

關於 ${}^2\text{H}$, ${}^3\text{H}$, ${}^3\text{He}$, ${}^4\text{He}$ 諸原子核 結合能之問題

第二部

金星南 張繼恒 彭桓武

(中國科學院近代物理研究所)

在這裏我們證明, 應用摩勒—羅森費德之原子核位能公式, 以 286 倍電子質量為介子之質量, 來計算 ${}^3\text{H}$, ${}^4\text{He}$ 之結合能, 所得之結果與實驗結果間之差異, 雖經修改波函數, 仍不能補救。

在本文第一部內, 應用摩勒—羅森費德之原子核位能, 計算 ${}^2\text{H}$, ${}^3\text{H}$, ${}^3\text{He}$, ${}^4\text{He}$ 之結合能。如取介子之質量為 200 電子質量, 則結果與實驗結果甚相符合。但如取介子之質量約為 300 電子質量, 結果與實驗結果相差很遠。今因擊撞原子核時所產生介子之質量約為 286 電子質量, 前述之差異當視為嚴重之問題, 所以在本文內, 欲修善對波函數之選擇, 以研討前文中之計算是否準確。本文接續第一部, 故所用未經註明之符號請參閱第一部。

${}^4\text{He}$ 內四個核子之自旋總和為零者, 可有兩種聯合方法, 其同位自旋亦如此。故本值為 $I=0, M_I=0, J=0, M_J=0$ 之自旋電荷本函數共四, 可取為

$$\left. \begin{aligned} u &= (41)^{-\frac{1}{2}} \det [P\alpha, P\beta, N\alpha, N\beta]; \\ v_1 &= (41)(23); \quad v_2 = (42)(31); \quad v_3 = (43)(12) \end{aligned} \right\} \quad (1)$$

u 即第一部中所用之行列式, v_1, v_2, v_3 內含兩兩核子之配合, 例如

$$(41) = \frac{\alpha(4)\beta(1) - \beta(4)\alpha(1)}{\sqrt{2}} \cdot \frac{P(4)N(1) - N(4)P(1)}{\sqrt{2}} \quad (2)$$

現在我們用普遍的 ${}^4\text{He}$ 波函數

$$\Psi = \varphi u + \chi_1 v_1 + \chi_2 v_2 + \chi_3 v_3 \quad (3)$$

來替代第一部內所用之特殊形狀 φu 。由於包里原則, $\varphi, \chi_1, \chi_2, \chi_3$ 之對稱須取決

於 u, v_1, v_2, v_3 之對稱情形。因此 $\varphi = \varphi(r_4, r_1, r_2, r_3)$ 對諸 r 完全對稱。 $\chi_1 = \chi(r_4, r_1; r_2, r_3)$ 對 r_4, r_1 反對稱, 對 r_2, r_3 反對稱, 對 r_4 與 r_1, r_2 與 r_3 之同時互換為對稱; 關於 χ_2, χ_3 之對稱情形, 可得自 r_1, r_2, r_3 之連環互換。

如 $A = 3$, 即 $^3\text{H}, ^3\text{He}$ 之波函數與上式相同, 可用同樣方法得到。僅 u, v_1, v_2, v_3 之形狀不同, 例如, 當 $M_I = \frac{1}{2}, M_J = \frac{1}{2}$

$$\left. \begin{aligned} u &= (3!)^{-\frac{1}{2}} \det [P\alpha, P\beta, N\alpha], & v_1 &= (23)P(1)\alpha(1), & v_2 &= (31)P(2)\alpha(2) \\ v_3 &= (12)P(3)\alpha(3) \end{aligned} \right\} \quad (4)$$

M_I, M_J 為其他值時亦相類似, 又 $\varphi = \varphi(r_1, r_2, r_3)$ 為完全對稱。 $\chi_1 = \chi(r_1; r_2, r_3)$ 對 r_2, r_3 反對稱, χ_2, χ_3 亦可由 r_1, r_2, r_3 之連環互換導出。

其變分積分

$$\frac{\int \psi^* H \psi d\tau}{\int \psi^* \psi d\tau} = \frac{\int \psi^* T \psi d\tau}{\int \psi^* \psi d\tau} + \frac{A(A-1)}{2} \cdot \frac{\int \psi^* V_{12} \psi d\tau}{\int \psi^* \psi d\tau}, \quad (5)$$

內之各項經計算可得

$$\int \psi^* \psi d\tau = \int \left\{ \varphi^2 + 3\chi_1^2 + \frac{3}{2}\chi_2\chi_1 \right\} d(r), \quad (6)$$

$$\int \psi^* T \psi d\tau = \int \left\{ \varphi T \varphi + 3\chi_1 T \chi_1 + \frac{3}{2}\chi_2 T \chi_1 \right\} d(r), \quad (7)$$

$$\int \psi^* V_{12} \psi d\tau = \int \frac{e^{-ir_{12}}}{r_{12}} \left\{ -(g_0^2 + 3f_0^2)\varphi^2 - \sqrt{6}g_0^2\varphi(\chi_2 - \chi_1) + \frac{3}{2}(g_0^2 + 3f_0^2)\chi_2\chi_1 + 3(-g_0^2 + 3f_0^2)\chi_3(\chi_1 + \chi_3) \right\} d(r) \quad (8)$$

$A = 3, A = 4$ 時上列方程式均可適用, T 表總動能, $\int \dots d(r)$ 表對整個形相空間之積分。

今 φ 即選為第一部 (11) 式所示之簡單形狀, 更依其對稱關係取

$$\chi(r_4, r_1; r_2, r_3) = \frac{1}{9\sqrt{40}} b (r_4 - r_1, r_2 - r_3) (2a^2\lambda^2) \varphi, \quad A = 4 \quad (9)$$

$$\chi(r_1; r_2, r_3) = \frac{1}{9\sqrt{7}} b \left(\frac{r_2 + r_3 - 2r_1}{3}, r_2 - r_3 \right) (2a^2\lambda^2) \varphi, \quad A = 3 \quad (10)$$

b 爲另一參變數。代入第一部內所引用之相對坐標，則 $A = 4$ 時

$$\chi_1 = \frac{1}{9\sqrt{40}} b \left\{ \frac{1}{2} q_1^2 - \frac{1}{2} q_2^2 - \frac{1}{\sqrt{3}} (q_1, q_2) + \sqrt{\frac{2}{3}} (q_1, q_3) - \sqrt{2} (q_2, q_3) \right\} (2 a^2 \lambda^2) \varphi \quad (11)$$

$$\chi_2 = \frac{1}{9\sqrt{40}} b \left\{ -\frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 - \frac{1}{\sqrt{3}} (q_1, q_2) + \sqrt{\frac{2}{3}} (q_1, q_3) + \sqrt{2} (q_2, q_3) \right\} (2 a^2 \lambda^2) \varphi \quad (12)$$

$$\chi_3 = \frac{1}{9\sqrt{40}} b \left\{ \frac{2}{\sqrt{3}} (q_1, q_2) - 2\sqrt{\frac{2}{3}} (q_1, q_3) \right\} (2 a^2 \lambda^2) \varphi \quad (13)$$

$A = 3$ 時，則將所有含 q_3 之項去掉，並換 $\sqrt{40}$ 爲 $\sqrt{7}$ 。經過一些與第一部內相同之計算，得下列能量式替代第一部內之 (22) 式與 (23) 式，(能量單位爲 $\hbar^2 \lambda^2 / M$)

$$\varepsilon = \varepsilon(a, b) = \frac{1}{4} a^2 - \frac{B + Cb + Db^2}{1 + b^2} \quad (14)$$

$A = 4$ 時

$$B = \frac{105}{128} (k + l) \frac{a^8}{(a+1)^5} \left(a^3 + \frac{47}{35} a^2 + \frac{5}{7} a + \frac{1}{7} \right), \quad (15)$$

$$C = \frac{105}{512} \sqrt{\frac{3}{5}} (k - l) \frac{a^8}{(a+1)^7} \left(a^5 + 7a^4 + 10a^3 + \frac{142}{21} a^2 + \frac{7}{3} a + \frac{1}{3} \right), \quad (16)$$

$$D = \frac{77}{2048} (k + l) \frac{a^8}{(a+1)^9} \left(\frac{27}{4} a^7 + \frac{10519}{308} a^6 + \frac{24139}{308} a^5 + \frac{29587}{508} a^4 + \frac{3099}{44} a^3 + \frac{4153}{132} a^2 + \frac{351}{44} a + \frac{39}{44} \right) - \frac{77}{128} l \frac{a^5}{(a+1)^8} \left(a^4 + \frac{104}{77} a^3 + \frac{62}{77} a^2 + \frac{8}{33} a + \frac{1}{33} \right); \quad (17)$$

$A = 3$ 時

$$B = \frac{1}{5\pi} (k + l) \frac{a^3}{(a^2 - 1)^3} \left(8a^4 + 9a^2 - 2 - 30 \frac{a^4}{\sqrt{a^2 - 1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right), \quad (18)$$

$$C = \frac{1}{30\pi} \sqrt{\frac{6}{7}} (k - l) \frac{a^3}{(a^2 - 1)^5} \left(16a^8 - 208a^6 - 165a^4 + 50a^2 - 8 \right. \\ \left. + 630 \frac{a^6}{\sqrt{a^2 - 1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right), \quad (19)$$

$$D = \frac{1}{7560\pi} (k + l) \frac{a^3}{(a^2 - 1)^7} \left(3200a^{12} + 7160a^{10} + 81560a^8 + 59857a^6 - 21674a^4 \right. \\ \left. + 5752a^2 - 720 - \frac{15120a^{12} + 25200a^{10} + 229950a^8}{\sqrt{a^2 - 1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right) \\ - \frac{4}{945\pi} l \frac{a^5}{(a^2 - 1)^6} \left(256a^8 + 2639a^6 + 690a^4 - 136a^2 + 16 \right. \\ \left. - \frac{1890a^8 + 5040a^6}{\sqrt{a^2 - 1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right); \quad (20)$$

其中引用簡式

$$k = \frac{M}{\hbar^2 \lambda} (3g_0^2 + 3f_0^2), \quad l = \frac{M}{\hbar^2 \lambda} (-g_0^2 + 3f_0^2) \quad (21)$$

對 b 取 (14) 式之極小值, 可得

$$\varepsilon = \varepsilon(a) = \frac{1}{4} a^2 - B - \Delta \quad (22)$$

此處

$$\Delta = \frac{1}{2} \left\{ \sqrt{(B - D)^2 + C^2} - (B - D) \right\} \quad (23)$$

Δ 代表因波函數內含 χ_1, χ_2, χ_3 項所生 ε 之改正項。再求 (22) 對 a 之極小值, 可得結合能 (單位為 $\hbar^2 \lambda^2 / M$), 但尚未計其負號。(22) 式相當複雜, 其對 a 之變化見下列各表。

第一表 $A=4$

a	B	$B-D$	C^2
3	$0.89(k+l)$	$0.63k + 0.91l$	$0.092(k-l)^2$
4	$1.49(k+l)$	$1.04k + 1.61l$	$0.226(k-l)^2$
5	$2.14(k+l)$	$1.50k + 2.41l$	$0.400(k-l)^2$
6	$2.84(k+l)$	$1.98k + 3.30l$	$0.648(k-l)^2$
7	$3.55(k+l)$	$2.48k + 4.25l$	$0.956(k-l)^2$
8	$4.30(k+l)$	$3.05k + 5.28l$	$1.312(k-l)^2$

第二表 $A=3$

a	B	$B-D$	C^2
1	$\cdot 073(k+l)$	$\cdot 059k + \cdot 066l$	$\cdot 00155(k-l)^2$
2	$\cdot 321(k+l)$	$\cdot 228k + \cdot 290l$	$\cdot 0284(k-l)^2$
3	$\cdot 667(k+l)$	$\cdot 514k + \cdot 713l$	$\cdot 160(k-l)^2$
4	$1.06(k+l)$	$\cdot 807k + 1.235l$	$\cdot 244(k-l)^2$
5	$1.51(k+l)$	$1.14k + 1.75l$	$\cdot 505(k-l)^2$

$I=1$ 之氫二核，如不計張量力，其電荷自旋本函數無須加以修正。只引用一因式 $(1 + c\sqrt{2} a l \rho)$ 作為對空間波函數之修改，式內含另一參變數 c 。求能量對 c 之極小值，則得下列列式方程，可用來確定 $\varepsilon = \varepsilon(a)$ ，

$$\begin{vmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon \end{vmatrix} = 0, \quad (24)$$

其中

$$\varepsilon_{11} = \frac{1}{4} a^2 - \frac{k a^3}{2(1+a)^2},$$

$$\varepsilon_{12} = -\frac{a^2}{2\sqrt{3}} - \sqrt{3} \left(-\frac{1}{2} + \frac{a}{3(1+a)} \right) \frac{k a^3}{(1+a)^2},$$

$$\varepsilon_{22} = \frac{7}{12} a^2 - \left\{ \frac{3}{2} - \frac{2a}{1+a} + \frac{a^2}{(1+a)^2} \right\} \frac{k a^3}{(1+a)^2},$$

然後再求對 a 之極小值,

在這裏我們不用第一部內所採用 $g_0^2/\hbar c$ 及 $f_0^2/\hbar c$ 之數值, 要從實驗所得 ${}^3\text{H}$, ${}^3\text{H}$, ${}^4\text{He}$ 之結合能來定 k, l 。下面說明用一較簡單之方法解決上述之問題。一如求 $\varepsilon = F(a, \hbar)$ 對 a 之極小值, 即得其能量, \hbar 視爲一參變數。我們應選一適當之 \hbar 值使計算所得之極小值與實驗之結合能 ε_0 相等。因此之故解方程式 $F(a, \hbar) - \varepsilon_0 = 0$ 求 \hbar 爲 a 之函數, $\hbar = K(a)$, 如再代入原式, 則得 a 之恒等式即 $F(a, K(a)) - \varepsilon_0 = 0$, 對 a 微分仍得 a 之恒等式, $\partial F/\partial a + \partial F/\partial \hbar \cdot dK/da = 0$ 。今假定 $\hbar = K(a)$ 在 $a = a_0$ 處有一極小值或極大值 \hbar_0 , 且 $\hbar = \hbar_0, a = a_0$ 時 $\partial F/\partial \hbar$ 爲有限值。則由上二恒等式可知如 $\hbar = \hbar_0$, 在 $a = a_0$ 處除 $F(a_0, \hbar_0) = \varepsilon_0$ 外, $\partial F(a, \hbar_0)/\partial a = 0$, 此即所求者。

下面我們用實驗所得之 286 電子質量作爲介子之質量, 故能量單位爲 $\hbar^2 \lambda^2/M = 22.8 \text{ Mev}$ 。(24) 式中 $\varepsilon = -0.096$, 此數值與結合能 2.19 Mev 相當。代入不同之 a 值, 求 \hbar , 則找到在 $a \doteq 1.5$ 時 \hbar 爲極小, $\hbar \doteq 2.43$ 。這樣便從 $I = 1$ 態之氦二核確定 \hbar 之值。同樣可從 ${}^3\text{H}$ 或 ${}^4\text{He}$ 之結合能確定參變數 l 。結果如下: (i) ${}^3\text{H}$ 之結合能爲 8.59 Mev, 則 $\varepsilon = -0.368$, 得到 $l \doteq 1.45, a \doteq 2.7$; (ii) ${}^4\text{He}$ 之結合能爲 29 Mev*, 則 $\varepsilon = -1.27$, 得到 $l \doteq 1.05, a \doteq 4.7$ 。其間 l 值之不同表示用摩勒-羅森費德之原子核位能來解釋 ${}^2\text{H}, {}^3\text{H}, {}^3\text{He}, {}^4\text{He}$ 之結合能實不可能。而且差異很大, 可見於下表, 表示用同一 l 值計算 ${}^3\text{H}, {}^4\text{He}$ 之結合能各如何。

第三表 結果之比較

l	$a({}^3\text{H})$	$\varepsilon({}^3\text{H})$	$a({}^4\text{He})$	$\varepsilon({}^4\text{He})$	$\varepsilon({}^4\text{He})/\varepsilon({}^3\text{H})$
1.05	2.1	-0.09	4.7	-1.27	14
1.25	2.6	-0.21	5.0	-1.62	7.7
1.45	2.7	-0.37	5.4	-2.07	5.6
實驗結果		-0.37		-1.27	3.5

作上述計算時注意到, 用 (3) 式修改波函數所得到的結合能量之改正爲小。 $A = 3, A = 4$, 因引用 χ_1, χ_2, χ_3 而得結合能之改正, 即 (25) 式, 可由下式取其近似值

*此數值因庫倫位能改正之故, 較實驗值約大 3% 見第一部。

$$A \doteq \frac{1}{4} C^2 / (B - D)$$

$A = 5$ 時, 約改正 10%, $A = 4$ 時約改正 2.5%, 如令 $l = 1.25$, 我們來比較這些百分數與計算能量及實驗能量相差之百分數, 立刻可以見到後者相差確實存在, 並且不能用本文所討論的修改波函數之方法補救之。

第二表

下列表內之數值係根據修正之波函數計算所得之能量值, 與實驗值比較, 其百分數之誤差如下表所示。由表可知, 修正後之能量值與實驗值之百分數誤差, 較修正前之百分數誤差為小, 且其誤差之符號亦與實驗值之百分數誤差符號相反, 此即表示修正後之波函數, 較修正前之波函數, 較接近於實驗值。

	$(H^2)_{\text{exp}}$	$(H^2)_{\text{calc}}$	$(H^2)_{\text{calc}}$	$(H^2)_{\text{calc}}$	$(H^2)_{\text{calc}}$	λ
其中	1.1	1.27	1.1	1.0	1.0	20.1
	1.7	1.63	1.7	1.6	1.6	1.52
	2.0	1.9	2.0	1.9	1.9	1.4
	2.3	2.3	2.3	2.3	2.3	果計算實

修正之波函數與實驗值之百分數誤差, 較修正前之百分數誤差為小, 且其誤差之符號亦與實驗值之百分數誤差符號相反, 此即表示修正後之波函數, 較修正前之波函數, 較接近於實驗值。

第一表及第二表之數值係根據修正之波函數計算所得之能量值, 與實驗值比較, 其百分數之誤差如下表所示。

ON THE BINDING ENERGIES OF THE ATOMIC NUCLEI

${}^2\text{H}$, ${}^3\text{H}$, ${}^3\text{He}$ and ${}^4\text{He}$

PART II.

By KING SING-NAN, CHANG CHI-HENG AND H. W. PENG

Institute of Modern Physics, Academia Sinica, Peking

(Received May 10, 1950)

ABSTRACT

It is shown that the discrepancy between the experimental values for the binding energies of ${}^3\text{H}$ and ${}^4\text{He}$ and the calculated values based on the Møller-Rosenfeld nuclear potential with a meson mass of 286 electron mass cannot be removed by further refinement in the choice of the form of the wave functions.

In Part I*, the binding energies of the nuclei ${}^2\text{H}$, ${}^3\text{H}$, ${}^3\text{He}$ and ${}^4\text{He}$ have been calculated by using the Møller-Rosenfeld nuclear potential. The results are in fair agreement with the experimental results if the mass of the meson is taken to be about 200 electron mass. However, the calculated and the experimental values differ widely for a meson mass of the order of 300 electron mass. Since the mass of the mesons now produced by nucleon impacts is about 286 electron mass, the above discrepancy is serious. We shall therefore examine the accuracy of the previous calculation in the present paper by making further refinement in the choice of the wave functions. This paper is a sequel to Part I, and for notation which is not explained here, please consult Part I.

For ${}^4\text{He}$, the spin of the four nucleons can combine with a resultant zero in two ways; and similarly for the isotopic spin. Hence corresponding to the eigenvalues $I=0$, $M_I=0$, $J=0$, $M_J=0$, there are altogether four spin-charge eigenfunctions which can be taken as

$$\left. \begin{aligned} u &= (4!)^{-\frac{1}{2}} \det [P\alpha, P\beta, N\alpha, N\beta]; \\ v_1 &= (41)(23), \quad v_2 = (42)(31), \quad v_3 = (43)(12). \end{aligned} \right\} \quad (1)$$

* The preceding paper.

Here u is the determinantal eigenfunction used in Part I, while v_1, v_2 and v_3 involve the pairing of two nucleons e.g.

$$(41) = \frac{\alpha(4)\beta(1) - \beta(4)\alpha(1)}{\sqrt{2}} \cdot \frac{P(4)N(1) - N(4)P(1)}{\sqrt{2}} \quad (2)$$

Instead of the particular form φu adopted in Part 1 we shall now take for the wave function for ${}^4\text{He}$ the general form

$$\psi = \varphi u + \chi_1 v_1 + \chi_2 v_2 + \chi_3 v_3. \quad (3)$$

Because of Pauli's principle, the symmetry property of φ, χ_1, χ_2 and χ_3 follows from that of u, v_1, v_2 and v_3 . Thus, $\varphi = \varphi(\mathbf{r}_4, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is totally symmetrical in the r 's; $\chi_1 = \chi(\mathbf{r}_4, \mathbf{r}_1; \mathbf{r}_2, \mathbf{r}_3)$ is antisymmetrical in \mathbf{r}_4 and \mathbf{r}_1 , also antisymmetrical in \mathbf{r}_2 and \mathbf{r}_3 , but symmetrical in simultaneously interchanging \mathbf{r}_4 with \mathbf{r}_1 and \mathbf{r}_2 with \mathbf{r}_3 ; further, χ_2 and χ_3 can be obtained from χ_1 by cyclic permutations of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$.

Similar consideration yields also the above form for the wave function for $A=3$, ${}^3\text{H}$ or ${}^3\text{He}$, but with different expressions for u, v_1, v_2 , and v_3 . For example, for $M_I = \frac{1}{2}, M_J = \frac{1}{2}$ we have

$$u = (3!)^{-\frac{1}{2}} \det [P\alpha, P\beta, N\alpha], \quad v_1 = (23)P(1)\alpha(1), \quad v_2 = (31)P(2)\alpha(2), \\ v_3 = (12)P(3)\alpha(3), \quad (4)$$

and similar expressions for other values of M_I and M_J . Also, $\varphi = \varphi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is totally symmetrical, $\chi_1 = \chi(\mathbf{r}_1; \mathbf{r}_2, \mathbf{r}_3)$ is antisymmetrical in \mathbf{r}_2 and \mathbf{r}_3 , and further, χ_2 and χ_3 can be obtained from χ_1 by cyclic permutations of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$.

For the variational integral

$$\frac{\int \psi^* H \psi d\tau}{\int \psi^* \psi d\tau} = \frac{\int \psi^* T \psi d\tau}{\int \psi^* \psi d\tau} + \frac{A(A-1)}{2} \cdot \frac{\int \psi^* V_{12} \psi d\tau}{\int \psi^* \psi d\tau}, \quad (5)$$

we get

$$\int \psi^* \psi d\tau = \int \left\{ \varphi^2 + 3\chi_1^2 + \frac{3}{2}\chi_2\chi_1 \right\} d(r), \quad (6)$$

$$\int \psi^* T \psi d\tau = \int \left\{ \varphi T \varphi + 3\chi_1 T \varphi + \frac{3}{2}\chi_2 T \chi_1 \right\} d(r), \quad (7)$$

$$\int \psi^* V_{12} \psi d\tau = \int \frac{e^{-\lambda r_{12}}}{r_{12}} \left\{ - (g_0^2 + 3f_0^2) \varphi^2 - \sqrt{6} g_0^2 \varphi (\chi_2 - \chi_1) \right. \\ \left. + \frac{3}{2} (g_0^2 + 3f_0^2) \chi_2 \chi_1 + 3 (-g_0^2 + 3f_0^2) \chi_3 (\chi_1 + \chi_3) \right\} d(r). \quad (8)$$

These equations hold for both $A=3$ and $A=4$, with T denoting the total kinetic energy and $\int \dots d(r)$ denoting integration over the whole configuration space.

We choose for φ the simple expression given by (11), Part I, and for χ we take in accordance with the symmetry property

$$\chi(\mathbf{r}_4, \mathbf{r}_1; \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{9\sqrt{40}} b (\mathbf{r}_4 - \mathbf{r}_1, \mathbf{r}_2 - \mathbf{r}_3) 2 a^2 \lambda^2 \varphi \quad \text{for } A=4, \quad (9)$$

$$\chi(\mathbf{r}_1; \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{27\sqrt{7}} b (\mathbf{r}_2 + \mathbf{r}_3 - 2\mathbf{r}_1, \mathbf{r}_2 - \mathbf{r}_3) 2 a^2 \lambda^2 \varphi \quad \text{for } A=3, \quad (10)$$

b being another parameter. In terms of the relative co-ordinates q introduced in Part I, we have

$$\chi_1 = \frac{1}{9\sqrt{40}} b \left\{ \frac{1}{2} q_1^2 - \frac{1}{2} q_2^2 - \frac{1}{\sqrt{3}} (q_1, q_2) + \sqrt{\frac{2}{3}} (q_1, q_3) \right. \\ \left. - \sqrt{2} (q_2, q_3) \right\} 2 a^2 \lambda^2 \varphi, \quad (11)$$

$$\chi_2 = \frac{1}{9\sqrt{40}} b \left\{ -\frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 - \frac{1}{\sqrt{3}} (q_1, q_2) + \sqrt{\frac{2}{3}} (q_1, q_3) \right. \\ \left. + \sqrt{2} (q_2, q_3) \right\} 2 a^2 \lambda^2 \varphi, \quad (12)$$

$$\chi_3 = \frac{1}{9\sqrt{40}} b \left\{ \frac{2}{\sqrt{3}} (q_1, q_2) - 2\sqrt{\frac{2}{3}} (q_1, q_3) \right\} 2 a^2 \lambda^2 \varphi, \quad (13)$$

for $A=4$; while for $A=3$, the terms involving q_3 are to be omitted and at the same time $\sqrt{40}$ is to be replaced by $\sqrt{7}$ in the above expressions.

Then, after some calculation which is similar to that of Part I, we get for the energy (in units of $\hbar^2 \lambda^2/M$) the following expression which replaces (22) and (25) of Part I:

$$\varepsilon = \varepsilon(a, b) = \frac{1}{4} a^2 - \frac{B + Cb + Db^2}{1 + b^2}. \quad (14)$$

Here for $A=4$ we have

$$B = \frac{105}{128} (k+l) \frac{a^3}{(a+1)^5} \left\{ a^3 + \frac{47}{35} a^2 + \frac{5}{7} a + \frac{1}{7} \right\} \quad (15)$$

$$C = \frac{105}{512} \sqrt{\frac{3}{5}} (k-l) \frac{a^3}{(a+1)^7} \left\{ a^5 + 7a^4 + 10a^3 + \frac{142}{21} a^2 + \frac{7}{3} a + \frac{1}{3} \right\} \quad (16)$$

$$D = \frac{77}{2048} (k+l) \frac{a^3}{(a+1)^9} \left\{ \frac{27}{4} a^7 + \frac{10519}{308} a^6 + \frac{24139}{308} a^5 + \frac{29587}{308} a^4 \right. \\ \left. + \frac{3099}{44} a^3 + \frac{4155}{132} a^2 + \frac{351}{44} a + \frac{39}{44} \right\} \\ - \frac{77}{128} l \frac{a^5}{(a+1)^8} \left\{ a^4 + \frac{104}{77} a^3 + \frac{62}{77} a^2 + \frac{8}{35} a + \frac{1}{35} \right\}; \quad (17)$$

and for $A=3$ we have

$$B = \frac{1}{5\pi} (k+l) \frac{a^3}{(a^2-1)^3} \left\{ 8a^4 + 9a^2 - 2 - \frac{50a^4}{\sqrt{a^2-1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right\}, \quad (18)$$

$$C = \frac{1}{30\pi} \sqrt{\frac{6}{7}} (k-l) \frac{a^3}{(a^2-1)^5} \left\{ 16a^8 - 208a^6 - 165a^4 + 50a^2 - 8 \right. \\ \left. + \frac{630a^6}{\sqrt{a^2-1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right\}, \quad (19)$$

$$D = \frac{1}{7560\pi} (k+l) \frac{a^3}{(a^2-1)^7} \left\{ 3200a^{12} + 7160a^{10} + 81560a^8 + 59857a^6 \right. \\ \left. - 21674a^4 + 5752a^2 - 720 - \frac{15120a^{12} + 25200a^{10} + 229950a^8}{\sqrt{a^2-1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right\}$$

$$-\frac{4}{945\pi} l \frac{a^5}{(a^2-1)^6} \left\{ 256 a^8 + 2639 a^6 + 690 a^4 - 136 a^2 + 16 - \frac{1890 a^8 + 5040 a^6}{\sqrt{a^2-1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right\}; \quad (20)$$

where we have used for abbreviation

$$k = \frac{M}{\hbar^2 \lambda} (3g_0^2 + 3f_0^2), \quad l = \frac{M}{\hbar^2 \lambda} (-g_0^2 + 3f_0^2). \quad (21)$$

Minimizing (14) with respect to b , we get

$$\epsilon = \epsilon(a) = \frac{1}{4} a^2 - B - A, \quad (22)$$

where

$$A = \frac{1}{2} \{ \sqrt{(B-D)^2 + C^2} - (B-D) \} \quad (23)$$

denotes the correction to ϵ due to the inclusion of χ_1, χ_2, χ_3 terms in the wave function. By further minimizing (22) with respect to a , we get the binding energy in units of $\hbar^2 \lambda^2 / M$, apart from the minus sign. The variation of (22) with a can be seen from the following tables which are obtained from the complicated expressions given above.

TABLE 1

a	B	$B - D$	C^2
3	0.89 ($k + l$)	0.63 k + 0.91 l	0.92 ($k - l$) ²
4	1.49 ($k + l$)	1.04 k + 1.61 l	2.26 ($k - l$) ²
5	2.14 ($k + l$)	1.50 k + 2.41 l	4.00 ($k - l$) ²
6	2.84 ($k + l$)	1.98 k + 3.30 l	6.48 ($k - l$) ²
7	3.55 ($k + l$)	2.48 k + 4.25 l	9.56 ($k - l$) ²
8	4.30 ($k + l$)	3.05 k + 5.28 l	13.12 ($k - l$) ²

TABLE 2

 $A = 3$

a	B	$B - D$	C^2
1	$\cdot 073 (k + l)$	$\cdot 059 k + \cdot 066 l$	$\cdot 00155 (k - l)^2$
2	$\cdot 321 (k + l)$	$\cdot 228 k + \cdot 290 l$	$\cdot 0284 (k - l)^2$
3	$\cdot 667 (k + l)$	$\cdot 514 k + \cdot 713 l$	$\cdot 160 (k - l)^2$
4	$1\cdot 06 (k + l)$	$\cdot 807 k + 1\cdot 235 l$	$\cdot 244 (k - l)^2$
5	$1\cdot 51 (k + l)$	$1\cdot 14 k + 1\cdot 75 l$	$\cdot 505 (k - l)^2$

For the triplet state of deuteron there is no need of improving the charge-spin eigenfunctions as long as we neglect tensor forces. We shall improve the radial wave function by introducing a factor of the form $1 + c \sqrt{2} a \lambda \rho$, involving another parameter c . By minimizing the energy with respect to c , we get the following secular equation which determines $\varepsilon = \varepsilon(a)$,

$$\begin{vmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon \end{vmatrix} = 0 \quad (24)$$

where

$$\varepsilon_{11} = \frac{1}{4} a^2 - \frac{k a^3}{2(1+a)^2},$$

$$\varepsilon_{12} = -\frac{a^2}{2\sqrt{3}} - \sqrt{3} \left(-\frac{1}{2} + \frac{1}{3} \frac{a}{1+a} \right) \frac{k a^3}{(1+a)^2}$$

$$\varepsilon_{22} = \frac{7}{12} a^2 - \left\{ \frac{5}{2} - \frac{2a}{1+a} + \frac{a^2}{(1+a)^2} \right\} \frac{k a^3}{(1+a)^2}$$

and $\varepsilon(a)$ has further to be minimized with respect to a .

Instead of using the numerical values for $g_0^2/\hbar c$ and $f_0^2/\hbar c$ adopted in Part I, we shall now try to determine these in the form of k and l from the binding energy data of ${}^2\text{H}$, ${}^3\text{H}$ or ${}^4\text{He}$. The following mathematical consideration shows how this can be done in a simple way. Suppose the energy is to be obtained by minimizing the function $\varepsilon = F(a, k)$ with respect to a , k being a parameter. And we want to choose k so that the calculated minimum should agree with the experimental value ε_0 , say.

For this purpose, let us solve the equation $F(a, k) - \varepsilon_0 = 0$ for k as a function of a , say $k = K(a)$. If we insert this back into the above equation, the result is of course an identity in a , namely $F(a, K(a)) - \varepsilon_0 \equiv 0$. By differentiating this with respect to a we get again an identity in a , $\frac{\partial F}{\partial a} + \frac{\partial F}{\partial k} \frac{dK}{da} \equiv 0$. Suppose that the function $k = K(a)$ possesses a minimum or a maximum k_0 at $a = a_0$, and $\partial F / \partial k$ is finite for $k = k_0$, $a = a_0$. Then the above identity shows that at $k = k_0$, $a = a_0$, we also have $\frac{\partial F(a, k_0)}{\partial a} = 0$ in addition to $F(a_0, k_0) = \varepsilon_0$, which is just what we desired.

In what follows, we shall adopt the experimental value 286 electron mass for the meson mass. We insert in (24) the value -0.096 for ε (which corresponds to a binding energy of 2.19 Mev), solve for k for various numerical values of a , and find the minimum $k \doteq 2.43$ at $a \doteq 1.5$. Having thus determined k from the binding energy of the deuteron, we can treat (22) similarly and determine now the parameter l from either the binding energy of ${}^3\text{H}$ or that of ${}^4\text{He}$. The results are: (i) by using $\varepsilon = -0.368$ which corresponds to a binding energy of 8.39 Mev for ${}^3\text{H}$, we get $l \doteq 1.45$ and $a \doteq 2.7$; (ii) by using $\varepsilon = -1.27$ which corresponds to a binding energy of 29 Mev* for ${}^4\text{He}$, we get $l \doteq 1.05$ and $a \doteq 4.7$. This difference in the values of l shows the inconsistency of explaining the binding energies of ${}^2\text{H}$, ${}^3\text{H}$, ${}^3\text{He}$ and ${}^4\text{He}$ by means of the Møller-Rosenfeld nuclear potential. The discrepancy is large, as is further illustrated in the following table where the same value for l is used for both ${}^3\text{H}$ and ${}^4\text{He}$.

TABLE 3
Comparison of results

l	$a({}^3\text{H})$	$\varepsilon({}^3\text{H})$	$a({}^4\text{He})$	$\varepsilon({}^4\text{He})$	$\varepsilon({}^4\text{He})/\varepsilon({}^3\text{H})$
1.05	2.1	-0.09	4.7	-1.27	14
1.25	2.6	-0.21	5.0	-1.62	7.7
1.45	2.7	-0.37	5.4	-2.07	5.6
Experimental		-0.37		-1.27	3.5

* This is about 3% larger than the real experimental value, thus allowing for the Coulomb correction. Cf. Part I.

It appears from the present calculation that the correction to the energy due to the improvement made on the wave function is rather small. For $A=3$ and $A=4$, the correction (25) to the energy due to the χ_1, χ_2, χ_3 terms of the wave function can be approximated by

$$\Delta \approx C^2/4(B - D),$$

and this amounts to a 10% correction for $A=3$ and a 2.5% correction for $A=4$. When we compare these figures with the percentage deviation of the calculated values for ϵ , say with $l=1.25$, from the experimental values, we immediately see that the latter discrepancy is real and cannot be removed by merely refining the form of the wave functions.

(which corresponds to a binding energy of 1.10 Mev) solve for ϵ for various numerical values of λ and find the minimum value of ϵ for each λ . Having thus determined ϵ from the binding energy of the deuteron we can treat (22) similarly and determine now the parameter λ from either the binding energy of ^3H or that of ^3He . The results are: (i) by using $\epsilon = -0.568$ which corresponds to a binding energy of 8.39 Mev for ^3H we get $\lambda = 1.45$ and $\alpha = 2.7$; (ii) by using $\epsilon = -1.37$ which corresponds to a binding energy of 29 Mev* for ^3He we get $\lambda = 1.05$ and $\alpha = 4.7$. This difference in the values of λ shows the inconsistency of explaining the binding energies of ^3H , ^3He and ^4He by means of the Miller-Rosenfeld nuclear potential. The discrepancy is large, as is further illustrated in the following table where the same value for λ is used for both ^3H and ^3He .

TABLE 3
Comparison of results

λ	ϵ (H)	ϵ (He)	ϵ (He)
1.05	-0.37	-0.37	-0.37
1.25	-0.37	-0.37	-0.37
1.45	-0.37	-0.37	-0.37

* This is about 3% lower than the real experimental value. This shows that the present correction of Part I.