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容變粘滯性之唯象理論

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試先構成一弛緩式之容變粘滯及彈性的唯象理論。設流體受壓力後所成之容變爲立成與緩成兩部分所組成，若以 s_0 表一壓力所能完成之總相對容縮。$\Delta V/V$, s_∞ 表其壓力加上時立即縮成之部分，s_r 表該力加上後漸次遞成之部分，則 $s_0=s_\infty+s_r$. 命 $p=p(\varrho)$ 爲任一密度 ϱ 時之靜壓力，$p_0=p_0(\varrho_0)$ 爲原有密度 ϱ_0 時之靜壓力，則此流體之總壓縮係數 β_0 應有下式之定義：

$$s_0=\beta_0(p-p_0), \tag{1}$$

而其立成及緩成之兩部分則有下列各式表之：

$$\beta_0=\beta_\infty+\beta_r, \qquad s_\infty=\beta_\infty(p-p_0), \qquad s_r=\beta_r(p-p_0). \tag{2}$$

若所加之壓力與時俱變，則

$$\frac{ds}{dt}=\frac{ds_\infty}{dt}+\left(\frac{ds}{dt}\right)_{\text{vis}} \tag{3}$$

由 (2) 知

$$\frac{ds_\infty}{dt}=\beta_\infty\frac{dp}{dt}. \tag{4}$$

穩定流動中之平均壓力 $\bar{p}$ 不變時即相當於 $p=p(\varrho)$, 故粘滯流體於穩流中之容變爲

$$s_0=\beta_0(\bar{p}-p_0). \tag{5}$$

若 $\bar{p}$ 變，則可用下式

$$s=\beta_0(p'-p_0) \tag{6}$$

以表容變，p' 遂為一有效之動態壓力。今試設容變粘滯係數之定義為

$$\eta_2 = (\bar{p} - p') \Big/ \left(\frac{ds}{dt}\right)_{\text{vis}} \tag{7}$$

則由以上各式得如下之容變粘滯彈性方程式

$$\frac{ds}{dt} = \beta_\infty \frac{dp}{dt} + \frac{s_0 - s}{\beta_0 \eta_2}. \tag{8}$$

由此式知弛緩容變之弛緩時間為

$$\tau_2 = \beta_0 \eta_2. \tag{9}$$

其次，試利用此方程式計算傳播於流體中之聲波，因距離阻尼而有之衰減。在聲波中，吾人可設

$$p - p_0 = A e^{i\omega t}, \quad \text{及} \quad s = B e^{i\omega t}$$

代入方程式 (8) 中得

$$s = \{\beta_\infty + \beta_r/(1 + i\omega\tau_2)\}(p - p_0),$$

故複壓縮係數為

$$\beta_c = \beta_\infty + \beta_r/(1 + i\omega\tau_2). \tag{10}$$

以 $s = (c e^{\alpha x}) e^{i\omega(t - x/v)}$ 表阻尼平面波，則

$$s = c e^{i\omega(t - x/v_c)} \tag{11}$$

式中之複速度為

$$v_c = \frac{v}{1 + (\alpha v/\omega)^2} + i \frac{\alpha v^2}{\omega + \alpha^2 v^2/\omega}.$$

實際情形下，$\omega \gg \alpha v$, 故

$$v_c \approx v + i(\alpha v^2/\omega). \tag{12}$$

由彈性理論知波式 (11) 之複速度應為

$$v_c = \left[\left(\frac{1}{\beta_c} + \frac{4}{3}\mu_c\right)/\varrho\right]^{\frac{1}{2}} \tag{13}$$

式中 μ_c 爲複切變彈性係數，可分兩類以求之：

(i) 氣體或似氣體之液體：實切變係數 μ 爲零，故其切應力與應變率之關係爲

$$t_{ij}=2\,\eta_1\frac{d\,s_{ij}}{d\,t}=2\,\eta_1\,i\,\omega\,s_{ij},$$

式中 η_1 爲普通熟知之切變粘滯係數，故 $\mu_c=i\,\omega\,\eta_1$.

(ii) 似固體之液體：μ 不爲零，依 Maxwell 之切變粘滯彈性方程式

$$2\frac{d\,s_{ij}}{d\,t}=\frac{1}{\mu}\frac{d\,t_{ij}}{d\,t}+\frac{1}{\eta_1}t_{ij}$$

得
$$\mu_c=\mu\,i\,\omega\,\tau_1/(1+i\,\omega\,\tau_1),$$

式中 $\tau_1=\eta_1/\mu$ 爲切變弛緩時間。

將 μ_c 及 β_c 代入 (13) 以比 (12) 即得音波之速度 v 及其振幅衰減係數 α 之公式，在實際常遇之情況下，$\omega\,\tau_1\ll 1$, $\omega\,\tau_2\ll 1$, 此二類均給下列之近似公式：

$$v=1/(\varrho\,\beta_0)^{\frac{1}{2}}, \tag{14}$$

$$\alpha=\tfrac{1}{2}\,\frac{\omega^2}{\varrho\,v^3}\left(\frac{4}{3}\,\eta_1+\frac{\beta_r}{\beta_0}\,\eta_2\right). \tag{15}$$

最後，將經典流體動力學加以改造後，亦能推得 (14) 及 (15) 兩近似公式。此改造之流體動力學包括容變粘滯性，其流動方程式爲

$$\varrho\,\frac{d\,\boldsymbol{v}}{d\,t}=\varrho\boldsymbol{f}-\nabla\,p'+(\frac{1}{3}\eta_1+\eta_2)\,\nabla\,\nabla\cdot\boldsymbol{v}+\eta_1\,\nabla^2\boldsymbol{v}+\eta_2\,\beta_\infty\,\nabla\,\frac{d\,p}{d\,t}.$$

乃來自
$$\begin{aligned}\bar{p}-p' &= (\lambda+\frac{2}{3}\eta_1)\left(\frac{d\,s}{d\,t}-\beta_\infty\frac{d\,p}{d\,t}\right)\\ &= (\lambda+\frac{2}{3}\eta_1)\left(\frac{d\,s}{d\,t}\right)_{\text{vis}}\\ &= -(\lambda+\frac{2}{3}\eta_1)\left(\nabla\cdot\boldsymbol{v}+\beta_\infty\frac{d\,p}{d\,t}\right)\end{aligned}$$

之假設也。

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PHENOMENOLOGICAL THEORIES OF BULK VISCOSITY

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ABSTRACT

A relaxational bulk visco-elastic theory is formulated, leading to an equation recently assumed by Hall. A derivation is given of an expression for the coefficient of sound attenuation resulting from both shearing and bulk viscosities. The result is compared with classical hydrodynamics generalized in a manner similar to that given by Tisza.

1. INTRODUCTION

It is recently well established that the experimental values of the absorption coefficients of ultrasonic waves in polyatomic gases and liquids exceed those classically calculated from the effect of ordinary (shearing) viscosity and that of heat conductivity. This excessive part of attenuation has been satisfactorily explained by certain molecular processes that are relaxational. In the case of gases[1,2], this interpretation is based on the assumption that a much larger number of collisions is required for the molecular rotation and vibration to attain the equilibrium state than the translation, so that the establishment of the steady state of the internal degrees of freedom lags behind that of the external ones. This time lag, if not small compared to the period of the sound wave, results in a dissipation of energy. In the case of liquids[3,4], variation of the structure of the liquid, which involves a rearrangement of the molecules and requires a certain activation energy, is assumed to accompany the uniform dilatational movement of the molecules, and it is this structure change that lags be-

1 Herzfeld K. F. and Rice, F.O. *Phys. Rev.* **31** (1929), 691
2 Richards, W. T. *Rev. Mod. Phys.* **11** (1939), 36.
3 Frenkel, J. *Kinetic Theory of Liquids* (Clarendon Press, Oxford, 1946), IV, 5, pp. 208-209.
4 Hall, L. *Phys. Rev.* **73** (1948), 775.

hind the otherwise uniform molecular motion. In both cases, the effect of relaxation may be represented phenomenologically by a second coefficient of viscosity, usually called the bulk or volume viscosity.

Another observable effect of the volume viscosity is the streaming flow of the fluid caused by a beam of sound. This phenomenon was first correctly interpreted by Eckart[5] and subsequently demonstrated by Liebermann[6] as furnishing another experimental method of determining the volume viscosity.

In this article, a visco-elastic theory is formulated and an expression for the coefficient of sound absorption due to both kinds of viscosity is derived. Tisza's generalization of classical hydrodynamics is examined, and the results of the two theories are compared.

2. RELAXATIONAL VISCO-ELASTIC THEORY

Assume that the dilatation of a fluid element due to a constant change of the applied pressure be composed of two parts: one that is brought about instantaneously and another that, on account of the bulk viscosity, takes time to accomplish. Then, denoting the relative compression, $-\Delta V/V$, by s, we have

$$s_0 = s_\infty + s_r,$$

where s_0 is the equilibrium value of s (ultimately) produced by a constant pressure $p=p\ (\varrho)$, and s_∞ and s_r are respectively the instantaneous and relaxational parts of it. We shall mean by $p=p\ (\varrho)$ the hydrostatic pressure when the density is ϱ, and by $p_0=p\ (\varrho_0)$ that at the initial (static) density ϱ_0, the hydrostatic pressure being supposed to be a function of density (and temperature) alone. Then, by definition,

$$s_0 = \beta_0 (p - p_0), \tag{1}$$

where β_0 is the static or total compressibility of the fluid, being the reciprocal of its static bulk modulus. Let β_∞ and β_r be respectively the

5 Eckart, C. *Phys. Rev.* **73** (1948), 68.

6 Liebermann, L. N. *Phys. Rev.* **75** (1949), 1415.

instantaneous and relaxational parts of this compressibility so that

$$\beta_0 = \beta_\infty + \beta_r, \tag{2}$$

$$s_\infty = \beta_\infty (p - p_0), \quad \text{and} \quad s_r = \beta_r (p - p_0).$$

When the applied pressure varies with the time, the rate of relative compression is

$$\frac{ds}{dt} = \frac{ds_\infty}{dt} + \left(\frac{ds}{dt}\right)_{\text{vis}} \tag{3}$$

By (2), we have

$$\frac{ds_\infty}{dt} = \beta_\infty \frac{dp}{dt}. \tag{4}$$

It remains to be formulated how the second part, $(ds/dt)_{\text{vis}}$, may be represented by a coefficient of volume viscosity.

In a viscous fluid in motion, the value of the pressure is, as is well known, dependent on the orientation of the surface considered. Let $\bar{p} = -\frac{1}{3}(t_{11} + t_{22} + t_{33})$ be the arithmetic mean of the pressures on any three mutually perpendicular surface elements at the point considered. As far as the static part of s is concerned, this mean dynamic pressure, when constant, is, in effect, to correspond to a static pressure p so that for a viscous fluid in steady flow we have

$$s_0 = \beta_0 (\bar{p} - p_0). \tag{5}$$

When the pressure varies with the time, we may introduce an effective dynamic pressure $p' = p'(\varrho)$ which determines the actual relative compression according to

$$s = \beta_0 (p' - p_0). \tag{6}$$

We shall now define the coefficient of volume viscosity, η_2, by the equation

$$\left(\frac{ds}{dt}\right)_{\text{vis}} = \frac{\bar{p} - p'}{\eta_2}. \tag{7}$$

As will be seen later, this definition of η_2 is in harmony with a natural generalization of the classical hydrodynamics. By (5) and (6), (7) may be written as

$$\left(\frac{ds}{dt}\right)_{\text{vis}} = \frac{s_0 - s}{\beta_0 \eta_2}. \tag{8}$$

Substituting (8) and (4) in (3), we arrive at the bulk visco-elastic equation

$$\frac{ds}{dt} = \beta_\infty \frac{dp}{dt} + \frac{s_0 - s}{\beta_0 \eta_2}, \tag{9}$$

an equation that has been assumed by Hall.

For a constant applied pressure, $\dfrac{dp}{dt} = 0$, and the relaxational part of (9) is

$$\frac{ds}{dt} = -\frac{s - s_0}{\beta_0 \eta_2}, \tag{10}$$

whose solution in a form appropriate to the present problem is

$$s - s_0 = (s_\infty - s_0) \exp(-t/\beta_0 \eta_2), \tag{11}$$

as may readily be proven by substitution. When the applied pressure is withdrawn, $p = p_0$ so that s_0 in (10) vanishes and the appropriate solution becomes

$$s = s_\infty \exp(-t/\beta_0 \eta_2). \tag{12}$$

The phenomena represented by (11) and (12) are respectively shown by the curves *AB* and *CD* in Fig. 1. Thus, the relaxation time τ_2 for this process, i.e., the time required for s_r to reach $(1/e)th$ of its equilibrium value under a constant p, is seen to be

$$\tau_2 = \beta_0 \eta_2.$$

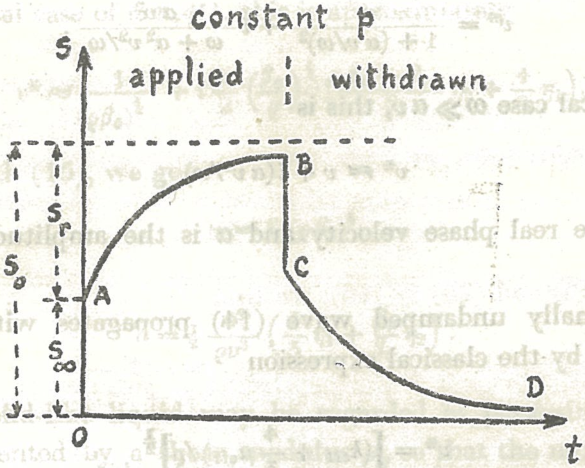

Fig. 1.

3. ATTENUATION OF SOUND

In an acoustic wave, we may assume

$$p - p_0 = A\, e^{i\omega t}, \quad \text{and} \quad s = B\, e^{i\omega t}.$$

Substituting in (9) and making use of (1), we have

$$s = \left(\beta_\infty + \frac{\beta_0 - \beta_\infty}{1 + i\,\omega\,\tau_2}\right)(p - p_0).$$

Hence, the effective coefficient of compressibility for a sound wave is

$$\beta_{\text{eff}} = \beta_\infty + \beta_r/(1 + i\,\omega\,\tau_2). \tag{13}$$

Representing the damped plane sound wave by

$$s = (Ce^{-\alpha x})\, e^{i\omega(t - x/v)},$$

we may write

$$s = Ce^{i\omega(t - x/v^*)}, \tag{14}$$

where the complex velocity is

$$v^* = \frac{v}{1 + (\alpha v/\omega)^2} + i \frac{\alpha v^2}{\omega + \alpha^2 v^2/\omega}.$$

For the practical case $\omega \gg \alpha v$, this is

$$v^* \approx v + i(\alpha v^2/\omega) \tag{15}$$

where v is the real phase velocity and α is the amplitude attenuation coefficient.

The formally undamped wave (14) propagates with a complex velocity given by the classical expression

$$v^* = \left[(k_{\text{eff}} + \frac{4}{3}\mu_{\text{eff}})/\varrho\right]^{\frac{1}{2}}, \tag{16}$$

where $k_{\text{eff}} = 1/\beta_{\text{eff}}$ is the complex effective bulk modulus and μ_{eff} the complex effective shear modulus. To find μ_{eff}, we shall distinguish two cases; both, however, lead to practically the same expression for α.

In the case of a gas or a more gas-like liquid, the real shear modulus may be taken to be zero so that the shearing stress is simply

$$t_{ij} = 2\eta_1 \frac{d s_{ij}}{d t} = \left(2\eta_1 \frac{d}{d t}\right) s_{ij},$$

where η_1 is the coefficient of shearing viscosity and $s_{ij} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j}\right)$ is the shearing strain. The operator $\eta_1 \frac{d}{d t}$ thus plays the role of the effective shearing modulus. For harmonic vibrations in sound waves, the stress and strain tensors depend on the time through the factor $e^{i\omega t}$ so that $\frac{d}{d t}$ is equivalent to multiplication by $i\omega$. Hence, we find for a gas or gas-like liquid

$$\mu_{\text{eff}} = i\omega\eta_1. \tag{17}$$

Substituting (13) and (17) in (16), we find

$$v^* = \frac{1}{\varrho^{\frac{1}{2}}}\left[\frac{\beta_0 + \beta_\infty \omega^2 \tau_2^2}{\beta_0^2 + \beta_\infty^2 \omega^2 \tau_2^2} + i\left(\frac{\omega \tau_2 (\beta_0 - \beta_\infty)}{\beta_0^2 + \beta_\infty^2 \omega^2 \tau_2^2} + \frac{4}{3}\omega\eta_1\right)\right]^{\frac{1}{2}},$$

For the practical case of $\omega\tau_2 \ll 1$, this is approximately

$$v^* \approx \frac{1}{(\varrho\beta_0)^{\frac{1}{2}}} + i\frac{\omega}{2}\left(\frac{\beta_0}{\varrho}\right)^{\frac{1}{2}}\left(\frac{\beta_0-\beta_\infty}{\beta_0}\eta_2 + \frac{4}{3}\eta_1\right).$$

Comparing with (15), we get

$$v = 1/(\varrho\beta_0)^{\frac{1}{2}},$$

and

$$\alpha = \tfrac{1}{2}\frac{\omega^2}{\varrho v^3}\left(\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_0}\eta_2\right). \tag{18}$$

A more solid-like liquid may be regarded as possessing appreciable rigidity represented by a shear modulus μ, so that the motion is in part due to elastic displacement, the remaining part being that of viscous flow. Then, one readily finds, as first given by Maxwell, the shearing visco-elastic equation

$$2\frac{d\,s_{ij}}{d\,t} = \frac{1}{\mu}\frac{d\,t_{ij}}{d\,t} + \frac{1}{\eta_1}t_{ij}, \tag{19}$$

from which it follows that when the motion is suddenly stopped, the shearing stress drops with time exponentially according to

$$t_{ij} = (t_{ij})_0\, e^{-(\mu/\eta_1)t},$$

with a time of relaxation given by

$$\tau_1 = \eta_1/\mu.$$

(19) may be written as

$$\left(1+\tau_1\frac{d}{d\,t}\right)t_{ij} = \left(\eta_1\frac{d}{d\,t}\right)2\,s_{ij}$$

so that, following Frenkel, the effective shear modulus may be represented by the operator

$$\mu_{\rm eff} = \eta_1 A\frac{d}{d\,t},$$

where A is an operator defined by

$$A\left(1+\tau_1\frac{d}{dt}\right)=1.$$

For harmonic vibrations, we then have

$$\mu_{\mathrm{eff}}=\mu\, i\,\omega\,\tau_1/(1+i\,\omega\,\tau_1). \tag{20}$$

Substituting (13) and (20) in (16), we find

$$v^*=\frac{1}{\varrho^{\frac{1}{2}}}\left[\left(\frac{\beta_0+\beta_\infty\omega^2\tau_2^2}{\beta_0^2+\beta_\infty^2\omega^2\tau_2^2}+\frac{\frac{4}{3}\mu\,\omega^2\tau_1^2}{1+\omega^2\tau_1^2}\right)+i\left(\frac{\omega\,\tau_2(\beta_0-\beta_\infty)}{\beta_0^2+\beta_\infty^2\omega^2\tau_2^2}+\frac{\frac{4}{3}\mu\,\omega\,\tau_1}{1+\omega^2\tau_1^2}\right)\right]^{\frac{1}{2}}.$$

For the practical case of $\omega\,\tau_2\ll 1$ and $\omega\,\tau_1\ll 1$, this is approximately

$$v^*\approx\left(\frac{k_0}{\varrho}\right)^{\frac{1}{2}}+i\,\frac{\omega}{2\,(\varrho\,k_0)^{\frac{1}{2}}}\left(\frac{\beta_0-\beta_\infty}{\beta_0}\,\eta_2+\frac{4}{3}\,\eta_1\right).$$

where $k_0=1/\beta_0$. Comparing with (15), we get, again,

$$v=1/(\varrho\,\beta_0)^{\frac{1}{2}},$$

and, again,

$$\alpha=\tfrac{1}{2}\,\frac{\omega^2}{\varrho\,v^3}\left(\frac{4}{3}\,\eta_1+\frac{\beta_r}{\beta_0}\,\eta_2\right).$$

4. GENERALIZATION OF CLASSICAL HYDRODYNAMICS

Tisza[7] has discussed at length how Stokes' relation in classical hydrodynamics must be modified in order to embody the effect of bulk viscosity. He has also defined the effective dynamic pressure p' as the pressure that would be produced by an adiabatic compression from the static density ϱ_0 to the actually existing density ϱ. This definition is not necessary, although it allows a somewhat simplified way of writing the equation of motion. Fox and Rock[8], instead, has simply taken the

7. Tisza, L. *Phys. Rev.* **71** (1942), 531.
8. Fox, F. E. and Rock, G. D. *Phys. Rev.* **70** (1946), 73.

hydrostatic pressure p for p'. From the view-point of our visco-elastic theory this is not permissible, as is also obvious from the fact that even for an ideal incompressible fluid p' is not the same as the hydrostatic pressure. It is, therefore, best to leave p' undefined. We may then generalize Newton's law of viscosity by assuming that the excess of stress over the effective dynamic pressure be proportional to the rate of strain due to viscous flow. Since the rate of instantaneous strain is, by (4), proportional to the rate of applied pressure dp/dt, our generalized Newton's law may be written as

$$t_{11} = -p' + (\lambda + 2\eta_1)\left(\frac{ds_{11}}{dt} - \gamma\frac{dp}{dt}\right) + \lambda\left(\frac{ds_{22}}{dt} - \gamma\frac{dp}{dt}\right) + \lambda\left(\frac{ds_{33}}{dt} - \gamma\frac{dp}{dt}\right),$$

$$t_{22} = -p' + \lambda\left(\frac{ds_{11}}{dt} - \gamma\frac{dp}{dt}\right) + (\lambda + 2\eta_1)\left(\frac{ds_{22}}{dt} - \gamma\frac{dp}{dt}\right) + \lambda\left(\frac{ds_{33}}{dt} - \gamma\frac{dp}{dt}\right),$$

$$t_{33} = -p' + \lambda\left(\frac{ds_{11}}{dt} - \gamma\frac{dp}{dt}\right) + \lambda\left(\frac{ds_{22}}{dt} - \lambda\frac{dp}{dt}\right) + (\lambda + 2\eta_1)\left(\frac{ds_{33}}{dt} - \gamma\frac{dp}{dt}\right).$$

Hence, we have

$$\bar{p} - p' = -\left(\lambda + \frac{2}{3}\eta_1\right)\nabla\cdot\boldsymbol{v} + 3\left(\lambda + \frac{2}{3}\eta_1\right)\gamma\frac{dp}{dt}$$
$$= \left(\lambda + \frac{2}{3}\eta_1\right)\frac{ds}{dt} + 3\left(\lambda + \frac{2}{3}\eta_1\right)\gamma\frac{dp}{dt}.$$

Comparing with (3), (4), and (7), we have

$$\eta_2 = \lambda + \frac{2}{3}\eta_1 \text{ and } \gamma = -\frac{\beta_\infty}{3}.$$

Our generalization of Newton's law of viscous flow may then be put as

$$[T] = -\left\{p' + \left(\frac{2}{3}\eta_1 - \eta_2\right)\nabla\cdot\boldsymbol{v} - \eta_2\beta_\infty\frac{dp}{dt}\right\}[I] + 2\eta_1\left[\frac{dS}{dt}\right], \tag{21}$$

where $[T]$, $\left[\frac{dS}{dt}\right]$, and $[I]$ are respectively the stress, rate of (pure) strain, and unit tensors or dyadics. The force on unit volume due to stress is the divergence of the stress tensor, which, by (21), is

$$-\nabla p' + \left(\frac{1}{3}\eta_1 + \eta_2\right) \nabla \nabla \cdot \boldsymbol{v} + \eta_1 \nabla^2 \boldsymbol{v} + \eta_2 \beta_\infty \nabla \frac{dp}{dt}.$$

Hence, the equation of motion becomes

$$\varrho \frac{d\boldsymbol{v}}{dt} = \varrho \boldsymbol{f} - \nabla p' + \left(\frac{1}{3}\eta_1 + \eta_2\right) \nabla \nabla \cdot \boldsymbol{v} + \eta_1 \nabla^2 \boldsymbol{v} + \eta_2 \beta_\infty \nabla \frac{dp}{dt}, \quad (22)$$

where $\boldsymbol{f}$ is the body force per unit mass.

For an acoustic wave propagating in the x-direction, equation (22), disregarding body force, becomes

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{dx} = -\frac{1}{\varrho}\frac{\partial p'}{\partial x} + \frac{1}{\varrho}\left(\frac{4}{3}\eta_1 + \eta_2\right)\frac{\partial^2 v_x}{\partial x^2} + \frac{\eta_2 \beta_\infty}{\varrho}\left(\frac{\partial^2 p}{\partial x \partial t} + v_x \frac{\partial^2 p}{\partial x^2}\right).$$

Even in supersonics, the wave length is so long that the product terms of $\frac{\partial v_x}{\partial x}$, v_x, s, etc., such as $v_x \frac{\partial v_x}{\partial x}$, are small compared to the individual terms and may be neglected as is usual in acoustic theory. Furthermore, since $v_x \approx \partial \xi / \partial t$, $s = -\partial \xi / \partial x$, and, by (6), $\frac{\partial p'}{\partial x} = \frac{1}{\beta_0}\frac{\partial s}{\partial x}$, and also, by neglecting $\omega \tau_2$ in (13),

$$\frac{\partial^2 p}{\partial x \partial t} \approx \frac{1}{\beta_0}\frac{\partial^2 s}{\partial x \partial t},$$

this equation may be written approximately

$$\frac{\partial^2 s}{\partial t^2} \approx \frac{k_0}{\varrho}\frac{\partial^2 s}{\partial x^2} + \frac{1}{\varrho}\left(\frac{4}{3}\eta_1 + \frac{\beta_0 - \beta_\infty}{\beta_0}\eta_2\right)\frac{\partial^3 s}{\partial x^2 \partial t}. \quad (23)$$

Representing the damped plane sound wave by

$$s = C e^{-\alpha' x} e^{i\omega t} \quad (24)$$

where the complex absorption coefficient is

$$\alpha^* = \alpha + i\frac{\omega}{v}, \tag{25}$$

and substituting (24) in (23), we have

$$\omega^2 \approx -\alpha^{*2}\left[\frac{k_0}{\varrho} - i\frac{\omega}{\varrho}\left(\frac{4}{3}\eta_1 + \frac{\beta r}{\beta_0}\eta_2\right)\right],$$

In the first approximation,

$$\alpha^* \approx \frac{1}{2}\beta_0\,\omega^2(\beta_0\varrho)^{\frac{1}{2}}\left(\frac{4}{3}\eta_1 + \frac{\beta r}{\beta_0}\eta_2\right) + i\,\omega\,(\beta_0\varrho)^{\frac{1}{2}},$$

Comparing with (25), we get, again,

$$v = 1/(\varrho\beta_0)^{\frac{1}{2}},$$

and

$$\alpha = \frac{1}{2}\,\frac{\omega^2}{\varrho v^3}\left(\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_0}\eta_2\right).$$

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