LARGE DEFLECTION OF A CIRCULAR CLAMPED PLATE UNDER UNIFORM PRESSURE

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ABSTRACT

The problem of large deflection of a clamped circular plate under uniform pressure is studied by the method of successive approximation in terms of the parameter representing the ratio of the center deflection to the thickness. The tedious numerical computations involved in Way's power series solution are thus avoided. The yielding condition at the edge checks very well with the experimental results given by McPherson, Ramberg and Levy. The method may be easily extended to any other boundary conditions and loading details.

I. INTRODUCTION

Although the solution of Kirchhoff's equation of small deflection of thin plates under various lateral loads and boundary conditions are well-known, there are very few satisfactory treatments of the large deflection theory based upon the famous Kármán equations. The difficulties in solving the Kármán equations are chiefly due to their non-linearity[1].

So far only two cases have been studied with numerical certainty, namely, a clamped circular plate under uniform pressure and a simply supported rectangular plate under combined uniform pressure. The case of circular plate was solved by S. Way[2] by the method of power series, and the case of rectangular plate was solved by S. Levy[3] in terms of trigonometric series. Both methods involve the numerical determination of an infinite number of coefficients for a given value of loading. Hence the numerical work involved is excessive.

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1. Th. von Kármán, "The Engineer Grapples with Non-linear Problems," *Bull. Amer. Math. Soc.* **46** (1940), 615-683.

2. S. Way, "Bending of Circular Plate with Large Deflection", *A.S.M.E. Transactions, Applied Mechanics*, **56** (1934), 627-636.

3. S. Levy, "Bending of Rectangular Plate with Large Deflections", *N.A.C.A. Report*, No. 737 (1942).

In this paper, we shall treat the case of clamped circular plate by means of the perturbation method based upon the smallness of center deflection. The results obtained by this method are equivalent to that of power series method, but in greatly simplified form. The extension of this method to any other conditions will not be difficult. The work culminates in the determination of yielding condition along the clamped edge.

II. ELASTIC CIRCULAR PLATE

For a circular plate of uniform thickness h and radius a under uniform pressure q, the equations of equilibrium can be written as follows:

$$\frac{1}{r}\frac{dM_t}{dr}-\frac{1}{r}\frac{d^2}{dr^2}(rM_r)=\frac{1}{r}\frac{d}{dr}\left(N_r r\frac{dw}{dr}\right)+q, \tag{1}$$

$$\frac{d}{dr}(rN_r)-N_t=0, \tag{2}$$

where M_t, M_r are tangential and radial bending moments, N_t, N_r are tangential and radial membrane stresses, r is the radial coordinate and w is the normal displacement. These equations are valid not only for the elastic plate, but also for the plate deformed beyond the elastic limit.

For an elastic plate, let u be the radial displacement, then the radial and circumferential strain in the middle surface are

$$\epsilon_r'=\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2, \qquad \epsilon_t'=\frac{u}{r}. \tag{3}$$

At a distance z from the middle surface, there are additional strain due to bending, namely

$$\epsilon_r''=-Z\frac{d^2w}{dr^2}, \qquad \epsilon_t''=-Z\frac{dw}{dr}\frac{1}{r}. \tag{4}$$

By Hooke's law, it follows that the stress at a distance z from the middle surface can be written as the sum of stretching stress σ_r', σ_t' and bending stress σ_r'', σ_t'':

$$\sigma_r=\sigma_r'+\sigma_r'', \qquad \sigma_t=\sigma_t'+\sigma_t'', \tag{5}$$

where

$$\sigma_r'=\frac{E}{1-\mu^2}\left[\frac{du}{dr}+\frac{1}{r}\left(\frac{dw}{dr}\right)^2+\mu\frac{u}{r}\right],\quad \sigma_t'=\frac{E}{1-\mu^2}\left[\frac{u}{r}+\mu\frac{1}{r}\left(\frac{dw}{dr}\right)^2+\mu\frac{du}{dr}\right] \quad (6)$$

$$\sigma_r''=-\frac{Ez}{1-\mu^2}\left(\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}\right),\qquad \sigma_t''=-\frac{Ez}{1-\mu^2}\left(\frac{1}{r}\frac{dw}{dr}+\mu\frac{d^2w}{dr^2}\right). \quad (7)$$

Consequently, the bending moment and membrane stress are

$$M_r=-D\left(\frac{d^2w}{dr^2}+\mu\frac{1}{r}\frac{dw}{dr}\right),\qquad M_t=-D\left(\frac{1}{r}\frac{dw}{dr}+\mu\frac{d^2w}{dr^2}\right), \quad (8)$$

$$N_r=h\sigma_r' \qquad , \qquad N_t=h\sigma_t'. \quad (9)$$

where D is the fluxial rigidity

$$D=\frac{Eh^3}{12(1-\mu^2)}. \quad (10)$$

An useful relation between u, N_t, X_r can be obtained by eliminating w from (6):

$$\frac{u}{r}=\frac{1}{E}(\sigma_t'-\mu\sigma_r')=\frac{1}{Eh}(N_t-\mu N_r). \quad (11)$$

Furthermore, eliminating u from equations (6), and using equation (2), we obtain the equation of compatibility:

$$r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)+\frac{Eh}{2}\left(\frac{dw}{dr}\right)^2=0, \quad (12)$$

By means of (8), the equation (1) may be simplified, after integration once, into the following form:

$$D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}=N_r\frac{dw}{dr}+\frac{qr}{2} \quad (13)$$

Equations (12), (13) are the two famous Kármán equations for the determination of the two unknowns w and N_r. The boundary conditions for a clamped plate are

$$\left.\begin{aligned}
&w=\frac{dw}{dr}=0 \quad \text{at } r=a,\\
&\frac{1}{r}\frac{dw}{dr}=\text{finite} \quad \text{at } r=0,\\
&Eh\frac{u}{r}=N_t-\mu N_r=\frac{d}{dr}(rN_r)-\mu N_r=0 \ \text{at } r=a,\\
&N_r=\text{finite at } r=0.
\end{aligned}\right\} \tag{14}$$

We begin by transforming these equations to a dimensionless form, in which the following notations are introduced:

$$W=\frac{w}{h}, \qquad N_r=\frac{Eh^3}{a^2}S_r, \qquad N_t=\frac{Eh^3}{a^2}S_t, \qquad P=\frac{a^4q}{h^4E}(1-\mu^2) \tag{15}$$

We introduce also a dimensionless variabie η,

$$\eta=1-\frac{r^2}{a^2}. \tag{16}$$

With these quantities, equations (2), (12), (13) become

$$S_t=S_r-2(1-\eta)\frac{dS_r}{d\eta}, \tag{17}$$

$$\frac{d^2}{d\eta^2}\Big[(1-\eta)S_r\Big]+\frac{1}{2}\left(\frac{dW}{d\eta}\right)^2=0, \tag{18}$$

$$-\frac{1}{4}\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW}{d\eta}\right]=\frac{3}{16}P-\frac{3}{4}(1-\mu^2)S_r\frac{dW}{d\eta}. \tag{19}$$

The boundary conditions are

$$W=\frac{dW}{d\eta}=2\frac{dS_r}{d\eta}-(1-\mu)S_r=0 \qquad \text{at edge}(\eta=0), \tag{20}$$

$$\frac{dW}{d\eta},\ S_r \text{ remain finite at center } (\eta=1). \tag{21}$$

Equations (18), (19) are the two equations for the determination of the two unknowns W, S_r under the boundary conditions given in (20), (21). From (17), S_t can be calculated when S_r is given.

III. SOLUTION OF KÁRMÁN EQUATIONS BY PERTURBATION METHOD

We shall now proceed to obtain the solution of Kármán equations by the perturbation method based upon the smallness of maximum deflection at center. Let

$$W_m = W_{\eta=1} = \left(\frac{w}{h}\right)_{r=0}. \qquad (22)$$

It is evident that

$$P = P(W_m), \quad W = W(W_m, \eta), \quad S_r = S_r(W_m, \eta), \quad S_t = S_t(W_m, \eta). \qquad (23)$$

For small W_m, or within the possible range of convergence, we may expand every quantity in ascending powers of W_m,

$$\begin{aligned} \frac{3}{16} P &= \alpha_1 W_m + \alpha_3 W_m^3 + \alpha_5 W_m^5 + \ldots, \\ W &= w_1(\eta) W_m + w_3(\eta) W_m^3 + w_5(\eta) W_m^5 + \ldots, \\ S_r &= f_2(\eta) W_m^2 + f_4(\eta) W_m^4 + f_6(\eta) W_m^6 + \ldots, \\ S_t &= g_2(\eta) W_m^2 + g_4(\eta) W_m^4 + g_6(\eta) W_m^6 + \ldots, \end{aligned} \qquad (24)$$

where α's are constants, and f, g, w's are functions of η to be determined. These expressions are valid only in the sense of asymptotic nature, and their convergences are not required.

We substitute the expressions (24) into (17), (18), (19), and also into the boundary conditions (20), (21), (22). By collecting terms of the successive order in W_m, we obtain a sequence of linear differential equations for α_1, w_1, f_2, g_2, α_3, w_3, f_4, g_4, etc. accompanied by the corresponding boundary conditions.

$$\left.\begin{aligned} &\text{For } \alpha_1, w_1, \text{ we find the following problem (Problem I):} \\ &-\frac{1}{4}\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_1}{d\eta}\right] = \alpha_1, \\ &w_1(1) = 1, \quad w_1(0) = w_1'(0) = 0, \quad \text{and } w_1'(1) \text{ remains finite.} \end{aligned}\right\} \qquad (25)$$

The solution of (I) is obvious:

$$w_1(\eta)=\eta^2, \qquad \alpha_1=1. \tag{26}$$

This is the well-known solution of a clamped plate with very small deflection (or the solution of the Kirchhoff theory). For $f_2(\eta)$, $g_2(\eta)$, we have the equations and the boundary conditions as follows (Problem II):

$$\left.\begin{aligned} &g_2(\eta)=f_2(\eta)-2(1-\eta)\frac{d}{d\eta}f_2(\eta)\ ,\\ &\frac{d^2}{d\eta^2}\Big[(1-\eta)f_2\Big]+\frac{1}{2}\left(\frac{dw_1}{d\eta}\right)^2=0\ ,\\ &g_2(0)-\mu f_2(0)=0,\quad f_2(1)\ \text{remains finite.}\end{aligned}\right\} \tag{27}$$

The solution of problem II is

$$f_2(\eta)=\frac{1}{6}\left(\frac{2}{1-\mu}+\eta+\eta^2+\eta^3\right),$$

$$g_2(\eta)=\frac{1}{6}\left(\frac{2\mu}{1-\mu}-\eta-\eta^2+7\eta^3\right). \tag{28}$$

The next approximation gives the equations (Problem III):

$$\left.\begin{aligned} &-\frac{1}{4}\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dw_3}{d\eta}\right]=\alpha_3-\frac{3}{4}(1-\mu^2)f_2(\eta)\frac{dw_1}{d\eta}\ ,\\ &w_3(0)=w_3'(0)=w_3(1)=0 \quad \text{and } w_3'(1)\ \text{remains finite.}\end{aligned}\right\} \tag{29}$$

where w_1, f_2 are given respectively in (26), (28). The solution of problem III gives

$$\alpha_3=\frac{1}{360}(1+\mu)(173-73\mu),$$

$$w_3(\eta)=\frac{1}{360}(1-\mu^2)\eta^2(1-\eta)\left(\frac{83-43\mu}{1-\mu}+23\eta+8\eta^2+2\eta^3\right). \tag{30}$$

For $f_4(\eta)$, $g_4(\eta)$, we find the following problem (Problem IV)

$$\left.\begin{aligned} &g_4(r_i) = f_4(r_i) - 2\frac{df_4}{dr_i}(1-r_i), \\ &\frac{d_2}{dr_i^2}\Big[(1-r_i)f_4\Big] + \frac{dw_1}{dr_i}\frac{dw_3}{dr_i} = 0, \\ &g_4(0) - \mu f_4(0) = 0, \text{ and } f_4(1) \text{ remains finite.} \end{aligned}\right\} \tag{31}$$

Making use of $w_1(r_i)$, $w_3(r_i)$ given in (26), (30), we find the solution of problem IV as follows:

$$f_4(r_i) = \frac{1}{7560}(1-\mu^2)\left\{\frac{160-104\mu}{(1-\mu)^2} + \frac{80-52\mu}{1-\mu}(r_i + r_i^2 + r_i^3) - \frac{501-249\mu}{1-\mu}r_i^4 - 123\,r_i^5 - 39r_i^6 - 9r_i^7\right\} \tag{32}$$

$$g_4(r_i) = \frac{1}{7560}(1-\mu^2)\left\{\frac{\mu(160-104\mu)}{(1-\mu)^2} - \frac{80-52\mu}{1-\mu}(r_i + r_i^2) + \frac{4568-2356\mu}{1-\mu}r_i^3 - \frac{3279-1011\mu}{1-\mu}r_i^4 - 885r_i^5 - 21r_i^6 - 135r_i^7\right\}.$$

We shall now be satisfied with the process of successive approximations up to the present stage. The results obtained above may be summarized as follows.

IV. DISCUSSION OF THE APPROXIMATE SOLUTION

The relation between center deflection and pressure for an elastic plate of uniform thickness with clamped edge is given by

$$\frac{3}{16}P = W_m + \alpha^3 W_m^3, \tag{33}$$

where from (30)

$$\alpha_3 = \frac{1}{360}(1+\mu)(173-73\mu). \tag{34}$$

For various values of μ, the constant α_3 in (34) takes the following values:

μ:	0.250	0.275	0.300	0.325	0.350
α_3:	0.536	0.540	0.544	0.548	0.551

The relation (33) has also been investigated approximately by a number of authors. Their results differ from each other in the values of α_3 as follows: For $\mu=0.300$,

Nádái[4], $\alpha_3=0.583$,

Timoshenko[5], $\alpha_3=0.488$,

Federhofer[6], $\alpha_3=\frac{1}{40}(1+\mu)(19-9\mu)=0.530$,

Waters[7], $\alpha_3=\frac{1}{56}(1+\mu)(23-9\mu)-0.474$,

McPherson, Ramberg. Levy[9], $\alpha_3=0.588$.

It should be noted both Nádái and Federhofer derive the relation by solving the von Kármán differential equations under certain assumed physical conditions. Nádái considers a plate subjected to a pressure which is only approximately uniform, while Federhofer assumes a suitable radial distribution of membrane displacement. Timoshenko aud Waters use the energy method based upon an assumed form of normal displacement. McPherson, Ramberg and Levy follow the procedure used by Föppl[8] for the treatment of square plate under normal pressure. Föppl makes the assumption that the total pressure is the sum of two parts, namely, the pressure resisting bending and the pressure carrying the membrane action. In addition to this, Föppl assumes also that the bending of the plate is proportional to that given by Kirchhoff's theory while the extension of the plate is proportional to that for a membrane.

A numerical solution of the differential equations based upon the power series method was first obtained by Way. His numerical results as well as the

4. A. Nádái, *Elastische Platten*, (Julius Springer, Berlin. 1925).

5. S. Timoshenko, *Theory of Plates and Shells*, pp. 333-337, 451 (McGraw Hill, 1940).

6. K. Federhofer, "Zur Berechnung der duennen Kreisplatte mit grosser Ausbiegung," *Forschung auf dem Gebiete des Ingenieurwesens Ausg. B*, Bd 7, Heft 3, VDI-Verlag G.m.b.H. (Berlin), p.148-151, (1936)

7. E.O. Waters, Discussion on S. Way's paper given in reference 2.

8. A. Föppl and L, Föppl, *Drang and Zwang*, vol I, (R. Oldenbourg, Munich, 1924).

9. A. McPherson, W. Ramberg, and S. Levy, "Normal Pressure Tests of Circular Plates with Clamped edges," *N.A.C.A. Report*, No. 744 (1942).

results given above are shown graphically in figure 1 for $\mu=0.300$. These five curves differ from the results obtained by the present method less than 8 percent. The present results not only agree perfectly well with Way's numerical solution, but also appears to be near to the average values of all the others. It is a well-known fact that, for low pressures, the experimental results agree closely with the theoretical curves, but for larger pressure, the experimental deflections consistently exceed the theoretical values by four to twelve percent or more. This disagreement can probably be explained by the partial yielding in the edge, and by the action of initial compressive stress set up by the clamping procedure.

The present results give also the stresses in the plate in explicit expressions. Let us denote the dimensionless form of the radial tensile stress σ_r' in the middle surface by $\Sigma_r'(r_i)$, and the dimensionless form of the radial bending stress σ_r'' at the convex side of the plate by $\Sigma_r''(r_i)$, or

$$\Sigma_r' = \frac{\sigma_r' a^2}{Eh^2}, \qquad \Sigma_r'' = \frac{\sigma_r'' a^2}{Eh^2}.$$

Hence we have the following useful results:

$\Sigma_r'(0)$=reduced radial tensile stress at edge

$$= \frac{W_m^2}{3(1-\mu)} \left\{ 1 + \frac{1}{630}(1+\mu)(40-26\mu) W_m^2 \right\}, \tag{35a}$$

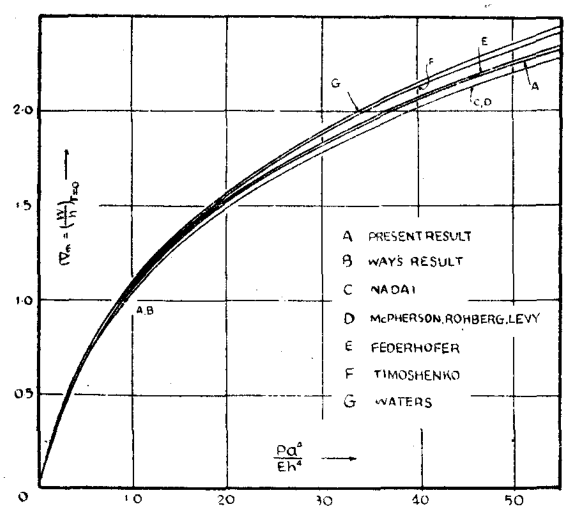

Fig. 1 Variation of Center Deflection with Pressure By Various Authors (μ=0.300)

$\Sigma_r'(1)$ = reduced radial tensile stress at center

$$=\frac{W_m^3}{6(1-\mu)}\left\{(5-3\mu)-\frac{1}{315}(1+\mu)(68-148\mu+66\mu^2)W_m^2\right\}, \tag{35b}$$

$\Sigma_r''(0)$ = reduced radial bending stress at edge

$$=\frac{4W_m}{1-\mu^2}\left\{1+\frac{1}{360}(1+\mu)\ (83-43\mu)\ W_m^3\right\}, \tag{36a}$$

$\Sigma_r''(1)$ = reduced radial bending stress at center

$$=\frac{2W_m}{1-\mu}\left\{1-\frac{1}{180}(1+\mu)(29-19\mu)\ W_m^2\right\}. \tag{36b}$$

These stresses for the case $\mu=0.300$ are shown in figure 2. They are numerically in perfect agreement with that obtained by Way. However, due to extremely

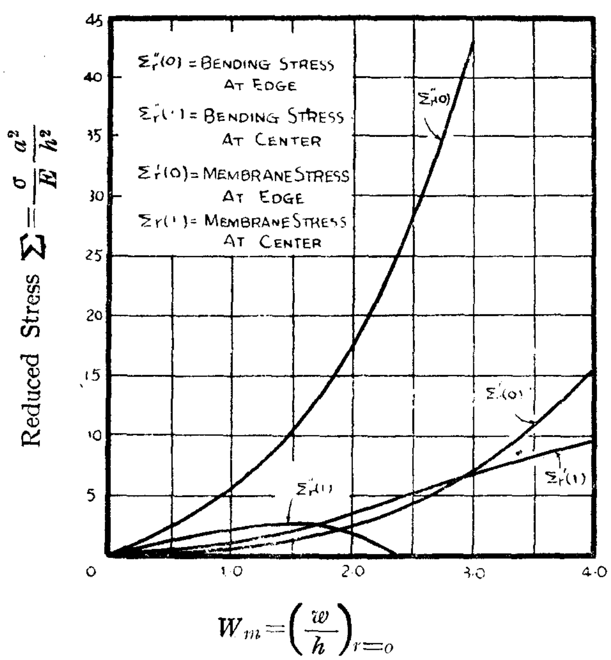

Fig. 2 Variation of Stresses with Center Deflection

slow convergence of the power series used by Way, further extension of power series method for larger deflections becomes very tedious. Since our solution is asymptotic in nature, the problem of convergence does not come into consideration.

V. YIELDING ALONG THE EDGE

From the figure 2, one observes that the radial stress in the extreme fiber along the edge increases with the increasing normal deflection in the center. When yielding stress is reached, the bending strength in the edge breaks gradually. Hence, once yielding occurs in the edge, the boundary conditions for a clamped plate no longer has any physical justification.

The condition of yielding along the edge can be computed from the assumption of von-Mises-Hencky theory of plastic failure. If σ_1, σ_2, σ_3 are three principle stresses, the yielding condition is

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2E_0\ , \tag{37}$$

where E_0 is the yielding tensile stress. At the edge of a circular plate, radial displacement vanishes. Hence

$$\sigma_1=\sigma_{re}\ , \qquad \sigma_2=\sigma_{te}=\mu\sigma_{re} \qquad \sigma_3\ll\sigma_{re} \tag{38}$$

where σ_{re}, σ_{te} are the extreme fiber stress on the convex side in radial and circumferential directions at the edge. By neglecting σ_3, we have

$$\sigma_{re}=\frac{E_0}{\sqrt{1-\mu+\mu^2}}\ , \qquad \sigma_{te}=\frac{\mu E_0}{\sqrt{1-\mu+\mu^2}}. \tag{39}$$

The condition of yielding at edge is therefore

$$\frac{a^2E_0}{h^2E}=\sqrt{1-\mu+\mu^2}\Big(\sum_r''(0)+\sum_r'(0)\Big). \tag{40}$$

Or by equations (35), (36), the yielding condition along the edge may be written as

$$\frac{a^2E_0}{h^2E}=\sqrt{1-\mu+\mu^2}\frac{W_m}{1-\mu^2}\Big\{4+\frac{1}{3}(1+\mu)W_m^2+\frac{1}{90}(1+\mu)(83-43\mu)W_m^2 + \frac{1}{1890}(1+\mu)^2(40-26\mu)W_m^3\Big\}. \tag{41}$$

The theoretical yielding condition at the center may be computed in a similar manner. At the center of the plate, the principle stresses are equal; hence

$$\sigma_1=\sigma_2=\sigma_{rc}\,, \qquad \sigma_3 \ll \sigma_{rc}\,, \tag{42}$$

where σ_{rc} is the extreme fibre stress in a radial direction at the center of plate on the convex side. By neglecting σ_3, we have similarly

$$\sigma_{rc}=E_0. \tag{43}$$

The condition of yielding at the center is therefore

$$\frac{a^2E_0}{h^2E}=\frac{W_m}{6(1-\mu)}\left\{12+(5-3\mu)W_m-\frac{1}{15}(1+\mu)(29-19\mu)W_m^2\right.$$
$$\left.-\frac{1}{315}(1+\mu)\,(68-148\mu+66\mu^2)\,W_m^3\right\}. \tag{44}$$

The equations (41), (44) are plotted in figure 3. The experimental points are obtained from the data published by McPherson, Ramberg, and Levy[9]. It can be seen that the present theory agrees very well with the experimental results.

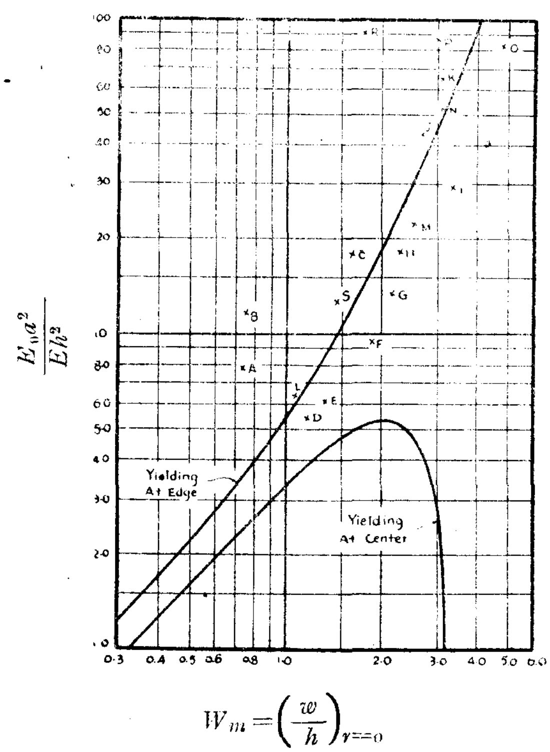

Fig. 3 Comparison Of Theoretical Yielding Center Deflections And Experimental Results From McPherson, Ramberg, And Levy's Tests