MEASUREMENT OF PHASE ANGLE BETWEEN FUNDAMENTAL COMPONENTS OF TWO NON-SINUSOIDAL PERIODIC WAVES

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ABSTRACT

This paper presents a simple method by which the phase angle between the fundamental components of two non-sinusoidal waves can be measured. Applications of this method in an electron tube amplifier and oscillator are discussed.

1. INTRODUCTION

It is often desired to measure the phase angle between the fundamental components of two periodic waves which are not necessarily sinusoidal. This kind of problem occurs very frequently in a physics or electrical engineering laboratory. This paper presents a simple method by which this phase angle can be directly measured on the screen of a cathode ray oscilloscope. If the two waves are electrical, the two voltages are applied either directly or indirectly, through a voltage divider, to the two pairs of deflecting plates of the cathode ray tube. If the waves are not electrical in nature, they can be translated into electrical voltages by means of electro-mechanical or electro-accoustical devices such as piezo-electric crystals, photo-cells, microphones, together with their auxiliary apparatus such as input circuits and amplifiers. It will be shown in the following that there is a definite relationship between this phase angle and the area enclosed by the open loop or loops of the Lissajous figure so formed on the screen of the oscilloscope. From this area, the phase angle can be readily measured.

2. RELATION BETWEEN THE PHASE ANGLE AND THE ENCLOSED AREA

Let us consider the general case in which neither one of the two waves is sinusoidal. Expressing them in the form of Fourier series, we have,

$$x = X_0 + \sum_{n=1}^{\infty} X_n \cos(n\omega t + \alpha_n), \tag{1}$$

$$y = Y_0 + \sum_{n=1}^{\infty} Y_n \cos(n\omega t + \beta_n), \tag{2}$$

where X_0 and Y_0 are the constant terms, X_n and Y_n are the amplitudes of the nth harmonic, and α_n and β_n the phase angles. If we consider the fundamental component of x as our reference, then $\alpha_1 = 0$ in equation (1). Let us assume that the Lissajous figure shown on the screen has the form shown in Fig. 1 which has two loop-areas A_1 and A_2. Let the cross point P in Fig. 1 be defined by

$$\omega t = \theta \text{ in the first half cycle}$$

and $$\omega t = \phi \text{ in the second half cycle.}$$

Note that at point 0 which corresponds to the instant $\omega t = 0$, x or y does not necessarily have a maximum value. The area of the loop is equal to,

$$\oint y\,dx = \oint y \cdot \frac{dx}{dt}\,dt. \tag{3}$$

The areas A_1 and A_2 are given by

$$A_1 = \int_{t=0}^{\theta} y\,dx + \int_{t=\phi}^{2\pi} y\,dx, \qquad A_2 = \int_{\omega t=\theta}^{\pi} y\,dx + \int_{\omega t=\pi}^{\phi} y\,dx. \tag{4}$$

The total net area A is equal to

$$A = A_1 + A_2 = \int_0^{2\pi} y \frac{dx}{dt}\,dt. \tag{5}$$

Differentiating equation (1) with respect to time and substituting in equation (5), we get

$$A = -\int_0^{2\pi} Y_0 \sum_{n=1}^{\infty} n X_n \sin(n\omega t + \alpha_n)\, d\omega t$$

$$-\int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n X_n Y_m \sin(n\omega t + \alpha_n) \cos(m\omega t + \beta_m)\, d\omega t$$

$$-\int_0^{2\pi} \sum_{n=1}^{\infty} n X_n Y_n \sin(n\omega t + \alpha_n) \cos(n\omega t + \beta_n)\, d\omega t, \qquad (6)$$

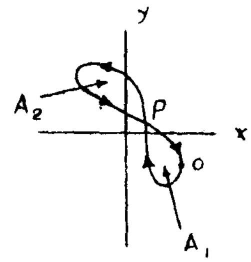

Fig. 1

where $n = 1,2,3,4, \cdots$

and $m = 1,2,3,4, \cdots$ with $m \neq n$.

Performing the integration in equation (6) and taking the proper limits, it is obvious that the first and the second terms in equation (6) vanish, giving,

$$A = \sum_{n=1}^{\infty} n\pi X_n Y_n \sin(\beta_n - \alpha_n). \qquad (7)$$

Equation (7) is a very important result. It gives an expression for the total net area enclosed by the loops in terms of the amplitudes of the voltages and the phase angles. Although we used a double-loop figure in deriving equation (7), it is evident that the result we obtained is quite general and is applicable to any shape of path of operation.

3. APPLICATION OF EQUATION (7) IN AN ELECTRON-TUBE AMPLIFIER

As an illustration in a rather special case, let us consider a vacuum-tube amplifier where the input voltage is sinusoidal. Here X corresponds to the grid voltage and Y the plate voltage. Under this condition equation (7) becomes

$$A = \pi X_1 Y_1 \sin \beta_1. \qquad (8)$$

We see from equation (8) that if $\beta_1 = 0$ or π, the area A equals zero.

And conversely, if the area A is zero, the phase angle must be either zero or 180 degrees. There are three possible cases:

Case 1: Path of operation is a straight line. This means not only that $A = 0$, but also that $A_1 = A_2 = 0$. The expressions for A_1 and A_2 may be obtained from equation (4) where θ now equals $\pi/2$, and ϕ equals $3\,\pi/2$. Substituting these values in equation (4) and integrating, we get,

$$A_1 = -\sum_{n=2}^{\infty} X_1 Y_n \left(\frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right)\sin\beta_n + (X_1 Y_1 \sin\beta_1).\,\pi/2$$

$$A_2 = +\sum_{n=2}^{\infty} X_1 Y_n \left(\frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right)\sin\beta_n + (X_1 Y_1 \sin\beta_1).\,\pi/2. \tag{9}$$

It can be seen from equation (9) that if $A_1 = A_2 = 0$, either

$$\beta_1 = \beta_2 = \cdots = \beta_n = 0 \text{ or } \pi,$$

or

$$\beta_1 = 0 \text{ or } \pi \text{ and } Y_2 = Y_3 = \cdots = 0.$$

It is obvious that the first condition ($\beta_1 = \beta_2 = \pi$ or 0) is never realized in practice. We can conclude, therefore, that $A_1 = A_2 = 0$ is a necessary and sufficient condition for zero harmonic output and zero phase shift.

Case 2: Path of operation is a symmetrical figure of eight. In this case, $A = 0$ but $A_1 = A_2 \neq 0$. A study of equation (9) shows that the above conditions can be satisfied if

$$\beta_1 = \pi \text{ and } Y_n \neq 0.$$

Equation (9) may be rewritten as,

$$|A_1| = |A_2| = \sum_{n=2}^{\infty} X_1 Y_n \left(\frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right)\sin\beta_n. \tag{10}$$

If we neglect the harmonics higher than the second, the above equation reduces to

$$|A_1| = |A_2| = \frac{4}{3} X_1 Y_2 \sin\beta_2. \tag{11}$$

It will be recalled from circuit theory that when the load circuit is tuned to resonance for the fundamental frequency, it offers almost a pure capacitive load to the higher harmonics. This means that $\sin \beta_2$ is nearly equal to unity, and the above equation can be used as a rough check of the amount of the harmonic content.

Case 3: Path of operation is an open or an unsymmetrical figure. This is an indication that the circuit is not in tune, for it is seen in equation (9) that this happens only when $\beta_1 \neq 0$ or π.

4. APPLICATION OF EQUATION (7) IN AN ELECTRON-TUBE OSCILLATOR

It has been shown previously that when neither the plate nor the grid voltage is sinusoidal, the total area of the loops is given by

$$A = \pi \sum_{n=1}^{\infty} n X_n Y_n \sin (\beta_n - \alpha_n). \tag{7}$$

For the sake of simplicity, let us consider only the first two terms in equation (7), neglecting the effect of the higher harmonics. We have then,

$$A = \pi [X_1 Y_1 \sin \beta_1 + 2 X_2 Y_2 \sin (\beta_2 - \alpha_2)]. \tag{12}$$

Imagine that the circuit of an oscillator has been so adjusted that the area A is zero. We want to know what information can be derived from equation (12):

(1) $\beta_1 = 0 \text{ or } \pi, \text{ and } (\beta_2 - \alpha_2) = 0 \text{ or } \pi,$

(2) $\beta_1 \neq 0 \text{ or } \pi, \text{ and } (\beta_2 - \alpha_2) \neq 0 \text{ or } \pi, \text{ but } X_1 Y_1 \sin \beta_1 = - X_2 Y_2 \sin (\beta_2 - \alpha_2),$

(3) $\beta_1 = 0 \text{ or } \pi, \text{ and } X_2 \text{ or } Y_2 = 0.$

The first two are only nathematically possible and cannot be realized physically. For, if the fundamental components of the plate and grid voltages are 180 degrees out of phase, the second harmonic voltages, in general, will not have the same phase difference. It can also be easily shown that, in general, $\sin (\beta_1 - \alpha_1)$ and $\sin (\beta_2 - \alpha_2)$ will always have the same sign, that is, the angles $(\beta_1 - \alpha_1)$ and $(\beta_2 - \alpha_2)$ are in the same quadrant. Therefore the first two possibilities are ruled out leaving only the last. We can conclude therefore that if the condition $A = 0$ is satisfied, we know that first, the phase angle must be 180 degrees, and secondly, the plate or grid voltage must be free from harmonics. This is the best adjustment for an oscillator.

5. MEASUREMENT OF PHASE ANGLE FROM CATHODE-RAY OSCILLOGRAMS

If one of the voltages X and Y is sinusoidal, equation (7) reduces to

$$A = \pi X_1 Y_1 \sin \beta_1$$

from which we may write,

$$\beta_1 = \sin^{-1} \frac{A}{\pi X_1 Y_1} . \tag{13}$$

By applying X and Y to the vertical and horizontal deflecting plates of a cathode-ray oscilloscope through proper high frequency voltage dividers, one can observe the path of operation on the screen of the oscilloscope and measure the quantities X_1, Y_1 and A. The area A is measured by a planimeter. The justification of the use of equation (13) depends upon the condition that either x or y is practically sinusoidal. The voltages can be roughly checked by measuring the positive and negative peaks of the loop and determining if the amplitudes on both sides of the axis are the same. The percentage distortion can be easily shown to be given approximately by,

$$\% \text{ harmonic content } \quad \frac{\frac{1}{2}(D_a + D_b) - D_a}{\frac{1}{2}(D_a + D_b)} \, 100 \tag{14}$$

where D_a is the maximum deflection above the axis and D_b is the maximum deflection below the axis.

The accuracy of equation (13) depends on how small is the harmonic content. If either x or y is free from harmonics, equation (13) is exact. If both x and y are non-sinusoidal, equation (13) is only approximate. Let us consider the effect of the second harmonic. Assuming, for example, that the second harmonic distortion is 5% in both the grid and plate voltages, and assuming furthur that the phase angle has a value that is most unfavorable, that is, when $\sin(\beta_2 - \alpha_2) = 1$, then the expression for A in this case is, from equation (12),

$$A = \pi X_1 Y_1 \sin\beta_1 + 2\pi X_2 Y_2 \sin(\beta_2 - \alpha_2).$$

Therefore, the phase angle is given by,

$$\beta_1 = \sin^{-1}\left[\frac{A}{\pi X_1 Y_1} - 0.0018\right].$$

For β_1 equal to about 2 degrees, the effect of the correction term is about 1/10 of a degree. For β_2 equal to about one degree, the correction term is about 5 minutes. It is obvious that for large values of β_1, the correction term has little effect. It follows from the above example that provided the harmonic content is not too large, equation (13) gives a very simple way of measuring the phase angle.

The author has used this method in measuring the phase angle between the grid and plate voltages of power oscillators and found it very convenient. Undoubtedly, it can be used for many other purposes.

中 文 提 要

兩個非正弦波中之基本頻率部份間相角之測量

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設有兩個非正弦波，共基本頻率相等，將此二波變爲兩電壓，接至陰極射線示波器之兩對偏轉板極，則產生一複雜之李氏圖形。此圖形之總面積，與二波中基本頻率之相角，有一極簡單之關係。本文指出此種關係之存在，並引用於電子管放大器及振盪器中相角之測量。