M 2009 *Chin. Phys.* B **18** 3155

[20] Zhang Y 2008 *Chin. Phys.* B **17** 4365

Generalized Pfaff-Birkhoff-d'Alembert principle and form invariance of generalized Birkhoff 's equations*

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Abstract

The generalized Pfaff-Birkhoff-d'Alembert principle and the form invariance of the generalized Birkhoff's equations under the infinitesimal transformation of group are studied. The criterion of the form invariance is given and the Noether conserved quantity and a new conserved quantity are obtained by the invariance.

Keywords: generalized Pfaff-Birkhoff-d'Alembert principle, generalized Birkhoff's equations, Noether conserved quantity, form invariance

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