ON THE HEISENBERG PICTURE IN QUANTUM ELECTRODYNAMICS

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ABSTRACT

The form of quantum theory of radiation introduced by Heisenberg is discussed from the point of view of the transformation theory of quantum electrodynamics. A general investigation of the connection between Heisenberg's method and Dirac's method of variation of parameters is given. The extension of Heisenberg's method to eigenvalue problems, which was first carried out by Weisskopf for the self energy of the electron, is presented in such a way as to show more clearly its quantum mechanical interpretation. A general proof of the equivalence of Weisskopf's method and the method of the perturbation theory of stationary states in quantum mechanics is given.

1. *Introduction.*

Heisenberg[1] introduced in 1931 a form of quantum theory of radiation which seems at first sight to rest on a theoretical basis essentially different from that of the original theory developed by Dirac[2] with the help of his method of variation of parameters. In Heisenberg's treatment of the interaction of the electrons and the electromagnetic field, no use is made of their Hamiltonians and the Schroedinger equations for their representatives. The theory is based on Maxwell's equations and Dirac's equation for the electromagnetic field and electron waves respectively, and these equations are integrated with respect to time in the same way as in the classical theory. The only difference of Heisenberg's theory from the classical theory is that the field variables are regarded as operators instead of ordinary numbers. The quantum conditions for these operators lead at once to the physical

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[1] Heisenberg, Ann. d. Phys. *9*, 338 (1931).

[2] Dirac, Proc. Roy. Soc. A. *114*, 243, 710 (1927).

interpretation of the theory, according to which the quantized waves are correlated to the emission and absorption operators for the particles arising from second quantization. It was shown by Heisenberg that his method gave the same results as Dirac's method for the phenomena of emission, absorption and scattering of radiation field by electrons.

Heisenberg's form of quantum theory of radiation has a much closer resemblance to the classical theory than Dirac's, especially when the classical theory is interpreted according to Bohr's Correspondence Principle. It is for this reason that Heisenberg's method provides a more convenient mathematical basis for showing the analogy between the classical and quantum theories of radiation, which is often a useful guide for further progress, and for justifying the Correspondence Principle from the theoretical point of view.

Heisenberg's method can also be applied to collision problems such as that of the retarded interaction of electrons. This was done by Møller[3], who obtained a collision cross section confirmed by later work based on the method of variation of parameters[4].

From the point of view of general theory it is important to study the connection between the two forms of quantum theory of radiation. For this purpose it is not sufficient to demonstrate that the two methods are equivalent merely on the ground that they always lead to identical results in practical problems. One has to go deeper into the physical interpretation and mathematical formalism of quantum electrodynamics, in order to establish the connection between the two methods on a more general basis. The situation may be compared with that in the early stage of development of quantum mechanics. The matrix mechanics introduced by Heisenberg and the wave mechanics introduced by Schroedinger appeared at first to be essentially different theories. A clear understanding of the equivalence of the two theories, after it had been established by Schroedinger, was first provided by the transformation theory of Dirac and Jordan. According to Dirac[5], the two theories are just two "pictures" of the same states of a dynamical system. Dirac showed that the connection between the two theories could be understood very satisfactorily with the help of the transformation theory.

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3. Møller, Zs. f. Phys. *70*, 786 (1931); Ann. d. Phys. *14*, 531 (1932).
4. Heitler, Quantum Theory of Radiation, 1936.
5. Dirac, The Principles of Quantum Mechanics, second edition, 1935.

The situation in the quantum theory of radiation is similar. It is clear that Heisenberg's theory of radiation, which deals directly with the variation of operators representing field variables of electron and electromagnetic waves, corresponds to the Heisenberg picture of quantum theory. As will be shown in §2, the connection between the two forms of quantum theory of radiation can be obtained with the help of the transformation theory on the same lines as Dirac's work.

In dealing with the phenomena of emission, absorption and scattering, there is no need to consider the Hamiltonian of the system of electrons and electromagnetic field. It becomes necessary, however, to incorporate the idea of the Hamiltonian into Heisenberg's mathematical scheme, when we wish to study the change in energy levels caused by the interaction between the electrons and electromagnetic field. This extension of Heisenberg's method was developed by Weisskopf[6] in his calculation of the self energy of the electron. Weisskopf calculated the self energy of the electron according to both the one-electron theory and the positron theory. This problem had previously been investigated by several authors on the one-electron theory, the most complete result for a free electron being that obtained by Waller[7]. As in the other cases mentioned above, Weisskopf's result for the one-electron theory was found to be in complete agreement with Waller's.

Like Heisenberg's treatment of radiation, Weisskopf's treatment of the self energy problem has the advantage of bringing out the analogy between the classical and quantum theories. It enables us to trace the origin of the divergencies of quantized field theory, which are mixed up in the works of the other authors. On the other hand, its physical meaning is not quite obvious from the point of view of the quantum theory, and its connection with the ordinary perturbation theory in quantum theory is not clear. It will be my object to discuss the quantum mechanical interpretation of Weisskopf's treatment and the mathematical equivalence of Weisskopf's method and the ordinary perturbation method in §3.

Throughout this paper we confine our attention to the transverse part of the electromagnetic field, which is related to the light quanta and is treated diffe-

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[6] Weisskopf, Zs. f. Phys. *89*, 27 (1934); *90*, 817 (1934); Phys. Rev. *56*, 72 (1939).

[7] Waller, Zs. f. Phys. *62*, 673 (1930).

rently in the classical and quantum theories. We shall leave out of account the classical longitudinal part of the electromagnetic field and the consequences that follow from its mathematical treatment.

Dirac's new bracket notation[8] will be used.

2. *Theory of Transitions.*

Heisenberg's theory of emission, absorption and scattering is applicable to a single electron as well as to several electrons. As the generalization from the case of a single electron to the case of several electrons does not involve any essential change in the mathematical method, we shall for convenience discuss only the case of one electron in the following considerations.

Heisenberg's theory of the interaction of an electron with the radiation field is based on the following fundamental assumptions:

1. The electron wave is described by a wave function with four components, $\psi(x, y, z, t)$, which satisfies the Dirac equation

$$\left\{ i\hbar\frac{\partial}{\partial t} + c\,\alpha\cdot\left(\frac{\hbar}{i}\nabla + \frac{e}{c}A\right) + \beta mc^2 \right\}\psi = 0. \tag{1}$$

Unlike the wave function in the original theory of Dirac, the function ψ is here regarded as an operator. If we expand ψ in the form

$$\psi = \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t}\,\varphi_n(r), \tag{2}$$

where the a_n's are operators and the φ_n's are ordinary Dirac wave functions

$$\varphi_n = \frac{1}{\sqrt{V}}\, u_n\, e^{i p_n \cdot r/\hbar} \tag{3}$$

that are normalized in a volume V and satisfy the relations

$$\int \varphi_n^{*}\, \varphi_m\, dv = \delta_{nm}, \tag{4}$$

then the expectation value of the operator $a_n^{*}(t)a_n(t)$ is interpreted as the probability of finding the electron in the state n at time t. The variations of

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[8] Dirac, Proc. Camb. Phil. Soc., 35, 416 (1939)

these probabilities with time give the transition probabilities between the different states of the electron.

2. The vector potential $\underline{A}$ of the electromagnetic field is determined by the current density of the electron wave, $\underline{j}$, according to the wave equation of the classical electromagnetic theory

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\underline{A} = \frac{4\pi}{c}\underline{j}, \tag{5}$$

where

$$\underline{j} = ec\psi^{\dagger}\underline{\alpha}\psi, \tag{6}$$

the quantities $\underline{A}$ and $\underline{j}$ being operators. The electric and magnetic field strengths $\underline{E}$ and $\underline{H}$ are derived from $\underline{A}$ as usual. In quantum theory it is only necessary to consider the transverse part of $\underline{A}$, which can be expanded in the form

$$\underline{A} = \sqrt{\frac{ch}{V}}\sum_n \left(b_n e^{-i\omega_n t} e^{i\underline{k}_n \cdot \underline{r}} + b_n^{\dagger} e^{i\omega_n t} e^{-i\underline{k}_n \cdot \underline{r}}\right)\underline{e}_n/\sqrt{k_n}, \tag{7}$$

where the b's and $b^{\dagger}$'s are operators and the $\underline{e}$'s are unit vectors in the directions of polarization. The expectation value of $b_n^{\dagger}(t)b_n(t)$ gives the probability of finding a light-quantum in the state n at time t.

Let us consider now these assumptions in accordance with the general principles of quantum theory. Let the Hamiltonian of the whole system be denoted by

$$H = H_a + H_b + H_I, \tag{8}$$

where H_a is the Hamiltonian of the electron, H_b the Hamiltonian of the radiation field and H_I the operator representing the perturbing energy arising from the interaction between the electron and the radiation field. We work with a Heisenberg representation for the unpertrubed system, i.e. a representation in which $H_a + H_b$ is diagonal, and take as basic states the states for which the electron has definite values of energy and momentum together with a definite direction of spin, and the radiation field consists of light quanta with definite values of energy and momentum and definite direction of polarization. We use a general set of α's and β's to label the representatives for the electron and radiation field respectively, so that a basic state is represented by a vector of the form $|\alpha'\beta'>$. The representatives of the operators H_a, H_b, H_I in this representation are

$$\left.\begin{aligned} <\alpha'\beta'\,|H_a|\,\alpha''\beta''> &= \delta_{\beta'\beta''}\int \varphi^{\dagger}_{\alpha'}(ic\hbar\alpha\cdot\nabla-\beta mc^2)\varphi_{\alpha''}\,dv,\\ <\alpha'\beta'\,|H_b|\,\alpha''\beta''> &= \delta_{\alpha'\alpha''}\,\delta_{\beta'\beta''}\,E_b(\beta'),\\ <\alpha'\beta'\,|H_I|\,\alpha''\beta''> &= -\frac{1}{c}\int \varphi^{\dagger}_{\alpha'}<\beta'|\,j\cdot A\,|\beta''>\varphi_{\alpha''}\,dv. \end{aligned}\right\} \quad (9)$$

The representatives of any state of the whole system vary with time according to the Schroedinger equations

$$\left.\begin{aligned} i\hbar\frac{d}{dt}\left(<\alpha'\beta'|>\right) &= \sum_{\alpha''\beta''}<\alpha'\beta'|H|\alpha''\beta''><\alpha''\beta''|>,\\ i\hbar\frac{d}{dt}\left(<|\alpha'\beta'>\right) &= -\sum_{\alpha''\beta''}<|\alpha''\beta''><\alpha''\beta''|H|\alpha'\beta'>. \end{aligned}\right\} \quad (10)$$

Dirac's method of variation of parameters consists in solving these differential equations for the representatives $<\alpha'\beta'|>$, $<|\alpha'\beta'>$. Alternatively we can determine the variation of operators with time in the Heisenberg picture, in which the vectors $>$, $<$ are fixed and the basic vectors $|\alpha'\beta'>$, $<\alpha'\beta'|$ vary with time according to the formulae

$$i\hbar\frac{d}{dt}|\alpha'\beta'> = -H|\alpha'\beta'>, \qquad i\hbar\frac{d}{dt}<\alpha'\beta'| = <\alpha'\beta'|H,$$

so that any operator ξ satisfies the equation

$$i\hbar\frac{d\xi}{dt} = [\xi, H], \quad (11)$$

with $$[\xi, H] = \xi H - H\xi.$$

Consider first Heisenberg's treatment of the electron waves. Heisenberg's method can be understood from two different points of view. We may apply to the electron waves the procedure of second quantization[9] and make use of the commutation relations for particles satisfying the Fermi-Dirac statistics, formu-

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[9] Pauli, Handbuch der Physik, XXIV/1. 208 (1933).

lated by Jordan and Wigner and by Heisenberg and Pauli.[10] We may also proceed on the basis of one-particle theory without considering the creation or annihilation of electron pairs. In order to compare with the theory of radiation as originally developed by Dirac before the positron theory came into existence, we shall adopt the second point of view.

For the purpose of passing to Heisenberg's theory we introduce a fictitious zero state for the electron, represented by a vector of the form $|o\beta'>$, and also emission and absorption operators $c^{\dagger}_{\alpha'}$, $c_{\alpha'}$ for the electron, with the matrix elements

$$<\alpha''\beta''|\,c_{\alpha'}\,|\alpha'''\beta'''> = <\alpha'''\beta'''|\,c^{\dagger}_{\alpha'}\,|\alpha''\beta''> = \delta_{\alpha''o}\,\delta_{\alpha'''\alpha'}\,\delta_{\beta'\beta''}\,. \tag{12}$$

It follows from (12) that the operators $c_{\alpha'}$, $c^{\dagger}_{\alpha'}$ may be written in the forms

$$c_{\alpha'} = \sum_{\beta'} |o\beta'><\alpha'\beta'|, \qquad c^{\dagger}_{\alpha'} = \sum_{\beta'} |\alpha'\beta'><o\beta'|, \tag{13}$$

the summation over β' ensuring that these operators do not give rise to any transition from one state of the radiation field to another. Introducing further operators $H_{\alpha'\alpha''}$ which have the representatives

$$<\alpha'''\beta'|\,H_{\alpha'\alpha''}\,|\alpha^{IV}\beta''> = \delta_{\alpha'''o}\,\delta_{\alpha^{IV}o}<\alpha'\beta'|H|\alpha''\beta''>, \tag{14}$$

we have the relation

$$<\alpha'''\beta'|\sum_{\alpha'\alpha''} c^{\dagger}_{\alpha'}\,H_{\alpha'\alpha''}\,c_{\alpha''}|\alpha^{IV}\beta''> = <\alpha'''\beta'|\,c^{\dagger}_{\alpha'''}\,H_{\alpha'''\alpha^{IV}}\,c_{\alpha^{IV}}\,|\alpha^{IV}\beta''>$$

$$= <\alpha'''\beta'|\,H\,|\alpha^{IV}\beta''>$$

or

$$H = \sum_{\alpha'\alpha''} c^{\dagger}_{\alpha'}\,H_{\alpha'\alpha''}\,c_{\alpha''}\,. \tag{15}$$

In the Heisenberg picture the operator $c_{\alpha'}$ varies with time according to equation (11) with $\xi = c_{\alpha'}$ namely

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[10] Jordan and Wigner, Zs. f. Phys. *47*, 631 (1928); Heisenberg and Pauli, Zs. f. Phys. *56*, 1 (1929).

$$i\hbar\frac{dc_{\alpha'}}{dt}=\left[c_{\alpha'},H\right]=\left[c_{\alpha'},\sum_{\alpha''\alpha'''}c^{\dagger}_{\alpha''}H_{\alpha''\alpha'''}c_{\alpha'''}\right].$$

On account of the relations

$$c_{\alpha'}c^{\dagger}_{\alpha''}c_{\alpha'''}=\delta_{\alpha'\alpha''}c_{\alpha'}c^{\dagger}_{\alpha'}c_{\alpha'''}=\delta_{\alpha'\alpha''}c_{\alpha'''}$$

and
$$c_{\alpha'''}c_{\alpha'}=0$$

we have
$$i\hbar\frac{dc_{\alpha'}}{dt}=\sum_{\alpha'''}H_{\alpha'\alpha''}c_{\alpha''}. \tag{16}$$

The operators $H_{\alpha'\alpha''}$ are constant operators, as may be easily verified from (11), (14) and (15). Equation (16) gives

$$i\hbar\frac{d}{dt}\sum_{\alpha'}c_{\alpha'}\varphi_{\alpha'}(r)=\sum_{\alpha'\alpha''}H_{\alpha'}H_{\alpha'\alpha''}c_{\alpha''}=H\sum_{\alpha''}c_{\alpha''}\varphi_{\alpha''}.$$

Hence, if we put
$$\psi=e^{\frac{i}{\hbar}H_b t}\sum_{\alpha'}c_{\alpha'}\varphi_{\alpha'}, \tag{17}$$

$$\begin{aligned} i\hbar\frac{d}{dt}\psi &= e^{\frac{i}{\hbar}H_b t}(-H_b+H)\sum_{\alpha'}c_{\alpha'}\varphi_{\alpha'} \\ &= \left(H_a+e^{\frac{i}{\hbar}H_b(t)}H_I\,e^{-\frac{i}{\hbar}H(b)t}\right)\psi, \end{aligned} \tag{18}$$

which is just (1). Comparing (17) with (2) we see that

$$c_{\alpha'}=a_{\alpha'}e^{-\frac{i}{\hbar}(H_a+H_b)t}. \tag{19}$$

The expectation value of the operator $a^{\dagger}_{\alpha'}a_{\alpha'}$ is

$$\begin{aligned} \langle a^{\dagger}_{\alpha'}a_{\alpha'}\rangle=\langle c^{\dagger}_{\alpha'}c_{\alpha'}\rangle &= \sum_{\beta'}\langle c^{\dagger}_{\alpha'}|0\beta'\rangle\langle 0\beta'|c_{\alpha'}\rangle \\ &= \sum_{\beta'}\left|\langle\alpha'\beta'|\rangle\right|^{2}. \end{aligned} \tag{20}$$

Now the expression $\sum_{\beta'} \left| \langle \alpha' \beta' | \rangle \right|^2$ represents the probability of finding the electron in the state α' in Dirac's theory. Hence $\langle a^{\dagger}_{\alpha'} a_{\alpha'} \rangle$ gives the same probability in the Heisenberg picture, in agreement with Heisenberg's first assumption.

Let us consider now Heisenberg's treatment of the radiation field. Equation (11) applied with $\xi = b_n e^{-i\omega_n t}$ gives

$$i\hbar \frac{d}{dt}(b_n e^{-i\omega_n t}) = [\, b_n e^{-i\omega_n t}, H \,] = [\, b_n e^{-i\omega_n t}, H_b \,] + [\, b_n e^{-i\omega_n t}, H_I \,].$$

From (7) and (9),

$$H_b = \sum_n b_n^{\dagger} b_n \hbar\omega_n \,.$$

$$H_I = -\sqrt{\frac{h}{cV}} \sum_n \frac{1}{\sqrt{k_n}} \Big\{ b_n e^{-i\omega_n t} \int \underline{e}_n \cdot \underline{j}\, e^{i\underline{k}_n \cdot \underline{r}}\, dv$$
$$+ b_n^{\dagger} e^{i\omega_n t} \int \underline{e}_n \cdot \underline{j}\, e^{-i\underline{k}_n \cdot \underline{r}}\, dv \Big\}. \qquad (21)$$

Hence, on account of the commutation relations

$$\left[b_n , b_m \right] = \left[b_n^{\dagger} , b_m^{\dagger} \right] = 0, \qquad \left[b_n , b_m^{\dagger} \right] = \delta_{nm} \,. \qquad (22)$$

we have

$$i\hbar \frac{d}{dt}(b_n e^{-i\omega_n t}) = \hbar\omega_n b_n e^{-i\omega_n t} - \sqrt{\frac{h}{V\omega_n}} \int \underline{e}_n \cdot \underline{j}\, e^{-i\underline{k}_n \cdot \underline{r}}\, dv. \qquad (23)$$

Similarly we obtain

$$i\hbar \frac{d}{dt}(b_{n'}^{\dagger} e^{i\omega_{n'} t}) = -\hbar\omega_n b_{n'}^{\dagger} e^{i\omega_{n'} t} + \sqrt{\frac{h}{V\omega_n}} \int \underline{e}_n \cdot \underline{j}\, e^{i\underline{k}_n \cdot \underline{r}}\, dv, \qquad (24)$$

where n' stands for the radiation oscillator with $\underline{k}_{n'} = -\underline{k}_n$. From (23) and (24) we obtain

$$i\hbar\frac{d}{dt}(b_n e^{-i\omega_n t}+b^\dagger_{n'}e^{i\omega_{n'}t})=\hbar\omega_n(b_n e^{-i\omega_n t}-b^\dagger_{n'}e^{i\omega_{n'}t}),$$

$$i\hbar\frac{d}{dt}(b_n e^{-i\omega_n t}-b^\dagger_{n'}e^{i\omega_{n'}t})=\hbar\omega_n(b_n e^{-i\omega_n t}+b^\dagger_{n'}e^{i\omega_{n'}t})$$
$$-2\sqrt{\frac{\hbar}{V\omega_n}}\int(e_n\cdot j)e^{ik_n\cdot r}\,dv,$$

whence

$$\left(\frac{d^2}{dt^2}+\omega_n^2\right)(b_n e^{-i\omega_n t}+b^\dagger_{n'}e^{i\omega_{n'}t})=2\sqrt{\frac{2\pi\omega_n}{\hbar V}}\int(e_n\cdot j)e^{-ik_n\cdot r}\,dv. \tag{25}$$

This is, however, just what one obtains from the wave equation (5) with the help of (7).

The equivalence of the two forms of quantum theory of radiation is thus completely proved.

3. *Eigenvalue Problems.*

As was first shown by Weisskopf for the self energy of the electron, it is possible to apply Heisenberg's method to calculate the change in energy levels of stationary states caused by a perturbation. We shall consider below a general type of problems of which the problem of self energy of the electron and that of the interaction between electrons through the medium of electromagnetic field are special cases.

The perturbation of stationary states in quantum theory can be considered either in the Heisenberg picture or in the Schroedinger picture. The connection between the two pictures has been discussed by Heisenberg[11]. We choose here the Schroedinger picture in which the operators are fixed and an eigen > representing a stationary state varies with time according to the simple harmonic law. A stationary state of the perturbed system is then represented by an eigen vector of the form $e^{-\frac{i}{\hbar}H't}|\gamma'>$, where H' is the eigenvalue of the total energy and $|\gamma'>$ is a constant vector satisfying the equation

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[11] Heisenberg, The Physical Principles of the Quantum Theory, 1930.

$$H|\gamma'> = H'|\gamma'>. \tag{26}$$

$|\gamma'>$ is assumed to be normalized.

We consider problems in which the Hamiltonian of the whole system is equal to

$$H = H_0 + \lambda H_1 , \tag{27}$$

where H_0 is the Hamiltonian of the unperturbed system, λH_1 the perturbing energy, λ a small parameter, H_0 and H_1 being independent of λ. It is assumed that H_1 is the product of two operators, A and B, namely

$$H_1 = A\,B , \tag{28}$$

and that all the matrix elements of H_1 refering to two stationary states of the unperturbed system with the same energy are equal to zero, *i.e.*

$$<\gamma', 0|H_1|\gamma'', 0> = 0 \qquad (H_0' = H_0''), \tag{29}$$

where $|\gamma', 0>$ denotes an eigen vector of the unperturbed system corresponding to the eigen vector $|\gamma'>$ in the perturbed system, satisfying the equation

$$H_0|\gamma', 0> = H_0'|\gamma', 0>. \tag{30}$$

We shall work in the Heisenberg representation whose basic states are represented by the vectors $|\gamma',0>$.

$|\gamma'>$ and H' can be expanded in powers of λ, thus

$$|\gamma'> = |\gamma', 0> + \lambda|\gamma', 1> + \lambda^2|\gamma', 2> + \cdots , \tag{31}$$

$$H' = H'^{(0)} + \lambda\, H'^{(1)} + \lambda^2\, H'^{(2)} + \cdots . \tag{32}$$

We shall calculate H' to the second order of accuracy. Proceeding as in the ordinary perturbation theory, we find

$$H'^{(0)} = H_0', \tag{33}$$

$$H'^{(1)} = <\gamma', 0|H_1|\gamma', 0> = 0 \tag{34}$$

by (29), and

$$H'^{(2)} = \sum_{\gamma''\neq\gamma'} \frac{<\gamma',0|H_1|\gamma'',0> <\gamma'',0|H_1|\gamma',0>}{H_0' - H_0''} . \qquad (35)$$

In the following we shall need also the first-order corrections for the representatives of the eigenvector $|\gamma'>$ and of any operator ξ, for which the following relations exist:

$$\left.\begin{aligned} \{<\gamma'',0|\gamma'>\}^{(1)} = <\gamma'',0|\gamma',1> &= \frac{<\gamma'',0|H_1|\gamma',0>}{H_0' - H_0''} \qquad (\gamma''\neq\gamma') \\ &= 0 \qquad\qquad\qquad\qquad (\gamma''=\gamma') , \end{aligned}\right\} \quad (36)$$

$$\{<\gamma'|\xi|\gamma''>\}^{(1)} = <\gamma',0|\xi|\gamma'',1> + <\gamma',1|\xi|\gamma'',0> . \qquad (37)$$

The calculations of self energy by Waller and others were based on equation (35). Proceeding on the same lines as Heisenberg in radiation problems, Weisskopf obtained another general formula, which was shown to lead to the same result as (35). In the following we shall first give a presentation of Weisskopf's method in such a way as to show more clearly its quantum mechanical interpretation, and then give a direct proof of the equivalence of the two methods.

The eigenvalue H' may be expanded according to Taylor's theorem, thus

$$H' = (H')_{\lambda=0} + \lambda \left(\frac{dH'}{d\lambda}\right)_{\lambda=0} + \frac{\lambda^2}{2}\left(\frac{d^2H'}{d\lambda^2}\right)_{\lambda=0} + \cdots \qquad (38)$$

Equating the right-hand sides of (32) and (38) we find

$$H'^{(0)} = (H')_{\lambda=0} , \qquad (39)$$

$$H'^{(1)} = \left(\frac{dH'}{d\lambda}\right)_{\lambda=0} , \qquad (40)$$

$$H'^{(2)} = \frac{1}{2}\left(\frac{d^2H'}{d\lambda^2}\right)_{\lambda=0} , \qquad (41)$$

etc. Now it follows from the condition of normalization of the eigenvectors that

$$<\gamma'|\gamma'> = 1 \qquad (42)$$

and

$$H' = <\gamma'|H|\gamma'> . \qquad (43)$$

On differentiating with respect to λ we find from these equations

$$\left.\begin{aligned} &\frac{d}{d\lambda}\left\{ < \gamma' | \gamma' > \right\} = 0, \\ &\frac{dH'}{d\lambda} = H' \frac{d}{d\lambda}\left\{ < \gamma' | \gamma' > \right\} + < \gamma' \left| \frac{dH}{d\lambda} \right| \gamma' > = < \gamma' | H_1 | \gamma' > , \end{aligned}\right\} \quad (44)$$

by (27), and, on account of (28),

$$\begin{aligned} \frac{d^2H'}{d\lambda^2} &= \frac{d}{d\lambda}\left\{ < \gamma' | AB | \gamma' > \right\} = \sum_{\gamma''} \frac{d}{d\lambda}\left\{ < \gamma' | A | \gamma'' > < \gamma'' | B | \gamma' > \right\} \\ &= \sum_{\gamma''}\left\{ \frac{d}{d\lambda} < \gamma' | A | \gamma'' > + < \gamma' | A \frac{d}{d\lambda} | \gamma'' > \right\} < \gamma'' | B | \gamma' > \\ &\quad + \sum_{\gamma''} < \gamma' | A | \gamma'' > \left\{ \frac{d}{d\lambda} < \gamma'' | B | \gamma' > + < \gamma'' | B \frac{d}{d\lambda} | \gamma' > \right\} \end{aligned} \quad (45)$$

It follows from (39), (40), (41), (44), (45) that

$$H'^{(0)} = H'_0 ,$$

$$H'^{(1)} = \left(\frac{dH'}{d\lambda}\right)_{\lambda=0} = < \gamma', 0 | H_1 | \gamma', 0 > = 0,$$

in agreement with (33), (34), and

$$\begin{aligned} H'^{(2)} &= \frac{1}{2}\left(\frac{d^2H'}{d\lambda^2}\right)_{\lambda=0} \\ &= \frac{1}{2}\sum_{\gamma''}\left\{ < \gamma',1 | A | \gamma'',0 > + < \gamma',0 | A | \gamma'',1 > \right\} < \gamma'',0 | B | \gamma',0 > \\ &\quad + \frac{1}{2}\sum_{\gamma''} < \gamma',0 | A | \gamma'',0 > \left\{ < \gamma'',1 | B | \gamma',0 > + < \gamma'',0 | B | \gamma',1 > \right\} \end{aligned} \quad (46)$$

Formula (46) corresponds to the general formula obtained by Weisskopf for the self energy of the electron.

It remains for us to prove directly the equivalence of (35) and (46). From (35) and (36) we have

$$H'^{(2)} = < \gamma', 0 \mid H_1 \mid \gamma', 1 >$$

and also

$$H'^{(2)} = < \gamma', 1 \mid H_1 \mid \gamma', 0 >$$

Hence

$$E'^{(2)} = \frac{1}{2} < \gamma',0|H_1|\gamma',1> + \frac{1}{2} < \gamma',1|H_1|\gamma',0>$$

$$= \frac{1}{2} \sum_{\gamma''} < \gamma',0|A|\gamma'',0> < \gamma'',0|B|\gamma',1>$$

$$+ \frac{1}{2} \sum_{\gamma''} < \gamma',1|A|\gamma'',0> < \gamma'',0|B|\gamma',0> . \qquad (47)$$

Now it follows from (36) that

$$\sum_{\gamma''} |\gamma'',1><\gamma'',0| + \sum_{\gamma''} |\gamma'',0><\gamma'',1|$$

$$= \sum_{\gamma'\gamma''} |\gamma',0> \frac{<\gamma',0|H_1|\gamma'',0>}{H_0'' - H_0'} <\gamma'',0|$$

$$+ \sum_{\gamma'\gamma''} |\gamma'',0> \frac{<\gamma'',0|H_1|\gamma',0>}{H_0'' - H_0'} <\gamma',0| = 0, \qquad (48)$$

whence

$$\frac{1}{2} \sum_{\gamma''} <\gamma',0|A|\gamma'',0><\gamma'',1|B|\gamma',0>$$

$$+ \frac{1}{2} \sum_{\gamma''} <\gamma',0|A|\gamma'',1><\gamma''0,|B|\gamma',0> = 0. \qquad (49)$$

(47) and (49) give at once (46), and so the equivalence of the two methods is proved.

The formula for the transverse energy of the electron can be obtained from our above result by writing e for λ, $-\frac{1}{c}\underline{j}$ for A, $\underline{A}$ for B, and carrying out all the summations and integrations. The detailed calculations will not be given here.