INVESTIGATIONS ON A NEW NUCLEAR POTENTIAL

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ABSTRACT

The potential between two nucleons suggested by K. C. Wang, viz.

$$V=V(r)=-Ae^{k/r} \quad \text{for } r \geqq a,$$
$$V=V(a) \quad \text{for } r < a,$$

is applied to find (1) the wave function of the ground state of the deuteron by the method of numerical integration, (2) the cross section of the neutron-proton scattering, and (3) the cross-section of the photo-disintegration of the deuteron by γ-rays. The results of calculation for the cross-sections are in fair agreement with the experimental values.

1. INTRODUCTION

It was pointed out by K. C. Wang[1] that the force between two nucleons may be related to the gravitational force. He took two alternative forms of the nuclear potential;

$$V=V(r)=-Ae^{k/r} \tag{1a}$$

$$V=V(r)=-\frac{B}{r}e^{k/r} \tag{1b}$$

where $K=\frac{\hbar}{mc}=3.84\times10^{-11}$ *cm*, A and B as determined by the gravitational constant are 4.78×10^{-45} and 1.84×10^{-55} respectively.

There are two ways of cut-off of the potential, viz.:

(a) zero cut-off: $V=o$ for $r \leq a$, $V=V(r)$ for $r \geq a$;

and (b) straight cut-off: $V=V(a)$ for $r \leq a$, $V=V(r)$ for $r \geq a$.

In the papers by M. H. Wang, the binding energy of the deuteron and the scattering cross-section of the neutron by the proton,[2] and the binding energies of

1. K. C. Wang and H. L. Tsao, *Nature* **155,** (1945) 512.
2. M. H. Wang, *Phys. Rev.*, **68,** (1945) 163.

H^3 and He^4 [3] have been calculated with this potential. The results are in good agreement with experiments.

The purpose of the present paper is to find (1) the wave function of the ground state of the deuteron by the method of numerical integration, (2) the cross-section of the neutron-proton scattering, and (3) the cross-section of the photo-disintegration of the deuteron by γ-rays. In these calculations, the potential (1 a) and the method of straight cut-off are employed. The calculations follow closely those of Bethe and Bacher[4].

2. THE WAVE FUNCTION OF THE GROUND STATE OF THE DEUTERON

The wave equation for the relative motion of two nucleons is

$$\triangle\psi+\frac{M}{\hbar^2}(E-\nabla)\psi=0. \tag{2}$$

Since the potential used is spherically symmetrical, (2) can thus be separated in polar coordinates r, θ, φ by putting

$$\psi(r,\theta,\varphi)=\frac{1}{r}U_l(r)P_{lm}(\theta)e^{im\varphi} \tag{3}$$

where $P_{lm}(\theta)$ is a spherical harmonic. Then

$$\frac{\hbar^2}{M}\left[\frac{d^2U_l}{dr^2}-\frac{l(l+1)}{r^2}U_l\right]+(E-V)U_l=0. \tag{4}$$

For the ground state of the deuteron, $l=0$, we have

$$\frac{d^2U_0}{dr^2}+\frac{M}{\hbar^2}[-\epsilon-V(r)]U_0=0 \tag{5}$$

where $\epsilon=-E=2.17$ Mev, is the binding energy of the deuteron.

For $r\leq a$, $V=Ae^{k/a}\equiv V_0$, equation (5) takes the form

$$\frac{d^2U_0}{dr^2}+\frac{M}{\hbar^2}(-\epsilon-V_0)U_0=0. \tag{6}$$

Its solution is

$$U_0=D\sin\alpha r, \tag{7}$$

3. M. H. Wang, *Phys. Rev.*, **70**, (1946), 492.
4. H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.*, **8**, (1936) 82.

where $$\alpha=[-\frac{M}{\hbar^2}(\epsilon+V_0)]^{\frac{1}{2}} ,$$

and $$\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{r=a} = \alpha \cot \alpha a \tag{8}$$

For sufficiently large values of r, $V=-Ae^{k/r}$ is negligible compared with ϵ. Hence (5) becomes

$$\frac{d^2 U_0}{d r^2}-\frac{M\epsilon}{\hbar^2} U_0=O. \tag{9}$$

Its solution is

$$U_0=C\, e^{-\beta r} \tag{10}$$

where $$\beta= \left[\frac{M\epsilon}{\hbar^2}\right]^{\frac{1}{2}}.$$

In general, the value of $V(r)$ is not negligible compared with ϵ; the values of U_0 can be calculated from the following equations:

$$\begin{cases} U_0(r-\triangle r)=U_0(r)-U_0'(r)\triangle r+\frac{1}{2}U_0''(r)\triangle r^2 \\ U_0'(r-\triangle r)=U_0'(r)-U_0''(r)\triangle r+\frac{1}{2}U_0'''(r)\triangle r^2 \\ U_0''(r)=\frac{M}{\hbar^2}[\epsilon+V(r)]U_0(r), \\ U_0'''(r)=\frac{M}{\hbar^2}[\epsilon+V(r)]U_0'(r)-\frac{M}{\hbar^2}\frac{K}{r^2}V(r)U_0(r), \end{cases} \tag{11}$$

provided that $\triangle r$ is sufficiently small.

In order that the wave function U_0 could be joined smoothly at $r=a$, the values of $\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{r=a}$ given by (8) and (11) must be equal, also for the values of $U_0(a)$ given by (7) and (11).

Now start from $r=4.70\times10^{-13}\ cm. \equiv b$.

$$\left|\frac{V(b)}{\epsilon}\right| \doteq 0.0004, \quad \therefore\ |\,V(b)\,| \ll \epsilon.$$

Therefore, for $r \geq b$, $U_0 = Ce^{-\beta r}$ and $\left(\frac{dU_0}{dr}\right) = -\beta U_0$.

From equations (11), the values of U_0, U_0' etc. are calculated for $a \leq r \leq b$. The results are tabulated in Table I:

Table I

r (in 10^{-13} cm.)	4.70	4.65	4.60	4.55	4.50	4.45	4.40
$U_0(r)/C$	.3336	.3374	.3413	.3453	.3493	.3534	.3575
$V(r)$ (in erg)	1.45×10^{-9}	3.503×10^{-9}	8.582×10^{-9}	2.151×10^{-8}	5.488×10^{-8}	1.433×10^{-7}	3.820×10^{-7}

4.35	4.31	4.28	4.25	4.23	4.22	4.21
.3615	.3651	.3676	.3701	.3718	.3726	.3735
1.041×10^{-6}	2.360×10^{-6}	4.405×10^{-6}	8.306×10^{-6}	1.273×10^{-5}	1.579×10^{-5}	1.963×10^{-5}

It is found that

$$\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{\text{outside}} = \left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{\text{inside}},$$

at $r = a = 4.215 \times 10^{-13}$ cm. This value of a gives $V_0 = V(a) = -11.1$ *Mev*, which is in good agreement with the experimental value of $V_0 = -10.5$ *Mev*.

Since the inside wave function must be equal to the outside wave function at $r = a$, therefore we should have

$$D \sin \alpha a = 0.3731C, \quad \therefore D = 0.4106C.$$

Hence the wave function U_0 is

$$\begin{cases} U_0 = 0.4106\ C \sin \alpha r, \text{ for } r \leq a, \\ U_0 = C \text{ numerical wave function (n.w.f.) given by Table I, for } a \leq r \leq b, \\ U_0 = C\, e^{-\beta r}, \text{ for } r \geq b. \end{cases} \tag{12}$$

To normalize the radial wave function, we evaluate the integral

$$C^2\left\{\int_0^a (0.4106 \sin\alpha\, r)^2 dr + \int_a^b (\text{n.w.f.})^2 dr + \int_b^\infty e^{-2\beta r}\, dr\right\} = 1.$$

We get $C = 3.723 \times 10^6$.

Finally, the wave function of the ground state of the deuteron is

$$U_0 = \frac{C}{\sqrt{4\pi}\ r} \times \begin{cases} 0.4106 \sin \alpha\, r, & \text{for } a \geq r \\ (\text{n.w.f.}) \text{ given by Table 1}, & \text{for } a \leq r \leq b \\ e^{-\beta r}, & \text{for } r \geq b \end{cases} \tag{13}$$

where $\frac{1}{\sqrt{4\pi}}$ is the normalizing factor for the θ- and φ- wave function.

3. THE SCATTERING OF NEUTRON BY PROTON

Let us denote by E the kinetic energy of a proton and a neutron in a coordinate system in which the center of gravity of the particle is at rest, which is equal to one half the kinetic energy of the incident neutron in a system at rest. The wave function U_l will satisfy equation (4). Asymptotically for large r, the solution is

$$U_l = C \sin (kr - \tfrac{1}{2}l\pi + \delta_l), \tag{14}$$

where $k^2 = \frac{ME}{\hbar^2}$.

The cross-section is given by the well-known formula

$$d\sigma = \frac{\pi}{2k^2} \left| \sum_l (2l+1)(P_l(\theta)(e^{2i\delta_l} - 1) \right|^2 \sin\theta d\theta \tag{15}$$

It has been shown[5] that if $\frac{1}{k} \ll a$, all phases δ_l's will be small except δ_0. Then

$$d\sigma = 2\pi k^{-2} \sin^2 \delta_0 \sin\theta\ d\theta$$

and

$$\sigma = \int d\sigma = 4\pi k^{-2} \sin^2 \delta_0. \tag{16}$$

We have already shown that, for the ground state of the deuteron,

$$\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{r=a} = \alpha \cot\alpha a.$$

5. Ref. (4), p. 115.

Now in the present case, E is positive. Therefore we should have

$$U_0 = B \sin \gamma r,$$

and

$$\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{r=a} = \gamma \cot \gamma a, \tag{17}$$

where

$$\gamma = \left[\frac{M}{\hbar^2}(E-V_0)\right]^{\frac{1}{2}}.$$

For large r,

$$\left(\frac{1}{U_0}\frac{dU_0}{dr}\right)_{r=a} = k \cot (ka+\delta_0) \tag{18}$$

In order to join the wave function at $r=a$ smoothly, the expressions given by (17) and (18) must be equal. Thus

$$\gamma \cot \gamma a = k \cot (ka+\delta_0) \tag{19}$$

Therefore δ_0 and hence σ can be found.

For $E=2.15$ Mev, $\sigma=2.16 \times 10^{-24}\ cm^2$.

For $E=1.05$ Mev, $\sigma=3.07 \times 10^{-24}\ cm^2$.

The experimental data quoted in reference (4) are

$\sigma=0.5-0.8 \times 10^{-24}\ cm^2$, for $E=2.15$ Mev,

$\sigma=1.1-1.5 \times 10^{-24}\ cm^2$, for $E=1.05$ Mev.

The calculated and the observed values of σ are in fair agreement. Better agreement is insignificant since we have neglected δ_l's except δ_0, which is a very rough approximation in our present case.

4. PHOTO-DISINTEGRATION OF THE DEUTERON BY γ-RAYS

The photo-disintegration of the deuteron by γ-rays is closely analogous to the photoelectric effect in atoms. The electric field of the γ-rays produces an optical transition of the deuteron from the ground state to a state of positive energy,

$$E = h\nu - \epsilon$$

E being the kinetic energies of the proton and neutron produced in the process and $h\nu$ the energy of the incident γ-rays. The cross-section for the photoelectric effect is given by the well-known formula

$$\sigma = 8\pi^3\nu \mid M^{el}_{0E} \mid^2 / c \tag{20}$$

where M^{el}_{0E} is the matrix element of the electric moment of the deuteron relative to its center of gravity and in the direction of polarization of the γ-ray, the matrix element referring to the transition from the ground state to the state of energy E. The coordinate of the proton relative to the center of gravity of the deuteron is $\frac{1}{2}r$. Hence we have

$$M^{el}_{0E} = \tfrac{1}{2} e \int U_0 \, z \, U_E \, d\tau \tag{21}$$

if the γ-ray is polarized in the z-direction.

Here U_0 is the wave function of the ground state as given by (13), and U_E is the wave function of the final state normalized per unit energy[6]:

$$U_E = \frac{3^{\frac{1}{2}}}{2\pi\hbar} \cos\theta \left(\frac{M}{k}\right)^{\frac{1}{2}} \frac{1}{kr^2} \, Re \, [e^{ikr} \ (-i-kr)], \tag{22}$$

"Re" denoting the real part. We assume the axis of our coordinate system to be parallel to z, so that

$$z = r \cos\theta.$$

Therefore we obtain

$$M^{el}_{0E} = \frac{1}{2} e \frac{3^{\frac{1}{2}}}{\hbar} \left(\frac{M}{k}\right)^{\frac{1}{2}} [J_1 + J_2 + J_3], \tag{23}$$

where $J_1 = \dfrac{C}{\sqrt{4\pi}} \times 0.4106 \displaystyle\int_0^a 4\pi r^3 dr \ \overline{\cos^2\theta} \ \frac{1}{kr^3} \ Re \, [e^{ikr} \ (-i-kr)] \times \sin\alpha r,$

$$J_2 = \frac{C}{\sqrt{4\pi}} \int_a^b 4\pi r^3 dr \ \overline{\cos^2\theta} \ \frac{1}{kr^3} \ Re \, [e^{ikr} \ (-i-kr)] \times (\text{n.w.f.}),$$

$$J_3 = \frac{C}{\sqrt{4\pi}} \int_b^\infty 4\pi r^3 dr \ \overline{\cos^2\theta} \ \frac{1}{kr^3} \ Re \, [e^{(ik-\beta)r} \ (-i-kr)],$$

where $\overline{\cos^2\theta} = \displaystyle\int_0^\pi \cos^2\theta \times \tfrac{1}{2} \sin\theta \, d\theta = \frac{1}{3}.$

6. Ref. (4), p. 124.

With $h\nu=2.62$ Mev, we get

$J_1=4.9\times 10^{-21}$,

$J_2=1.9\times 10^{-21}$,

$J_3=2.12\times 10^{-19}$,

and $\sigma=1.6\times 10^{-27}\ cm^2$.

The experimental value of σ quoted in reference (4) is $\sigma=0.5\times 10^{-27}\ cm^2$, with an uncertainty of a factor of two.

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