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A new type of adiabatic invariant for fractional order non-conservative Lagrangian systems

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分数阶非保守 Lagrange 系统的一类新型绝热不变量*

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为了更加准确地描述复杂非保守系统的动力学行为, 将 Herglotz 变分原理推广到分数阶模型, 研究分数阶非保守 Lagrange 系统的绝热不变量. 首先, 基于 Herglotz 变分问题, 导出分数阶非保守 Lagrange 系统的 Herglotz 型微分变分原理并进一步得到分数阶非保守 Lagrange 系统的运动微分方程; 其次, 引进无限小单参数变换, 由等时变分和非等时变分的关系, 导出了分数阶非保守 Lagrange 系统的 Herglotz 型精确不变量; 再次, 研究小扰动对分数阶 Lagrange 系统的影响, 建立了基于 Caputo 导数的分数阶 Lagrange 系统的绝热不变量存在的条件, 得到了该系统的 Herglotz 型绝热不变量; 最后, 举例说明结果的应用.

关键词: 非保守 Lagrange 系统, Herglotz 广义变分原理, 不变量, 分数阶微积分

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1 引 言

Herglotz 广义变分原理是由 Herglotz[1] 提出来的, 它的作用量是由微分方程定义的. 与经典的变分原理相比, 它有如下几点特征: 其一, 给出了非保守动力学过程的变分描述. 然而, 经典变分原理不能将非保守系统表示为泛函的极值; 其二, 经典的哈密顿原理是 Herglotz 广义变分原理的一个特例. 因此, Herglotz 广义变分原理不仅可以描述所有可以用经典变分原理描述的物理过程, 还可以描述经典变分原理不能应用的问题; 其三, Herglotz 广义变分原理将保守过程和非保守过程统一为同一动力学模型, 从而能够系统地处理实际动力学问题. 由于这一优势, Herglotz 广义变分原理被广泛地应用于研究非保守系统和耗散系统的 Noether 定理. Georgieva 和 Gueuther[2] 和 Georgieva 等[3] 基于 Herglotz 广义变分原理得到了 Noether 定理. Santos 等[4,5] 研究了高阶 Herglotz 变分问题和含时滞的 Herglotz 变分问题的 Noether 定理. Zhang 和 Tian[6–12] 基于 Herglotz 广义变分原理在非保守非完整系统、Birkhoff 系统、非保守 Lagrange 系统、相空间以及分数阶模型上分别研究了 Noether 对称性与守恒量. 但是关于 Herglotz 型绝热不变量的研究还处于起步阶段, 尚未引起重视.

研究非保守或非线性动力学的对称性和不变量具有重要的意义, 也是分析力学的前沿研究领域. 当力学系统受到小扰动时, 系统的对称性和守恒量都会发生改变, 我们称之为对称性摄动与绝热不变量. 近年来, 关于绝热不变量的研究已经取得了许多成果, 包括 Noether 型[13–16]、Hojman 型[17–20] 和 Mei 型[21] 的绝热不变量. 最近, 绝热不变量的

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研究还被推广到了分数阶微积分的框架下[22–24]. 可以发现这些绝热不变量都是通过研究对称性得到的. 实际上, 绝热不变量也可以通过微分变分原理得到. 本文将基于 Herglotz 型微分变分原理, 给出分数阶非保守 Lagrange 系统的一类新型绝热不变量, 并证明该绝热不变量存在的条件及其形式.

2 分数阶导数

在这一节中, 回顾本文中所用到的分数阶导数的一些基本定义和性质, 可参考文献 [25].

左 Riemann-Liouville 分数阶导数定义如下:

$$ {}_a^{\mathrm{RL}}D_t^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\int_a^t(t-\xi)^{k-\alpha-1}f(\xi)\,\mathrm{d}\xi, \tag{1}$$

右 Riemann-Liouville 分数阶导数定义如下:

$$ {}_t^{\mathrm{RL}}D_b^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\int_t^b(\xi-t)^{k-\alpha-1}f(\xi)\,\mathrm{d}\xi, \tag{2}$$

左 Caputo 分数阶导数定义如下:

$$ {}_a^{\mathrm{C}}D_t^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_a^t(t-\xi)^{k-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^k f(\xi)\,\mathrm{d}\xi, \tag{3}$$

右 Caputo 分数阶导数定义如下:

$$ {}_t^{\mathrm{C}}D_b^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_t^b(\xi-t)^{k-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^k f(\xi)\,\mathrm{d}\xi, \tag{4}$$

其中 $\Gamma(*)$ 是 Euler-Gamma 函数, 阶 α 满足 $k-1\leqslant\alpha<k$. 如果 α 为整数, 上述分数阶导数成为整数阶导数

假设函数 $f(\xi)$ 和 $g(\xi)$ 在区间 (a,b) 上是连续可积的, 则 Caputo 导数下的分数阶分部积分公式为

$$\int_a^b g(t){}_a^{\mathrm{C}}D_t^{\alpha}f(t)\mathrm{d}t=\int_a^b f(t){}_t^{\mathrm{RL}}D_b^{\alpha}g(t)\,\mathrm{d}t+\sum_{j=0}^{k-1}{}_t^{\mathrm{RL}}D_b^{\alpha+j-k}g(t)\,D^{k-1-j}f(t)\,|_a^b\,, \tag{5}$$

$$\int_a^b g(t){}_t^{\mathrm{C}}D_b^{\alpha}f(t)\mathrm{d}t=\int_a^b f(t){}_a^{\mathrm{RL}}D_t^{\alpha}g(t)\,\mathrm{d}t+\sum_{j=0}^{k-1}(-1)^{k+j}{}_a^{\mathrm{RL}}D_t^{\alpha+j-n}g(t)\,D^{k-1-j}f(t)\,|_a^b\,. \tag{6}$$

3 分数阶非保守 Lagrange 系统的 Herglotz 型微分变分原理

基于 Caputo 导数的分数阶 Herglotz 变分问题为: 确定函数 $q_s(t)$, 使由微分方程:

$$\dot z(t)=L\left(t,q_s(t),{}_a^{\mathrm{C}}D_t^{\alpha}q_s(t),z(t)\right),\ \alpha\in(0,1), \tag{7}$$

定义的泛函 z, 在给定的边界条件:

$$q_s(t)|_{t=a}=q_a,\ q_s(t)|_{t=b}=q_b,\ (s=1,2,\cdots,n), \tag{8}$$

及初始条件:

$$z(a)=z_a, \tag{9}$$

下, $z(t)$ 取得极值. 其中 $L\left(t,q_s(t),{}_a^{\mathrm{C}}D_t^{\alpha}q_s(t),z(t)\right)$ 可称为 Herglotz 意义下的分数阶 Lagrange函数; $q_s\,(s=1,2,\cdots,n)$ 为系统的广义坐标; q_a, q_b 和 z_a 均为固定常数. 考虑到黏弹性体的力学性质是介于弹性体和黏性流体之间, 其本构关系应为 $\sigma(t)\sim\mathrm{d}^{\beta}\varepsilon(t)/\mathrm{d}t^{\beta}\ (0<\beta<1)$, 而黏性和弹性则为黏弹性的两个极限状态[26], 因此这里将 α 的范围取为 $(0,1)$.

可称由 (7) 式确定的泛函 z 为 Hamilton-Herglotz 作用量, 上述变分问题称为分数阶 Herglotz 变分原理.

对 (7) 式取等时变分, 有

$$\delta\dot z=\delta L=\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^{\alpha}q_s}\delta{}_a^{\mathrm{C}}D_t^{\alpha}q_s+\frac{\partial L}{\partial z}\delta z, \tag{10}$$

由交换关系:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z=\delta\dot z, \tag{11}$$

则 (10) 式可表为

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z=A+\frac{\partial L}{\partial z}\delta z, \tag{12}$$

其中

$$A=\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^{\alpha}q_s}\delta{}_a^{\mathrm{C}}D_t^{\alpha}q_s, \tag{13}$$

由 (9) 式, 则 $\delta z(a)=0$, 所以上述初值问题的解为

$$\delta z(t)\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_a^t A\exp\left(-\int_a^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau. \tag{14}$$

并考虑到 $z(t)$ 在 $t=b$ 取得极值, 因此有

$$\delta z(b)=0. \tag{15}$$

由于 (14) 式对任意 $t\in[a,b]$ 上都成立, 若取

$t=b$, 则有

$$\int_a^b A\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}t=0. \tag{16}$$

将 (13) 式代入 (16) 式, 可得到

$$\int_a^b \exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\times\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\delta{}_a^{\mathrm{C}}D_t^\alpha q_s\right)\mathrm{d}t=0. \tag{17}$$

当 $0<\alpha<1$ 时, 根据 Caputo 导数下的分数阶分部积分公式 ((5) 式)、边界条件 $\delta q_s(t)|_{t=a}=\delta q_s(t)|_{t=b}=0$ 以及交换关系 $\delta{}_a^{\mathrm{C}}D_t^\alpha q_s={}_a^{\mathrm{C}}D_t^\alpha\delta q_s$, 对含 $\delta{}_a^{\mathrm{C}}D_t^\alpha q_s$ 的项进行分部积分运算:

$$\begin{aligned}&\int_a^b \exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\delta{}_a^{\mathrm{C}}D_t^\alpha q_s\mathrm{d}t\\
=&\int_a^b \exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\left({}_a^{\mathrm{C}}D_t^\alpha\delta q_s\right)\mathrm{d}t\\
=&\int_a^b \delta q_s{}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\mathrm{d}t\\
&+{}_t^{\mathrm{RL}}D_b^{\alpha-1}\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\delta q_s\Big|_a^b\\
=&\int_a^b \delta q_s{}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\mathrm{d}t.\end{aligned} \tag{18}$$

由 (18) 式, 则 (17) 式成为

$$\int_a^b\left\{\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial q_s}+{}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\right\}\delta q_s\mathrm{d}t=0. \tag{19}$$

由积分区间的任意性, 得到

$$\left\{\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial q_s}+{}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\right\}\delta q_s=0,$$
$$(0<\alpha<1,\ s=1,2,\cdots,n). \tag{20}$$

(20) 式是我们得到的分数阶非保守 Lagrange 系统的 Herglotz 型微分变分原理.

由 δq_s 的独立性, 则导出系统的运动微分方程为

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial q_s}+{}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\times\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]=0,(s=1,2,\cdots,n). \tag{21}$$

如果 Lagrange 函数不显含 z, 即 $L=L(t,q_s(t),{}_a^{\mathrm{C}}D_t^\alpha q_s)$, 则 (21) 式成为

$${}_t^{\mathrm{RL}}D_b^\alpha\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}+\frac{\partial L}{\partial q_s}=0. \tag{22}$$

(22) 式是经典分数阶的非保守 Lagrange 系统的运动微分方程.

当 $\alpha\to1$ 时, 则 (7) 式成为

$$\dot z(t)=L(t,q_s(t),\dot q_s(t),z(t)). \tag{23}$$

(21) 式成为

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial\dot q_s}\right)=0. \tag{24}$$

(24) 式是非保守 Lagrange 系统的 Herglotz 型运动微分方程[2].

4 分数阶非保守 Lagrange 系统的 Herglotz 型精确不变量

引进时间 t 和广义坐标 q_s 的单参数无穷小变换:

$$t^*=t+\Delta t,\ \ q_s^*(t^*)=q_s(t)+\Delta q_s,\ \ (s=1,2,\cdots,n), \tag{25}$$

或其展开式:

$$\begin{aligned}&vt^*=t+\varepsilon\tau^0\left(t,q_k,{}_a^{\mathrm{C}}D_t^\alpha q_k\right),\\
&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s^0\left(t,q_k,{}_a^{\mathrm{C}}D_t^\alpha q_k\right),\\
&(s,k=1,2,\cdots,n).\end{aligned} \tag{26}$$

对任意的函数 F, 等时变分 δF 和非等时变分 ΔF 之间存在如下关系[27]:

$$\Delta F=\delta F+\dot F\Delta t, \tag{27}$$

则可以得到

$$\delta q_s=\varepsilon\left[\xi_s^0\left(t,q_k,{}_a^{\mathrm{C}}D_t^\alpha q_k\right)-\dot q_s\tau^0\left(t,q_k,{}_a^{\mathrm{C}}D_t^\alpha q_k\right)\right],$$
$$(s,k=1,2,\cdots,n), \tag{28}$$

其中 ε 为无限小参数; τ^0 和 ξ_s^0 称为无穷小变换的生成函数.

值得指出, 在整数阶微积分的框架下, 从理论物理和微分几何的角度, 生成函数一般取为时间和广义坐标的函数, 即 $\tau^0(t,q_k)$ 和 $\xi_s^0(t,q_k)$, 这样的变换构成一个 Lie 群, 且变换是保几何结构的[27]. Sarlet 和 Cantrijn[28] 曾详细讨论生成函数的函数依赖性问题. 由于我们现在研究的是分数阶非保守系统及其不变量, 从应用的角度考虑生成函数也依赖于分数阶导数项, 这样拓宽了生成函数的取值范围.

将 (28) 式代入 (20) 式, 整理得

$$\varepsilon\left\{\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial q_s}+{}_t^{\mathrm{RL}}D_b^\alpha\right.$$
$$\left.\times\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right]\right\}\left(\xi_s^0-\dot q_s\tau^0\right)=0,\tag{29}$$

由于

$$\frac{\mathrm{d}}{\mathrm{d}t}L=\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot q_s+\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{C}}D_t^\alpha q_s\right)+\frac{\partial L}{\partial z}L,\tag{30}$$

在 (29) 式中加上并减去 $\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[G^0\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]$ 函数得

$$\varepsilon\left\{\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{C}}D_t^\alpha q_s\right)\tau^0\right.\right.\right.$$
$$\left.+{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)\right)+L\dot\tau^0+\frac{\partial L}{\partial t}\tau^0+\frac{\partial L}{\partial q_s}\xi_s^0+\dot G^0$$
$$\left.-\frac{\partial L}{\partial z}G^0\right]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\tau^0+G^0\right)\right]$$
$$-\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)$$
$$\left.+\left(\xi_s^0-\dot q_s\tau^0\right){}_t^{\mathrm{RL}}D_b^\alpha\left(\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right)\right\}=0,\tag{31}$$

其中 $G^0=G^0\left(t,q_s,{}_a^{\mathrm{C}}D_t^\alpha q_s\right)$称为规范函数. (31) 式是分数阶非保守 Lagrange 系统的 Herglotz 型微分变分原理不变性条件的变换. 我们得到如下定理:

定理 1 对于分数阶非保守 Lagrange 系统 (21), 如果存在规范函数 G^0 使无限小生成元 τ^0 和 ξ_s^0 满足如下条件:

$$\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{C}}D_t^\alpha q_s\right)\tau^0+{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)\right]+L\dot\tau^0$$
$$+\frac{\partial L}{\partial t}\tau^0+\frac{\partial L}{\partial q_s}\xi_s^0+\dot G^0-\frac{\partial L}{\partial z}G^0=0,\tag{32}$$

则系统存在守恒量

$$I=\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\tau^0+G^0\right)$$
$$+\int_a^t\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)\right.$$
$$\left.-\left(\xi_s^0-\dot q_s\tau^0\right){}_t^{\mathrm{RL}}D_b^\alpha\left(\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s}\right)\right]\mathrm{d}\theta.\tag{33}$$

守恒量 ((33) 式) 也称作精确不变量. 当 $\alpha\to1$, (33) 式退化为经典非保守 Lagrange 系统的 Herglotz 型守恒量[9]:

$$I=\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial\dot q_s}\xi_s^0\right.$$
$$\left.+\left(L-\dot q_s\frac{\partial L}{\partial\dot q_s}\right)\tau^0+G^0\right]=\mathrm{const.}\tag{34}$$

如果令

$$H\left(t,q_s,p_s,z\right)=p_s{}_a^{\mathrm{C}}D_t^\alpha q_s-L\left(t,q_s,{}_a^{\mathrm{C}}D_t^\alpha q_s,z\right),\tag{35}$$

$$p_s=\frac{\partial L}{\partial{}_a^{\mathrm{C}}D_t^\alpha q_s},\quad(s=1,2,\cdots,n),\tag{36}$$

则 (21) 式可写成

$$\frac{\partial H}{\partial p_s}={}_a^{\mathrm{C}}D_t^\alpha q_s,\quad\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\frac{\partial H}{\partial q_s}$$
$$={}_t^{\mathrm{RL}}D_b^\alpha\left[\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)p_s\right].\tag{37}$$

取无穷小变换为

$$\begin{aligned}&t^*=t+\varepsilon\tau^0\left(t,q_k,p_k\right),\\&q_s^*\left(t^*\right)=q_s\left(t\right)+\varepsilon\xi_s^0\left(t,q_k,p_k\right),\\&p_s^*\left(t^*\right)=p_s\left(t\right)+\varepsilon\eta_s^0\left(t,q_k,p_k\right),\\&\left(s,k=1,2,\cdots,n\right).\end{aligned}\tag{38}$$

得到如下定理:

定理 2 对于分数阶非保守 Hamilton 系统 (37), 如果无限小变换生成元 ξ_s^0, η_s^0, τ^0 和规范函数 G^0 满足下列条件:

$$\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left\{\eta_s^0{}_a^{\mathrm{C}}D_t^\alpha q_s+p_s\left[{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)\right.\right.$$
$$\left.+\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{C}}D_t^\alpha q_s\right)\tau^0\right]+\left(p_s{}_a^{\mathrm{C}}D_t^\alpha q_s-H\right)\dot\tau^0-\frac{\partial H}{\partial t}\tau^0$$
$$\left.-\frac{\partial H}{\partial q_s}\xi_s^0-\frac{\partial H}{\partial p_s}\eta_s^0+\frac{\partial H}{\partial z}G^0+\dot G^0\right\}=0,\tag{39}$$

则存在守恒量:

$$I=\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\left(p_s{}_a^{\mathrm{C}}D_t^\alpha q_s-H\right)\tau^0+G^0\right]$$
$$+\int_a^t\left[\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)p_s{}_a^{\mathrm{C}}D_t^\alpha\left(\xi_s^0-\dot q_s\tau^0\right)\right.$$
$$\left.-\left(\xi_s^0-\dot q_s\tau^0\right){}_t^{RL}D_b^\alpha\left(\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)p_s\right)\right]\mathrm{d}\theta.\tag{40}$$

当 $G^0=0$ 时, 得到了文献 [12] 的结果.

5　分数阶非保守 Lagrange 系统的 Herglotz 型绝热不变量

根据动力学系统绝热不变量的概念[13], 我们给出分数阶非保守系统的高阶绝热不变量的定义.

如果 $I_m\left(t,q_s,{}^{\rm C}_aD^\alpha_t q_s,z,\nu\right)$是分数阶非保守 Lagrange 系统的一个含有$\nu$的最高次幂为 m 的物理量, 它对时间 t 的一阶导数正比于ν^{m+1}, 那么 I_m 称为该系统的 m 阶绝热不变量.

假设分数阶非保守 Lagrange 系统 (21) 受到了一个小扰动νQ_s的作用, 则 (21) 式成为

$$\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\frac{\partial L}{\partial q_s}+{}^{\rm RL}_tD^\alpha_b\times\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\right]=\nu Q_s.\tag{41}$$

由于小扰动νQ_s的作用, 该系统原有的对称性和不变量都会发生改变. 假设受扰系统的无限小生成函数$\tau\left(t,q_k,{}^{\rm C}_aD^\alpha_t q_k\right)$, $\xi_s\left(t,q_k,{}^{\rm C}_aD^\alpha_t q_k\right)$可表示为

$$\tau=\tau^0+\nu\tau^1+\nu^2\tau^2+\cdots,\ \xi_s=\xi_s^0+\nu\xi_s^1+\nu^2\xi_s^2+\cdots,\tag{42}$$

并满足

$$\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\left\{\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\left[{}^{\rm C}_aD^\alpha_t\left(\xi_s-\dot q_s\tau\right)+\frac{\rm d}{{\rm d}t}\left({}^{\rm C}_aD^\alpha_t q_s\right)\tau\right]+\frac{\partial L}{\partial t}\tau+\frac{\partial L}{\partial q_s}\xi_s+L\dot\tau-\frac{\partial L}{\partial z}G+\dot G\right\}-\nu Q_s\left(\xi_s-\dot q_s\tau\right)=0,\tag{43}$$

其中 G 为规范函数, 记为

$$G=G^0+\nu G^1+\nu^2G^2+\cdots.\tag{44}$$

定理 3　对于受到小扰动作用的分数阶非保守 Lagrange 系统 (21), 如果无穷小变换的生成函数τ^j, ξ_s^j满足

$$\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\left\{\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\left[{}^{\rm C}_aD^\alpha_t\left(\xi_s^j-\dot q_s\tau^j\right)+\frac{\rm d}{{\rm d}t}\left({}^{\rm C}_aD^\alpha_t q_s\right)\tau^j\right]+\frac{\partial L}{\partial t}\tau^j+\frac{\partial L}{\partial q_s}\xi_s^j+L\dot\tau^j-\frac{\partial L}{\partial z}G^j+\dot G^j\right\}-Q_s\left(\xi_s^{j-1}-\dot q_s\tau^{j-1}\right)=0,\tag{45}$$

则系统存在 m 阶绝热不变量:

$$I_m=\sum_{j=0}^m\nu^j\left\{\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\left(L\tau^j+G^j\right)+\int_a^t\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\times{}^{\rm C}_aD^\alpha_t\left(\xi_s^j-\dot q_s\tau^j\right)-\left(\xi_s^j-\dot q_s\tau^j\right){}^{\rm RL}_tD^\alpha_b\times\left(\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\right)\right]{\rm d}\theta\right\}.\tag{46}$$

证 明　令 $\lambda(t)=\exp\left(-\int_a^t\frac{\partial L}{\partial z}{\rm d}\theta\right)$, 根据 (45) 式和 (41) 式可得

$$\begin{aligned}\frac{{\rm d}I_m}{{\rm d}t}=&\sum_{j=0}^m\nu^j\left\{\lambda(t)\left[\left(-\frac{\partial L}{\partial z}\right)L\tau^j+\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot q_s+\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\frac{\rm d}{{\rm d}t}\left({}^{\rm C}_aD^\alpha_t q_s\right)+\frac{\partial L}{\partial z}L\right)\tau^j+\dot G^j\right.\right.\\&\left.\left.+L\dot\tau^j+\left(-\frac{\partial L}{\partial z}\right)G^j+\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}{}^{\rm C}_aD^\alpha_t\left(\xi_s^j-\dot q_s\tau^j\right)-\left(\xi_s^j-\dot q_s\tau^j\right){}^{\rm RL}_tD^\alpha_b\left(\lambda(t)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\right)\right]\right\}\\=&-\left[\lambda(t)\frac{\partial L}{\partial q_s}+{}^{\rm RL}_tD^\alpha_b\left(\lambda(t)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\right)\right]\left(\xi_s^j-\dot q_s\tau^j\right)+\lambda(t)\frac{\partial L}{\partial q_s}\left(\xi_s^j-\dot q_s\tau^j\right)\\&+\lambda(t)\left(-\frac{\partial L}{\partial z}\right)L\tau^j+\lambda(t)\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot q_s+\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\frac{\rm d}{{\rm d}t}\left({}^{\rm C}_aD^\alpha_t q_s\right)+\frac{\partial L}{\partial z}L\right)\tau^j\\&+\lambda(t)\left(-\frac{\partial L}{\partial z}\right)G^j+\lambda(t)L\dot\tau^j+\lambda(t)\dot G^j+\lambda(t)\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}{}^{\rm C}_aD^\alpha_t\left(\xi_s^j-\dot q_s\tau^j\right)\Big]\\=&\sum_{j=0}^m\nu^j\left\{-\nu Q_s\left(\xi_s^j-\dot q_s\tau^j\right)+\lambda(t)\left[\frac{\partial L}{\partial q_s}\xi_s^j+\frac{\partial L}{\partial t}\tau^j+\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}\frac{\rm d}{{\rm d}t}\left({}^{\rm C}_aD^\alpha_t q_s\right)\tau^j+L\dot\tau^j\right.\right.\\&\left.\left.-\frac{\partial L}{\partial z}G^j+\dot G^j+\frac{\partial L}{\partial{}^{\rm C}_aD^\alpha_t q_s}{}^{\rm C}_aD^\alpha_t\left(\xi_s^j-\dot q_s\tau^j\right)\right]\right\}\\=&\sum_{j=0}^m\nu^j\left[-\nu Q_s\left(\xi_s^j-\dot q_s\tau^j\right)+Q_s\left(\xi_s^{j-1}-\dot q_s\tau^{j-1}\right)\right]=-\nu^{m+1}Q_s\left(\xi_s^m-\dot q_s\tau^m\right).\end{aligned}\tag{47}$$

因此, I_m是一个 m 阶绝热不变量.

(46) 式是基于 Herglotz 微分变分原理导出的一类新型绝热不变量. 特别地, 当$m=0$时, 绝热不变量为精确不变量.

当$\alpha\to 1$, 绝热不变量 (46) 式退化为经典非保守 Lagrange 系统的 Herglotz 型绝热不变量[29]:

$$I_m=\sum_{j=0}^{m}\nu^j\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\times\left[\frac{\partial L}{\partial\dot q_s}\xi_s^j+\left(L-\frac{\partial L}{\partial\dot q_s}\dot q_s\right)\tau^j+G^j\right].\quad(48)$$

类似地, 我们有:

定理 4 对于受到小扰动作用的分数阶非保守 Hamilton 系统 (37) 式, 如果无穷小变换的生成函数τ^j, ξ_s^j和η_s^j满足:

$$\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left\{\eta_s^j{}_a^{\mathrm C}D_t^\alpha q_s+p_s\left[{}_a^{\mathrm C}D_t^\alpha\left(\xi_s^j-\dot q_s\tau^j\right)+\frac{\mathrm d}{\mathrm dt}\left({}_a^{\mathrm C}D_t^\alpha q_s\right)\tau^j\right]-\frac{\partial H}{\partial t}\tau^j-\frac{\partial H}{\partial q_s}\xi_s^j+\left(p_s{}_a^{\mathrm C}D_t^\alpha q_s-H\right)\dot\tau^j-\frac{\partial H}{\partial p_s}\eta_s^j+\frac{\partial H}{\partial z}G^j+\dot G^j\right\}+Q_s\left(\xi_s^{j-1}-\dot q_s\tau^{j-1}\right)=0,\quad(49)$$

则系统存在 m 阶绝热不变量:

$$I_m=\sum_{j=0}^{m}\nu^j\left\{\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm d\theta\right)\left[\left(p_s{}_a^{\mathrm C}D_t^\alpha q_s-H\right)\tau^j+G^j\right]+\int_a^t\left[\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm d\theta\right)p_s\times{}_a^{\mathrm C}D_t^\alpha\left(\xi_s^j-\dot q_s\tau^j\right)-\left(\xi_s^j-\dot q_s\tau^j\right){}_t^{\mathrm{RL}}D_b^\alpha\left(\exp\left(\int_a^t\frac{\partial H}{\partial z}\mathrm d\theta\right)p_s\right)\right]\mathrm d\theta\right\}.\quad(50)$$

6 算 例

作为例子, 研究分数阶线性阻尼振子[30,31]. 分数阶振子是粘弹性阻尼系统的分数阶模型, 其动力学方程含有分数阶导数项. 由于分数阶模型具有记忆效应和空间全域性等, 它能更准确地描述系统的动力学行为, 因而分数阶振子的研究得到了广泛关注[32–38].

首先根据文献 [39] 的方法, Herglotz 意义下分数阶振子的 Lagrange 函数为

$$L=\frac12m\left({}_a^{\mathrm C}D_t^\alpha q\right)^2-\frac12kq^2-\frac{c}{m}z,\quad(51)$$

其中 m 为质点的质量, k 为弹性系数, c 为阻尼系数, m,k,c 为常量; 泛函 z 满足微分方程:

$$\dot z=\frac12m\left({}_a^{\mathrm C}D_t^\alpha q\right)^2-\frac12kq^2-\frac{c}{m}z.\quad(52)$$

由 (21) 式, 得到其运动微分方程:

$${}_t^{\mathrm{RL}}D_b^\alpha\left[m\cdot\mathrm e^{\frac{c}{m}(t-a)}\cdot\left({}_a^{\mathrm C}D_t^\alpha q\right)\right]+\mathrm e^{\frac{c}{m}(t-a)}(-kq)=0,\quad(53)$$

其中${}_a^{\mathrm C}D_t^\alpha q$在物理上可解释为时间加权广义速度. 显然在忽略记忆特征的情况下, 也就是$\alpha\to1$时, 方程 (53) 退化为整数阶线性阻尼振子方程:

$$m\ddot q+c\dot q+kq=0.\quad(54)$$

根据 (32) 式, 则有

$$m{}_a^{\mathrm C}D_t^\alpha q\left[\frac{\mathrm d}{\mathrm dt}\left({}_a^{\mathrm C}D_t^\alpha q\right)\tau^0+{}_a^{\mathrm C}D_t^\alpha\left(\xi^0-\dot q\tau^0\right)\right]+L\dot\tau^0-kq\xi^0+\dot G^0+\frac{c}{m}G^0=0.\quad(55)$$

方程 (55) 有解:

$$\tau^0=1,\quad \xi^0=-\frac{c}{2m}q,\quad G^0=\frac{c}{m}z.\quad(56)$$

由定理 1, 该系统的一个精确不变量为

$$I=\mathrm e^{\frac{c}{m}(t-a)}\left[\frac12m\left({}_a^{\mathrm C}D_t^\alpha q\right)^2-\frac12kq^2\right]+\int_a^t\left\{\mathrm e^{\frac{c}{m}(t-a)}\left[m\left({}_a^{\mathrm C}D_t^\alpha q\right)\times{}_a^{\mathrm C}D_t^\alpha\left(-\frac{c}{2m}q-\dot q\right)\right]-\left(-\frac{c}{2m}q-\dot q\right){}_t^{\mathrm{RL}}D_b^\alpha\times\left(m\cdot\mathrm e^{\frac{c}{m}(t-a)}\cdot\left({}_a^{\mathrm C}D_t^\alpha q\right)\right)\right\}\mathrm d\theta=\mathrm{const}.\quad(57)$$

当$\alpha\to1$时, 得到整数阶线性阻尼振子的精确不变量:

$$I=\mathrm e^{\frac{c}{m}(t-a)}\left(-\frac12m\dot q^2-\frac{c}{2}q\dot q-\frac12kq^2\right).\quad(58)$$

下面研究系统的绝热不变量. 假设系统受到的小扰动为

$$\nu Q=-2\nu q\mathrm e^{\frac{c}{m}(t-a)}.\quad(59)$$

方程 (45) 给出

$$\mathrm e^{\frac{c}{m}(t-a)}\left\{m{}_a^{\mathrm C}D_t^\alpha q\left[\frac{\mathrm d}{\mathrm dt}\left({}_a^{\mathrm C}D_t^\alpha q\right)\tau^1+{}_a^{\mathrm C}D_t^\alpha\left(\xi^1-\dot q\tau^1\right)\right]+L\dot\tau^1-kq\xi^1+\dot G^1+\frac{c}{m}G^1\right\}=Q\left(\xi^0-\dot q\tau^0\right).\quad(60)$$

方程 (60) 有解

$$\tau^1=1,\quad \xi^1=0,\quad G^1=q^2. \tag{61}$$

由定理 2, 则该系统有如下一阶绝热不变量:

$$\begin{aligned}I=&\mathrm{e}^{\frac{c}{m}(t-a)}\left[\frac{1}{2}m\left({}_a^{\mathrm{C}}D_t^{\alpha}q\right)^2-\frac{1}{2}kq^2\right]\\&+\int_a^t\left\{\mathrm{e}^{\frac{c}{m}(t-a)}\left[m\left({}_a^{\mathrm{C}}D_t^{\alpha}q\right)\right.\right.\\&\times{}_a^{\mathrm{C}}D_t^{\alpha}\left(-\frac{c}{2m}q-\dot{q}\right)\Big]-\left(-\frac{c}{2m}q-\dot{q}\right){}_t^{\mathrm{RL}}D_b^{\alpha}\\&\times\left(m\cdot\mathrm{e}^{\frac{c}{m}(t-a)}\cdot\left({}_a^{\mathrm{C}}D_t^{\alpha}q\right)\right)\Big\}\,\mathrm{d}\theta\\&+\nu\mathrm{e}^{\frac{c}{m}(t-a)}\left[\frac{1}{2}m\left({}_a^{\mathrm{C}}D_t^{\alpha}q\right)^2+q^2-\frac{1}{2}kq^2-\frac{c}{m}z\right].\end{aligned} \tag{62}$$

当$\alpha\to1$时, 得到整数阶线性阻尼振子的绝热不变量:

$$\begin{aligned}I=&\mathrm{e}^{\frac{c}{m}(t-a)}\left(-\frac{1}{2}m\dot{q}^2-\frac{c}{2}q\dot{q}-\frac{1}{2}kq^2\right)\\&+\nu\mathrm{e}^{\frac{c}{m}(t-a)}\left(\frac{1}{2}m\dot{q}^2+q^2-\frac{1}{2}kq^2-\frac{c}{m}z\right),\end{aligned} \tag{63}$$

可以进一步地求得系统的更高阶绝热不变量.

7 结 论

Herglotz 广义变分原理为研究非保守系统动力学提供了一种新的思路. 本文建立了分数阶非保守 Lagrange 系统的 Herglotz 型微分变分原理, 基于该原理给出了分数阶非保守 Lagrange 系统的精确不变量和绝热不变量. 主要结果是文中给出的原理 (20) 和 4 个定理.

当$\alpha\to1$时, 分数阶非保守 Lagrange 系统的 Herglotz 微分变分原理 (20) 退化为整数阶的 Herglotz 微分变分原理, 其方程 (21) 退化为经典的 Heglotz 型运动微分方程 (24), 与之相应的精确不变量 (33) 也退化为经典的 Herglotz 型精确不变量 (34). 当受到小扰动时, 根据高阶绝热不变量的定义, 得到了分数阶非保守 Lagrange 系统的 Herglotz 型绝热不变量 (46). 若 Lagrange 函数不显含 z, 则问题退化为经典分数阶 Lagrange 系统的变分问题, 方程 (21) 退化为经典分数阶 Lagrange系统的运动微分方程 (22).

最近, 文献 [40] 综述了 Herglotz 广义变分原理及其 Noether 对称性与守恒量理论研究的近期发展, 并提出了有待进一步研究的若干问题. 例如, 一般情形下 Herglotz 型 Lagrange 函数的物理解释; 对于一般非保守动力学系统, Herglotz 型 Lagrange 函数的构建问题等.

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A new type of adiabatic invariant for fractional order non-conservative Lagrangian systems*

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Abstract

The Herglotz variational problem is also known as Herglotz generalized variational principle whose action functional is defined by differential equation. Unlike the classical variational principle, the Herglotz variational principle gives a variational description of a holonomic non-conservative system. The Herglotz variational principle can describe not only all physical processes that can be described by the classical variational principlen, but also the problems that the classical variational principle is not applicable for. If the Lagrangian or Hamiltonian does not depend on the action functional, the Herglotz variational principle reduces to the classical integral variational principle. In this work, in order to describe the dynamical behavior of complex non-conservative system more accurately, we extend the Herglotz variational principle to the fractional order model, and study the adiabatic invariant for fractional order non-conservative Lagrangian system. Firstly, based on the Herglotz variational problem, the differential variational principle of Herglotz type and the differential equations of motion of the fractional non-conservative Lagrangian system are derived. Secondly, according to the relationship between the isochronal variation and the nonisochronal variation, the transformation of invariance condition of Herglotz differential variational principle is established and the exact invariants of the system are derived. Thirdly, the effects of small perturbations on fractional non-conservative Lagrangian systems are studied, the conditions for the existence of adiabatic invariants for the Lagrangian systems of Herglotz type based on Caputo derivatives are established, and the adiabatic invariants of Herglotz type are obtained. In addition, the exact invariant and adiabatic invariant of fractional non-conservative Hamiltonian system can be obtained by Legendre transformation. When $\alpha \to 1$, the Herglotz differential variational principle for fractional non-conservative Lagrangian system degrades into classical Herglotz differential variational principle, and the corresponding exact invariants and adiabatic invariants also degenerate into the classical exact invariants and adiabatic invariants of Herglotz type. At the end of the paper, the fractional order damped oscillator of Herglotz type is discussed as an example to demonstrate the results.

Keywords: non-conservative Lagrangian system, Herglotz generalized variational principle, invariants, fractional calculus

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