HYDRODYNAMIC THEORY OF LUBRICATION FOR PLANE SLIDERS OF FINITE WIDTH

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ABSTRACT

The hydrodynamic theory of viscous lubrication is studied from Navier-Stokes differential equations by the method of successive approximation based upon the smallness of the film thickness. It is found that the first approximation gives the Reynolds equation of viscous lubrication. To simplify the numerical nature of the solution of Reynolds equation, the equivalent variational problem is formulated. The approximate solution obtained from the variational problem involves only a very small error, but much less amount of numerical work.

INTRODUCTION

When two plane surfaces of finite width separated by a lubricant film slide against each other, there will be a positive fluid pressure developed in the film, as pointed out by Reynolds[1]. This pressure supports the load, floats the moving surface, and under favorable conditions will entirely prevent metallic contact. Because the plane slider of finite width is essentially nothing else than the thrust block bearing so widely used in industry, it is desirable to have a more comprehensive treatment of its lubrication properties.

The theory of the viscous lubrication of plane sliders was first given by Reynolds, in 1886. Reynolds' theory is based upon the following simplifying assumptions: (*a*) The lubricant be a Newtonian liquid. (*b*) The flow be viscous laminar flow. (*c*) The inertia effects of lubricant be neglected. (*d*) The lubricant be incompressible. (*e*) The fluid pressure be constant with respect to depth in the film. (*f*) The component of flow

1. Reynolds, O. On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments determination of the viscosity of olive oil. *Phil. Trans. Roy. Soc.*, **177**, pt. 1 (1866), 157-234.

normal to the film be neglected. (g) The viscosity of the lubricant be uniform throughout the film. Reynolds derived his famous differential equation for pressure distribution on the basis of these assumptions. The solution of the viscous lubrication of plane sliders of infinite width was given by Reynolds as illustration. The theory of the viscous lubrication of plane sliders of finite width, as based upon the Reynolds equation, was first given by A.G.M. Michell[2] in 1905. Except for some additional numerical calculations carried out by means of Michell's formulae by Martin[3] and Boswall[4], no extensions or improvements have been made in the theory. Duffing[5] has developed a solution for the mathematically simpler case where film viscosity is proportional to the film thickness, and has pointed out that in some particular cases the first derivative term in the Reynolds equation does not greatly affect the final solution. The disadvantage in numerical nature of Michell's method of solution was greatly reduced by M. Muskat, F. Morgan, and M. W. Meres[6] where the explicit analytic solution of Reynolds equation is given in a form which is more convenient for numerical calculation than those of Michell.

In the following we shall establish the hydrodynamic theory of viscous lubrication from Navier-Stokes differential equations by the method of successive approximation based upon the smallness of the film thickness. We shall only make the following assumptions. (a) The lubricant be a Newtonian liquid. (b) The flow be viscous laminar flow. (c) The lubricant be incompressible. (d) The viscosity of the lubricant be uniform throughout the film. It is found that the first approximation gives Reynolds equation of viscous lubrication.

To simplify the numerical nature of the solution of Reynolds differential equation of viscous lubrication, the equivalent variational problem has been formulated. It is found that the approximate solution

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2. Michell, A.G.M. The lubrication of plane surfaces. *Zeits. f. Math. Physik*, **52** (1905), 123-137.

3. Martin, H.M. The theory of Michell thrust bearing. *Engineering*, **109** (1920), 233-236.

4. Boswall, R.O. *The theory of film lubrication*. Longmans, Green and Co., 1928.

5. Duffing, G. *Handbuch der Physik, Tech. Mach.*, **5** (1931), 839.

6. Muskat, M., Morgan, F., Meres. M.W. Studies in lubrication. VII The lubrication of plane sliders of finite width. *Jour. of Applied Physics*, **11** (1940), 208-219.

obtained from the variational problem involves only a very small error, but much less amount of numerical work. It is evident that the same variational method can be applied to the lubrication theory of other bearings, such as journal bearing.

THE HYDRODYNAMIC EQUATIONS APPLIED TO LUBRICATION

In figure 1, let ABCD represent a bearing surface of any shape, and XYO its journal surface (slider). As we are concerned only with the relative motion, we may consider the journal at rest and suppose a velocity u_0 be imparted to the bearing in the direction x. The clearance h of the bearing is a linear function of x.

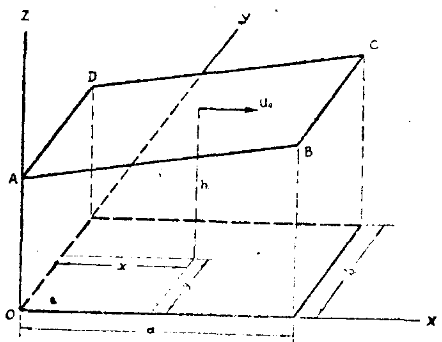

Fig. 1

Assuming constant viscosity, and neglecting the effect of body force on the motion of fluid, the Navier-Stokes differential equations of steady flow in the space bounded by the plane ABCD and XOY are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\varrho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (1)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\varrho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}=0\,, \qquad (2)$$

where u, v, w are the components of flow velocity, ϱ the density, p the pressure and $\nu = \mu/\varrho$ the kinematic viscosity of the fluid which is assumed to be uniform throughout the fluid.

The boundary conditions of the flow are

$$u = u_0,\ v = 0,\ w = 0 \text{ at ABCD, or at } z = h(x, y) \tag{3a}$$

$$u = v = w = 0 \qquad \text{at XOY,} \quad \text{or at } z = 0 \tag{3b}$$

$p = 0$ on all the free surfaces bounding the film, of which the length is a, width is b, and the thickness is h. (3c)

The present problem is therefore to find the unknowns u, v, w, p from (1) and (2) satisfying the boundary conditions (3a, b, c). We shall now introduce the following dimensionless quantities:

$$R = \text{Reynolds number} = u_0\, a/\nu, \tag{4a}$$

$$U = u/u_0,\ V = v/u_0,\ W = w/u_0, \tag{4b}$$

$$\xi = x/a,\ \eta = y/a,\ \zeta = z/a, \tag{4c}$$

$$P = p/\tfrac{1}{2}\varrho\, u_0^2. \tag{4d}$$

The Navier-Stokes differential equations and equation of continuity can be written in the following dimensionless form

$$U\frac{\partial U}{\partial \xi} + V\frac{\partial U}{\partial \eta} + W\frac{\partial U}{\partial \zeta} = -\frac{1}{2}\frac{\partial P}{\partial \xi} + \frac{1}{R}\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2}\right), \tag{5a}$$

$$U\frac{\partial V}{\partial \xi} + V\frac{\partial V}{\partial \eta} + W\frac{\partial V}{\partial \zeta} = -\frac{1}{2}\frac{\partial P}{\partial \eta} + \frac{1}{R}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \zeta^2}\right), \tag{5b}$$

$$U\frac{\partial W}{\partial \xi} + V\frac{\partial W}{\partial \eta} + W\frac{\partial W}{\partial \zeta} = -\frac{1}{2}\frac{\partial P}{\partial \zeta} + \frac{1}{R}\left(\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} + \frac{\partial^2 W}{\partial \zeta^2}\right); \tag{5c}$$

$$\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \zeta} = 0. \tag{5d}$$

The boundary conditions are

$$U = 1,\ V = 0,\ W = 0 \text{ at } \zeta = h(x, y)/a = h^*(\xi, \eta), \tag{6a}$$

$$U = V = W = 0 \text{ at } \zeta = 0, \tag{6b}$$

$P = 0$, on all free surfaces bounding the film, each of which contains respectively the lines

$$\text{(a) } \zeta = h^*, \xi = 0, \qquad \text{(b) } \zeta = h^*, \xi = 1,$$
$$\text{(c) } \zeta = h^*, \eta = 0, \qquad \text{(d) } \zeta = h^*, \eta = b/a. \tag{6c}$$

In the following we shall seek solutions of (5a, b, c, d) in power series form of the coordinates ζ. Let us assume that

$$P = P_0 + P_1 \zeta + P_2 \zeta^2 + \cdots, \tag{7a}$$

$$U = U_0 \zeta + U_1 \zeta^2 + U_2 \zeta^3 + \cdots, \tag{7b}$$

$$W = W_0 \zeta^2 + W_1 \zeta^3 + W_2 \zeta^4 + \cdots, \tag{7c}$$

$$V = V_0 \zeta + V_1 \zeta^2 + V_2 \zeta^3 + \cdots, \tag{7d}$$

where the coefficients P_i, U_i, V_i, W_i are functions of ξ, η only. It is easily verified that the boundary condition (6b) is satisfied. In the following we shall show that the coefficients

$$P_1, P_2, P_3, \cdots, U_1, U_2, U_3, \cdots, V_1, V_2, V_3, \cdots, W_0, W_1, W_2, \cdots$$

can be expressed in terms of the coefficients U_0, V_0, and P_0.

We substitute P, U, V, W from (7a, b, c, d) into (5a, b, c, d). By collecting terms of equal orders in ζ, we obtain a sequence of equations as follows:

For the terms of ζ^0 in (5a, b, c) and the terms of ζ^1 in (5d), we have

$$-\frac{1}{2}\frac{\partial P_0}{\partial \xi} + \frac{2}{R} U_1 = 0, \tag{8a}$$

$$-\frac{1}{2}\frac{\partial P_0}{\partial \eta} + \frac{2}{R} V_1 = 0, \tag{8b}$$

$$-\frac{1}{2} P_1 + \frac{2}{R} W_0 = 0, \tag{8c}$$

$$\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta} + 2 W_0 = 0. \tag{8d}$$

These equations determine the coefficients W_0, P_1, U_1, V_1 in terms of U_0, V_0, P_0:

$$W_0 = -\frac{1}{2}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right), \tag{9a}$$

$$P_1 = -\frac{2}{R}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right), \tag{9b}$$

$$V_1 = \frac{R}{4}\frac{\partial P_0}{\partial \eta}, \tag{9c}$$

$$U_1 = \frac{R}{4}\frac{\partial P_0}{\partial \xi}. \tag{9d}$$

From the terms of ζ^1 in (5a, b, c) and the terms of ζ^2 in (5d), we obtain

$$-\frac{1}{2}\frac{\partial P_1}{\partial \xi} + \frac{1}{R}\left(\frac{\partial^2 U_0}{\partial \xi^2} + \frac{\partial^2 U_0}{\partial \eta^2} + 6\,U_2\right) = 0, \tag{10a}$$

$$-\frac{1}{2}\frac{\partial P_1}{\partial \eta} + \frac{1}{R}\left(\frac{\partial^2 V_0}{\partial \xi^2} + \frac{\partial^2 V_0}{\partial \eta^2} + 6\,V_2\right) = 0, \tag{10b}$$

$$-P_2 + \frac{6}{R}W_1 = 0, \tag{10c}$$

$$\frac{\partial U_1}{\partial \xi} + \frac{\partial V_1}{\partial \eta} + 3\,W_1 = 0. \tag{10d}$$

Using P_1, V_1, U_1 given in (9b, c, d), we obtain, from (10a, b, c, d) the expressions of W_1, P_2, V_2, U_2

$$W_1 = -\frac{R}{12}\left(\frac{\partial^2 P_0}{\partial \xi^2} + \frac{\partial^2 P_0}{\partial \eta^2}\right), \tag{11a}$$

$$P_2 = -\frac{1}{2}\left(\frac{\partial^2 P_0}{\partial \xi^2} + \frac{\partial^2 P_0}{\partial \eta^2}\right), \tag{11b}$$

$$U_2 = -\frac{1}{6}\frac{\partial}{\partial \xi}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) - \frac{1}{6}\left(\frac{\partial^2 U_0}{\partial \xi^2} + \frac{\partial^2 U_0}{\partial \eta^2}\right), \tag{11c}$$

$$V_2 = -\frac{1}{6}\frac{\partial}{\partial \eta}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) - \frac{1}{6}\left(\frac{\partial^2 V_0}{\partial \xi^2} + \frac{\partial^2 V_0}{\partial \eta^2}\right). \tag{11d}$$

From the terms of ζ^2 in (5a, b, c) and the terms of ζ^3 in (5d), we obtain

$$U_0\frac{\partial U_0}{\partial \xi} + V_0\frac{\partial U_0}{\partial \eta} + W_0 U_0 = -\frac{1}{2}\frac{\partial P_2}{\partial \xi} + \frac{1}{R}\left(\frac{\partial^2 U_1}{\partial \xi^2} + \frac{\partial^2 U_1}{\partial \eta^2} + 12\, U_3\right), \tag{12a}$$

$$U_0\frac{\partial V_0}{\partial \xi} + V_0\frac{\partial V_0}{\partial \eta} + W_0 V_0 = -\frac{1}{2}\frac{\partial P_2}{\partial \eta} + \frac{1}{R}\left(\frac{\partial^2 V_1}{\partial \xi^2} + \frac{\partial^2 V_1}{\partial \eta^2} + 12\, V_3\right), \tag{12b}$$

$$0 = -\frac{3}{2}P_3 + \frac{1}{R}\left(\frac{\partial^2 W_0}{\partial \xi^2} + \frac{\partial^2 W_0}{\partial \eta^2} + 12\, W_2\right), \tag{12c}$$

$$\frac{\partial U_2}{\partial \xi} + \frac{\partial V_2}{\partial \eta} + 4\, W_2 = 0. \tag{12d}$$

From these relations, we obtain, with the aid of (9) and (11), the expressions for W_2, P_3, V_3, U_3.

$$W_2 = \frac{1}{12}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right), \tag{13a}$$

$$P_3 = \frac{1}{3R}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right), \tag{13b}$$

$$V_3 = \frac{R}{24}\left(V_0\frac{\partial V_0}{\partial \eta} - V_0\frac{\partial U_0}{\partial \xi} + 2\, U_0\frac{\partial V_0}{\partial \xi}\right) - \frac{R}{24}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\frac{\partial P_0}{\partial \eta}, \tag{13c}$$

$$U_3 = \frac{R}{24}\left(U_0\frac{\partial U_0}{\partial \xi} - U_0\frac{\partial V_0}{\partial \eta} + 2\, V_0\frac{\partial U_0}{\partial \eta}\right) - \frac{R}{24}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\frac{\partial P_0}{\partial \xi}. \tag{13d}$$

From the terms of ζ^3 in (5a, b, c) and the terms of ζ^4 in (5d), the expressions for W_3, P_4, V_4, U_4 can be obtained. We shall only give the final expressions for W_3 and P_4 as follows:

$$W_3 = -\frac{R}{120}\left\{\left(\frac{\partial U_0}{\partial \xi}\right)^2 + U_0\frac{\partial^2 U_0}{\partial \xi^2} + \left(\frac{\partial V_0}{\partial \eta}\right)^2 + V_0\frac{\partial^2 V_0}{\partial \eta^2} - 2\frac{\partial U_0}{\partial \xi}\frac{\partial V_0}{\partial \eta}\right.$$

$$\left. + 4\frac{\partial V_0}{\partial \xi}\frac{\partial U_0}{\partial \eta} + V_0\frac{\partial^2 U_0}{\partial \xi \partial \eta} + U_0\frac{\partial^2 V_0}{\partial \xi \partial \eta}\right\} + \frac{R}{120}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2 P_0, \quad (14a)$$

$$P4 = -\frac{1}{12}\left\{4\left(\frac{\partial U_0}{\partial \xi}\right)^2 + 4\frac{\partial U_0}{\partial \xi}\frac{\partial V_0}{\partial \eta} + 4\left(\frac{\partial V_0}{\partial \eta}\right)^2 - 2U_0\frac{\partial^2 U_0}{\partial \xi^2} - 2U_0\frac{\partial^2 V_0}{\partial \eta \partial \xi}\right.$$

$$\left. - 2V_0\frac{\partial^2 U_0}{\partial \eta \partial \xi} - 4\frac{\partial U_0}{\partial \eta}\frac{\partial V_0}{\partial \xi} - 2V_0\frac{\partial^2 V_0}{\partial \eta^2}\right) + \frac{1}{24}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2 P_0. \quad (14b)$$

By similar steps we can express all the other coefficients P_i, U_i, V_i, W_i in terms of P_0, U_0, V_0. However, for the purpose of the first approximation in the theory of thin film lubrication, the knowledge of the required expressions of $P_1, P_2, P_3, P_4, V_1, V_2, V_3, U_1, U_2, U_3, W_0, W_1, W_2, W_3$ is sufficient.

We have now succeeded in expressing P_i, V_i, U_i, W_i in terms of three quantities U_0, V_0, P_0. Hence if U_0, V_0, P_0 are given, the velocities of flow, and the pressure distribution throughout the film are determined. There remains, however, the task of determining U_0, V_0, P_0.

These three unknowns can be determined by the boundary conditions on the bearing surface ABCD, or the conditions (6a). They are, in the explicit form,

$$1 = U_0 h^* + \frac{R}{4}\frac{\partial P_0}{\partial \xi}h^{*2} - \frac{1}{6}\left\{\frac{6}{\partial \xi}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) + \nabla^2 U_0\right\}h^{*3}$$

$$+ \frac{R}{12}\left\{V_0\frac{\partial U_0}{\partial \eta} - \frac{1}{2}U_0\frac{\partial V_0}{\partial \eta} + \frac{1}{2}U_0\frac{\partial U_0}{\partial \xi} - \frac{1}{2}\nabla^2\frac{\partial P_0}{\partial \xi}\right\}h^{*4} + \cdots, \quad (15a)$$

$$0 = V_0 h^* + \frac{R}{4}\frac{\partial P_0}{\partial \eta}h^{*2} - \frac{1}{6}\left\{\frac{\partial}{\partial \eta}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) + \nabla^2 V_0\right\}h^{*3}$$

$$+ \frac{R}{12}\left\{U_0\frac{\partial V_0}{\partial \xi} - \frac{1}{2}V_0\frac{\partial U_0}{\partial \xi} + \frac{1}{2}V_0\frac{\partial V_0}{\partial \eta} - \frac{1}{2}\nabla^2\frac{\partial P_0}{\partial \eta}\right\}h^{*4} + \cdots, \quad (15b)$$

$$0 = -\frac{1}{2}\left(\frac{\partial U_0}{\partial \xi}+\frac{\partial V_0}{\partial \eta}\right)h^{*2}-\frac{R}{12}(\nabla^2 P_0)h^{*3}+\frac{1}{12}\nabla^2\left(\frac{\partial U_0}{\partial \xi}+\frac{\partial V_0}{\partial \eta}\right)\cdot h^{*4}$$

$$+\frac{R}{120}\nabla^2\nabla^2 P_0\cdot h^{*5}-\frac{R}{120}\left\{\left(\frac{\partial U_0}{\partial \xi}\right)^2+U_0\frac{\partial^2 U_0}{\partial \xi^2}+\left(\frac{\partial V_0}{\partial \eta}\right)^2\right.$$

$$\left.+V_0\frac{\partial^2 V_0}{\partial \eta^2}-2\frac{\partial U_0}{\partial \xi}\frac{\partial V_0}{\partial \eta}+4\frac{\partial V_0}{\partial \xi}\frac{\partial U_0}{\partial \eta}+V_0\frac{\partial^2 U_0}{\partial \xi\partial \eta}+U_0\frac{\partial^2 V_0}{\partial \eta\partial \xi}\right\}h^{*5}+\cdots. \tag{15c}$$

The pressure distribution throughout the interior of the film is given by the following expression:

$$P = P_0-\frac{2}{R}\left(\frac{\partial U_0}{\partial \xi}+\frac{\partial V_0}{\partial \eta}\right)\zeta-\frac{1}{2}(\nabla^2 P_0)\zeta^2+\frac{1}{3R}\nabla^2\left(\frac{\partial U_0}{\partial \xi}+\frac{\partial V_0}{\partial \eta}\right)\zeta^3$$

$$+\frac{1}{24}\nabla^2\nabla^2 P_0\cdot\zeta^4-\frac{1}{12}\left\{4\left(\frac{\partial U_0}{\partial \xi}\right)^2+4\frac{\partial U_0}{\partial \xi}\frac{\partial V_0}{\partial \eta}+4\left(\frac{\partial V_0}{\partial \eta}\right)^2-2U_0\frac{\partial^2 U_0}{\partial \xi^2}\right.$$

$$\left.-2U_0\frac{\partial^2 V_0}{\partial \eta\partial \xi}-2V_0\frac{\partial^2 U_0}{\partial \eta\partial \xi}-2V_0\frac{\partial^2 V_0}{\partial \eta^2}-4\frac{\partial U_0}{\partial \eta}\frac{\partial V_0}{\partial \xi}\right\}\zeta^4+\cdots. \tag{16}$$

In all the above expressions, the Laplacian oparator is taken with respect ξ, η, or

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}. \tag{17}$$

Equations (15a, b, c) are the three equations for the determination of the three unknowns, U_0, V_0, P_0, under the boundary condition (6c). These differential equations are too complicated to be handled without any approximation. In the following we shall introduce certain systematic approximations based upon the thinness of the film, so as to obtain, for various order of approximation, differential equations in simple forms.

APPROXIMATION BASED UPON THE THICKNESS OF FILM

We consider a family of infinite thin film of the same lubricant, bounded by stators and sliders of identical form, but of different average thickness. We assign to each film a value ε, so that the thickness of all films can be represented by

$$h^* = \varepsilon\, t(\xi, \eta) \tag{18}$$

where $0 < \varepsilon < \varepsilon_1$ and the function t is the same for all the films; for thin films, ε_1 is supposed to be small, but the basic idea of the method is that we seek solutions valid for all ε in the range $0 < \varepsilon < \varepsilon_1$. We may suppose ε chosen equal to the ratio of the average thickness to a selected lateral dimension of the film, say the length of the bearing surface a in the case of sliding bearing lubrication.

It is important to observe that ε is the only parameter involved. Except the Reynolds number R, all the quantities occurring in (15a, b, c) are functions of ε. We shall now suppose that all these quantities are regular in ε. In other words, we seek solutions of following forms:

$$U_0 = \frac{1}{\varepsilon}\{U_{00} + U_{01}\varepsilon^2 + U_{02}\varepsilon^4 + \cdots\}, \tag{19a}$$

$$V_0 = \frac{1}{\varepsilon}\{V_{00} + V_{01}\varepsilon^2 + V_{02}\varepsilon^4 + \cdots\}, \tag{19b}$$

$$P_0 = \frac{1}{\varepsilon^2}\{P_{00} + P_{01}\varepsilon^2 + P_{02}\varepsilon^4 + \cdots\}. \tag{19c}$$

We now substitute (19a, b, c) and (18) into (15a, b, c). The lowest power of ε occurring is ε^0 in (15a, b) and ε^1 in (15c). The corresponding coefficients give rise to equations of lubrication in the first approximation as follows:

$$1 = U_{00}\, t + \frac{R}{4}\frac{\partial P_{00}}{\partial \xi} t^2, \qquad 0 = V_{00}\, t + \frac{R}{4}\frac{\partial P_{00}}{\partial \eta} t^2, \tag{20a, b}$$

$$0 = -\frac{1}{2}\left(\frac{\partial U_{00}}{\partial \xi} + \frac{\partial V_{00}}{\partial \eta}\right) t^2 - \frac{R}{12}\left(\frac{\partial^2 P_{00}}{\partial \xi^2} + \frac{\partial^2 P_{00}}{\partial \eta^2}\right) t^3. \tag{20c}$$

We may remark that all the quantities in the above equations are finite, *i. e.*, independent of ε. These equations form a set of three equations for the determination of the three unknowns U_{00}, V_{00}, P_{00}. We may solve (20a, b) for U_{00}, V_{00} in terms of P_{00}, or

$$U_{00}=\frac{1}{t}-\frac{R}{4}\frac{\partial P_{00}}{\partial \xi}t, \qquad V_{00}=-\frac{R}{4}\frac{\partial P_{00}}{\partial \eta}t. \tag{21a, b}$$

Substituting (21a, b) in (20c), and regrouping the terms, we obtain an equation for the determination of P_{00},

$$\frac{\partial}{\partial \xi}\left(t^3\frac{\partial P_{00}}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(t^3\frac{\partial P_{00}}{\partial \eta}\right)=-\frac{12}{R}\frac{\partial t}{\partial \xi}. \tag{22}$$

This is the famous Reynolds equation for lubricant film between plane sliders, the solution of which has been studied by Muskat, Morgan and Meres in great details.

For the next approximation, we have the terms of ε^2 in (15a, b), and ε^3 in (15c). They are

$$U_{01}t+\frac{R}{4}\frac{\partial P_{01}}{\partial \xi}t^2-\frac{1}{6}\left\{\frac{\partial}{\partial \xi}\left(\frac{\partial U_{00}}{\partial \xi}+\frac{\partial V_{00}}{\partial \eta}\right)+\nabla^2 U_{00}\right\}t^3$$
$$+\frac{R}{12}\left\{V_{00}\frac{\partial U_{00}}{\partial \eta}-\frac{1}{2}U_{00}\frac{\partial V_{00}}{\partial \eta}+\frac{1}{2}U_{00}\frac{\partial U_{00}}{\partial \xi}-\frac{1}{2}\nabla^2\frac{\partial P_{00}}{\partial \xi}\right\}t^4=0, \tag{23a}$$

$$V_{01}t+\frac{R}{4}\frac{\partial P_{01}}{\partial \eta}t^2-\frac{1}{6}\left\{\frac{\partial}{\partial \eta}\left(\frac{\partial U_{00}}{\partial \xi}+\frac{\partial V_{00}}{\partial \eta}\right)+\nabla^2 V_{00}\right\}t^3$$
$$+\frac{R}{12}\left\{U_{00}\frac{\partial V_{00}}{\partial \xi}-\frac{1}{2}V_{00}\frac{\partial U_{00}}{\partial \xi}+\frac{1}{2}V_{00}\frac{\partial V_{00}}{\partial \eta}-\frac{1}{2}\nabla^2\frac{\partial P_{00}}{\partial \eta}\right\}t^4=0, \tag{23b}$$

$$-\frac{1}{2}\left(\frac{\partial U_{01}}{\partial \xi}+\frac{\partial V_{01}}{\partial \eta}\right)t^2-\frac{R}{12}(\nabla^2 P_{01})t^3+\frac{1}{12}\nabla^2\left(\frac{\partial U_{00}}{\partial \xi}+\frac{\partial V_{00}}{\partial \eta}\right)t^4$$
$$+\frac{R}{120}\nabla^2\nabla^2 P_{00}t^5-\frac{R}{120}\left\{\left(\frac{\partial U_{00}}{\partial \xi}\right)^2+U_{00}\frac{\partial^2 U_{00}}{\partial \xi^2}+\left(\frac{\partial V_{00}}{\partial \eta}\right)^2+V_{00}\frac{\partial^2 V_{00}}{\partial \eta^2}\right.$$
$$\left.-2\frac{\partial U_{00}}{\partial \xi}\frac{\partial V_{00}}{\partial \eta}+4\frac{\partial V_{00}}{\partial \xi}\frac{\partial U_{00}}{\partial \eta}+V_{00}\frac{\partial^2 U_{00}}{\partial \xi\partial \eta}+U_{00}\frac{\partial^2 V_{00}}{\partial \xi\partial \eta}\right\}t^5=0. \tag{23c}$$

From (23a, b) we may solve for U_{01}, V_{01} in terms of P_{01}, U_{00}, V_{00}:

$$U_{01} = -\frac{R}{4}\frac{\partial P_{01}}{\partial \xi} t + \frac{1}{6}\left\{\frac{\partial}{\partial \xi}\left(\frac{\partial U_{00}}{\partial \xi} + \frac{\partial V_{00}}{\partial \eta}\right) + \nabla^2 U_{00}\right\} t^2$$

$$-\frac{R}{12}\left\{V_{00}\frac{\partial U_{00}}{\partial \eta} - \frac{1}{2} U_{00}\frac{\partial V_{00}}{\partial \eta} + \frac{1}{2} U_{00}\frac{\partial U_{00}}{\partial \xi} - \frac{1}{2}\nabla^2\frac{\partial P_{00}}{\partial \xi}\right\} t^3, \quad (24a)$$

$$V_{01} = -\frac{R}{4}\frac{\partial P_{01}}{\partial \eta} t + \frac{1}{6}\left\{\frac{\partial}{\partial \eta}\left(\frac{\partial U_{00}}{\partial \xi} + \frac{\partial V_{00}}{\partial \eta}\right) + \nabla^2 V_{00}\right\} t^2$$

$$-\frac{R}{12}\left\{U_{00}\frac{\partial V_{00}}{\partial \xi} - \frac{1}{2} V_{00}\frac{\partial U_{00}}{\partial \xi} + \frac{1}{2} V_{00}\frac{\partial V_{00}}{\partial \eta} - \frac{1}{2}\nabla^2\frac{\partial P_{00}}{\partial \eta}\right\} t^3. \quad (24b)$$

Substituting (24a, b) into (23c), we obtain the equation for the determination of P_{01},

$$\frac{\partial}{\partial \xi}\left(t^3\frac{\partial P_{01}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(t^3\frac{\partial P_{01}}{\partial \eta}\right) = \frac{\partial}{\partial \xi}(t^3 M) + \frac{\partial}{\partial \eta}(t^3 N), \quad (25)$$

where M and N are expressions involving only known quantities U_{00}, V_{00}, P_{00}, t

$$M = \frac{t}{R}\left(2\frac{\partial^2 U_{00}}{\partial \xi^2} + \frac{\partial^2 V_{00}}{\partial \eta\,\partial \xi} + \frac{\partial^2 U_{00}}{\partial \eta^2}\right) - \frac{3}{10}t^2\left(2 V_{00}\frac{\partial U_{00}}{\partial \eta} - U_{00}\frac{\partial V_{00}}{\partial \eta}\right.$$

$$\left. + U_{00}\frac{\partial U_{00}}{\partial \xi} - \nabla^2\frac{\partial P_{00}}{\partial \xi}\right), \quad (26a)$$

$$N = \frac{t}{R}\left(2\frac{\partial^2 V_{00}}{\partial \eta^2} + \frac{\partial^2 U_{00}}{\partial \xi\,\partial \eta} + \frac{\partial^2 V_{00}}{\partial \xi^2}\right) - \frac{3}{10}t^2\left(2 U_{00}\frac{\partial V_{00}}{\partial \xi} - V_{00}\frac{\partial U_{00}}{\partial \xi}\right.$$

$$\left. + V_{00}\frac{\partial V_{00}}{\partial \eta} - \nabla^2\frac{\partial P_{00}}{\partial \eta}\right). \quad (26b)$$

Equation (25) gives the solution of P_{01}, which is the correction term to the first approximation of P_0.

The process of successive approximation can be carried on as far as required. However, we shall, for the time being, be satisfied with the second approximation.

SOLUTION OF REYNOLDS EQUATIONS BY VARIATIONAL METHOD

The solution of the Reynolds equation of type (22) or (25) depends on the boundary conditions of the pressure P as given in (6c). Or in particular, the conditions of the pressure P on the boundary lines AB, BC, CD, DA give for $\eta = 0,\ b/a$. or $\xi = 0,\ 1$,

$$0 = P_0 - \frac{2}{R}\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) h^* - \frac{1}{2}(\nabla^2 P_0) h^{*2} + \frac{1}{3R}\nabla^2\left(\frac{\partial U_0}{\partial \xi} + \frac{\partial V_0}{\partial \eta}\right) h^{*3}$$

$$+ \frac{1}{24}\nabla^2\nabla^2 P_0 \cdot h^{*4} - \frac{1}{12}\left\{4\left(\frac{\partial U_0}{\partial \xi}\right)^2 + 4\frac{\partial U_0}{\partial \xi}\frac{\partial V_0}{\partial \eta} + 4\left(\frac{\partial V_0}{\partial \eta}\right)^2 - 2U_0\frac{\partial^2 U_0}{\partial \xi^2}\right.$$

$$\left. - 2U_0\frac{\partial^2 V_0}{\partial \eta \partial \xi} - 2V_0\frac{\partial^2 U_0}{\partial \eta \partial \xi} - 2V_0\frac{\partial^2 V_0}{\partial \eta^2} - 4\frac{\partial U_0}{\partial \eta}\frac{\partial V_0}{\partial \xi}\right\} h^{*4} + \cdots. \qquad (27)$$

Substituting (18), (19a, b, c) into (27), we obtain in successive order of approximation, the boundary conditions for P_{00}, P_{01}, and so on. They are for $\eta = 0,\ b/a$ or $\xi = 0,\ 1$,

$$P_{00} = 0, \qquad (28a)$$

$$P_{01} = \frac{2}{R}\left(\frac{\partial U_{00}}{\partial \xi} + \frac{\partial V_{00}}{\partial \eta}\right) t, \qquad (28b)$$

and so on.

The present problem is therefore to find the solution of Reynolds equations in successive orders of approximation, (22), (25), under respective boundary conditions (26a, b). The analytic solution of Reynolds equation (22) for plane sliders of finite width was given by Muskat, Morgan, Meres[6]. The engineering application of these results is handicaped by the tediousness of numerical computation involved.

We shall here give an approximate method of solution of Reynolds equation based upon the variational method which is quite familiar in the theory of elasticity.

The following is the fundamental theorem to be proved:

Of all pressures P_{00} satisfying given boundary conditions (28a), those that satisfy the Reynolds equation (22) make the following integral H_0 a minimum:

$$H_0 = \iint_A \frac{R}{12}\left[t^3\left(\frac{\partial P_{00}}{\partial \xi}\right)^2 + t^3\left(\frac{\partial P_{00}}{\partial \eta}\right)^2 + \frac{24}{R}t\frac{\partial P_{00}}{\partial \xi}\right] d\xi\, d\eta, \tag{29}$$

where the integrals are taken over the area A bounded by the lines $\xi = 0, 1$, $\eta = 0, b/a$.

To prove this theorem, we merely have to find the variation of H_0 due to an arbitrary infinitesimal variation of P_{00}. Let ΔH_0 be the variation of H_0 due to the arbitrary variation ΔP_{00}. Hence from (29)

$$H_0 + \triangle H_0 = \iint_A \frac{R}{12}\left[t^3\left(\frac{\partial P_{00}}{\partial \xi} + \frac{\partial \triangle P_{00}}{\partial \xi}\right)^2 + t^3\left(\frac{\partial P_{00}}{\partial \eta} + \frac{\partial \triangle P_{00}}{\partial \eta}\right)^2 + \frac{24}{R}t\left(\frac{\partial P_{00}}{\partial \xi} + \frac{\partial \triangle P_{00}}{\partial \xi}\right)\right] d\eta\, d\xi. \tag{30}$$

The difference of (29) and (30) gives

$$\triangle H_0 = \iint_A \frac{R}{6}\left[t^3\left(\frac{\partial P_{00}}{\partial \xi}\frac{\partial \triangle P_{00}}{\partial \xi}\right) + t^3\left(\frac{\partial P_{00}}{\partial \eta}\frac{\partial \triangle P_{00}}{\partial \eta}\right) + \frac{12}{R}t\frac{\partial \triangle P_{00}}{\partial \xi}\right] d\eta\, d\xi + \iint_A \frac{R}{12}t^3\left[\left(\frac{\partial \triangle P_{00}}{\partial \xi}\right)^2 + \left(\frac{\partial \triangle P_{00}}{\partial \eta}\right)^2\right] d\eta\, d\xi. \tag{31}$$

The first integral can be integrated by parts. By means of the boundary conditions (28a) on P_{00}, or ΔP_{00}, we obtain

$$\triangle H_0 = -\iint_A \frac{R}{6}\Big[\frac{\partial}{\partial \xi}\Big(t^3\frac{\partial P_{00}}{\partial \xi}\Big)+\frac{\partial}{\partial \eta}\Big(t^3\frac{\partial P_{00}}{\partial \eta}\Big)+\frac{12}{R}\frac{\partial t}{\partial \xi}\Big]\triangle P_{00}\,d\eta\,d\xi$$

$$+\iint_A \frac{R}{12}t^3\Big[\Big(\frac{\partial \triangle P_{00}}{\partial \xi}\Big)^2+\Big(\frac{\partial \triangle P_{00}}{\partial \eta}\Big)^2\Big]d\eta\,d\xi. \tag{32}$$

It is evident that the first integral vanishes by the Reynolds equation, and the second integral is a positive second order terms in ΔP_{00}. Hence

$$\triangle H_0 = 0 + \text{Positive second order terms in } \triangle P_{00}. \tag{33}$$

Equation (32) expresses the above stated theorem of minimum H_0.

Similarly we can easily prove the following theorem for P_{01}:

Of all pressures P_{01} satisfying the given boundary conditions (28b), *those that satisfy the Reynolds equation* (25) *make the following integral H_1 a minimum:*

$$H_1 = \iint_A \frac{R}{12}\Big[t^3\Big(\frac{\partial P_{01}}{\partial \xi}\Big)^2 + t^3\Big(\frac{\partial P_{01}}{\partial \eta}\Big)^2 - 2t^3M\frac{\partial P_{01}}{\partial \xi} - 2t^3N\frac{\partial P_{01}}{\partial \eta}\Big]d\xi\,d\eta \tag{34}$$

where the integrals are taken over the area A bounded by the lines $\xi = 0$, 1, $\eta = 0$, b/a.

According to the theorem of minimum H_0, the exact solution of P_{00} is to be found by examining all possible functions satisfying the boundary condition (28a) on the lines $\xi = 0, 1, \eta = 0, b/a$ where the values of P_{00} are specified, and selecting only those that minimize the integral H_0. Since this procedure is, in general, very difficult one might hope to obtain an approximate solution by selecting from the set of all admissible functions a certain subset. One may assume that the function P_{00} can be represented with sufficient accuracy by the approximate expressions

$$P_{00}^{(n)} = \sum_{i=1}^{n} a_i f_i(\xi, \eta). \tag{35}$$

The functions $f_i(\xi, \eta)$ are assumed to satisfy the same boundary conditions as do the function P_{00}, but are otherwise unrestricted. If the approximate expressions (35) are inserted in the integrals H_0, as given in (29), the latter becomes a quadratic function $H_0^{(n)}$ of the parameters a_i. The minimizing conditions

$$\frac{\partial H_0^{(n)}}{\partial a_i} = 0 \qquad (i = 1, 2, 3, \cdots n) \tag{36}$$

are therefore linear equations for the determination of the unknown constants a_i. The approximate expression $P_{00}^{(n)}$ can thus be obtained.

It is evident from the theorem of minimum H_0, that the approximate value of the integral (29)

$$H_0^{(n)} = \iint_A \frac{R}{12} \left[t^3 \left(\frac{\partial P_{00}^{(n)}}{\partial \xi} \right)^2 + t^3 \left(\frac{\partial P_{00}^{(n)}}{\partial \eta} \right)^2 + \frac{24}{R} t \frac{\partial P_{00}^{(n)}}{\partial \xi} \right] d\xi \, d\eta \tag{37}$$

always exceed the true minimum H_0^* of the integral (29), or

$$H_0^{(n)} \geqq H_0^*. \tag{38}$$

This method of approximation can be used similarly for P_{01}.

It should be noted that the total load carried by the support of the bearing surface is borne by the combined action of the resultant hydrodynamic pressure Q_1 and the total viscous friction F_1. (Fig. 2) We shall now show that *the absolute value of the true minimum H_0^* of (29) represents the component S of the hydrodynamic force Q_1 in the direction of the motion of the slider surface.*

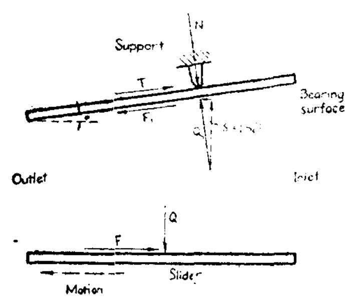

Fig 2

Let us denote this component of force in dimensionless form by S, or in the first approximation

$$S = \text{force}/\left(\frac{1}{2}\varrho u_0^2 a^2\right) = \iint_A \frac{\partial t}{\partial \xi} P_{00}\, d\xi\, d\eta. \tag{39}$$

Now the integral (29) can be integrated by parts with the aid of Green's theorem and simplified through the application of the boundary conditions (28a). The result is

$$H_0^* = -\iint_A \frac{R}{12}\left[\frac{\partial}{\partial \xi}\left(t^3 \frac{\partial P_{00}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(t^3 \frac{\partial P_{00}}{\partial \eta}\right) + \frac{24}{R}\frac{\partial t}{\partial \xi}\right] P_{00}\, d\xi\, d\eta. \tag{40}$$

Since P_{00} satisfies the Reynolds equation (25) in the region A, we have at once

$$H_0^* = -\iint_A \frac{\partial}{\partial \xi} P_{00}\, d\eta\, d\xi. \tag{41}$$

Hence the component of resultant hydrodynamic force S in the direction of the motion of the slider surface in the first approximation and the minimum value H_0 of the integral (29) are related by the expression

$$H_0^* = -S. \tag{42}$$

APPLICATION TO PLANE SLIDERS OF INFINITE WIDTH

We shall apply the above method of approximation to the case of plane sliders of infinite width, or slider bearing without side leakage. This is a one dimensional problem, in which

$$P_{00} = P_{00}(\xi), \qquad t = t_0(1 + \lambda\xi), \tag{43}$$

where t_0 and λ are given numerical constants. The Reynolds differential equation for the pressure distribution in this case can easily be solved. The result can be written as follows:

$$P_{00} = \frac{12}{R}\frac{\lambda}{\lambda + 2}\frac{\xi(1-\xi)}{t^2}. \tag{44}$$

This solution can also be obtained from the variational method stated above. Let the function P_{00} in (35) be of the form

$$P^{*}_{00} = a_1 \frac{\xi(1-\xi)}{t^2}, \tag{45}$$

where a_1 is a numerical constant to be determined. Substituting (45) into the integral (37), we find for unit width in η that

$$H^{*}_0 = \frac{R}{6\lambda t_0}\left[2 - \frac{\lambda+2}{\lambda}\log(1+\lambda)\right]\left(\frac{12}{R}a_1 - \frac{\lambda+2}{2\lambda}a_1^2\right). \tag{46}$$

From the minimizing condition $dH^{*}_0/da_1 = 0$, it follows that

$$a_1 = \frac{12}{R}\frac{\lambda}{\lambda+2}. \tag{47}$$

Hence we have the same solution for P_{00} as in (44), and the following expression for the minimum value of H_0

$$H^{*}_0 = \frac{12}{(\lambda+2)t_0 R}\left[2 - \frac{\lambda+2}{\lambda}\log(1+\lambda)\right]. \tag{48}$$

The friction force produced on the slider acting in the opposite direction of motion (Fig. 2) can be calculated from the shearing stress s_0 at $\zeta = 0$,

$$S_0 = \frac{\mu u_0}{a}\left(\frac{\partial U}{\partial \zeta}\right)_{\zeta=0} = \frac{\mu u_0}{a} U_0 \doteq \frac{\mu u_0}{a} U_{00}. \tag{49}$$

By means of (21a), we have

$$S_0 = \frac{\mu u_0}{a}\left(\frac{1}{t} - \frac{R}{4}\frac{\partial P_{00}}{\partial \xi} t\right). \tag{50}$$

The total frictional force on the slider is therefore in dimensionless form as follows:

$$F = \frac{\text{total frictional force}}{\frac{1}{2}\varrho u_0^2 a^2} = \frac{1}{\frac{1}{2}\varrho u_0^2}\iint_A S_0\, d\eta\, d\xi$$

$$= \frac{2}{R}\iint_A \frac{1}{t}\, d\xi\, d\eta - \frac{1}{2}\iint_A \frac{\partial P_{00}}{\partial \xi}\, t\, d\xi\, d\eta. \qquad (51)$$

The second integral can be integrated by parts, giving

$$F = \frac{2}{R}\iint_A \frac{1}{t}\, d\xi\, d\eta + \frac{1}{2}\iint_A \frac{\partial t}{\partial \xi} P_{00}\, d\xi\, d\eta = \frac{2}{R}\iint_A \frac{1}{t}\, d\xi\, d\eta - \frac{1}{2} H_0^*. \qquad (52)$$

The total normal pressure Q on the slider is given by, in dimensional form,

$$Q = \iint_A P_{00}\, d\xi\, d\eta. \qquad (53)$$

In the case of plane sliders of infinite width, we have from (48), the total frictional force and total normal pressure on the slider per unit width in η,

$$F = \frac{4}{t_0 R (2+\lambda)\lambda}\,[2(\lambda+2)\log(1+\lambda) - 3\lambda], \qquad (54)$$

$$Q = \frac{12}{t_0^2 R \lambda^2 (2+\lambda)}\,[(2+\lambda)\log(1+\lambda) - 2\lambda]. \qquad (55)$$

The coefficient of friction is therefore equal to

$$f = \frac{F}{Q} = \frac{\lambda t_0}{3}\left[\frac{2(2+\lambda)\log(1+\lambda) - 3\lambda}{(2+\lambda)\log(1+\lambda) - 2\lambda}\right]. \qquad (56)$$

APPLICATION TO PLANE SLIDERS OF FINITE WIDTH

We shall now proceed to apply the above method of approximation to the case of sliders of finite width, or with side leakage. Let us take as a first approximation,

$$P_{00}^{(1)} = a_1 \frac{\xi(1-\xi)}{t^2} \eta (k - \eta), \tag{57}$$

where a_1 is an unknown constant to be determined by the minimizing condition of H_0, and

$$t = t_0(1 + \lambda \xi), \qquad k = b/a. \tag{58}$$

It is evident that P_{00} in (57) satisfies the boundary conditions (28a). Substituting (57) into the integral (37), we find

$$H_0^{(1)} = a_1^2 \frac{R k^3}{36 t_0 \lambda^5} \left\{ \frac{k^2}{10} \lambda^2 (2+\lambda) [(2+\lambda) \log(1+\lambda) - 2\lambda] \right.$$

$$\left. + [(1+\lambda)^2 \log(1+\lambda) + \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6)] \right\}$$

$$- a_1 \frac{k^3}{3 t_0 \lambda^2} [(\lambda + 2) \log(1+\lambda) - 2\lambda]. \tag{59}$$

From the minimizing condition $dH_0^{(1)}/da_1 = 0$, it follows that

$$a_1 = \frac{6\lambda^3}{R} \frac{(2+\lambda)\log(1+\lambda) - 2\lambda}{(1+\lambda)^2 \log(1+\lambda) + \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6) + \frac{k^2}{10}\lambda^2(2+\lambda)[(2+\lambda)\log(1+\lambda) - 2\lambda]}. \tag{60}$$

Hence the minimum value of H_0 is

$$H_0^{(1)} = -\frac{k^3 \lambda}{R t_0} \frac{[(2+\lambda)\log(1+\lambda) - 2\lambda]^2}{(1+\lambda)^2 \log(1+\lambda) + \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6) + \frac{k^2}{10}\lambda^2(2+\lambda)[(2+\lambda)\log(1+\lambda) - 2\lambda]}. \tag{61}$$

In this case, the total normal load Q and the total frictional force on the slider can be found respectively from (53) and (52). They are

$$Q = \frac{k^3}{R t_0^2} \frac{1}{\Gamma} [(\lambda + 2) \log(1+\lambda) - 2\lambda]^2, \tag{62}$$

$$F = \frac{k}{2\lambda R t_0} \frac{1}{\Gamma} \left\{ \lambda^2 k^2 [(2+\lambda)\log(1+\lambda) - 2\lambda][\frac{7}{5}(2+\lambda)\log(1+\lambda) - 2\lambda] \right.$$

$$\left. + 4\log(1+\lambda)[(1+\lambda)^2\log(1+\lambda) + \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6)] \right\}, \quad (63)$$

where Γ represents the expression involving λ and k as follows

$$\Gamma = \frac{k^2}{10}\lambda^2(2+\lambda)[(\lambda+2)\log(1+\lambda) - 2\lambda] + (1+\lambda)^2\log(1+\lambda)$$

$$+ \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6). \quad (64)$$

Hence the coefficient of friction is

$$f = \frac{F}{Q} = \frac{t_0}{2\lambda k^2} \left\{ \lambda^2 k^2 \frac{[\frac{7}{5}(2+\lambda)\log(1+\lambda) - 2\lambda]}{[(2+\lambda)\log(1+\lambda) - 2\lambda]} \right.$$

$$\left. + 4\log(1+\lambda)\frac{[(1+\lambda)^2\log(1+\lambda) + \frac{1}{12}(2+\lambda)\lambda(\lambda^2 - 6\lambda - 6)]}{[(2+\lambda)\log(1+\lambda) - 2\lambda]^2} \right\}. \quad (65)$$

In this case, the exact value of the coefficient of friction has been calculated through laborious numerical computation by Muskat, Morgan, Meres[6]. The results was given in a series of graphical charts, all in dimensionless form. The approximate values of $f/(\lambda t_0)$ given by (65) and the corresponding exact values taken from M-M-M's graphs are compared in the following table:

λ	k	$f/(\lambda t_0)$ by (65)	$f/(\lambda t_0)$ by M-M-M's graphs
1	∞	5.7	5
1	2	6.9	6.5
1	1	10.7	11
1	1/2	25.5	24.5
1	1/3	50.2	46
2/3	∞	6.7	8

2/3	2	12.2	12
2/3	1	18.4	19
2/3	1/2	44.0	44
1/2	∞	15.2	12.5
1/2	2	19.2	18.5
1/2	1	30.9	29.5

The agreement between the approximate values and exact values of $f/(\lambda t_0)$ is evident.

中 文 提 要

長方滑板間之滑潤理論

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本文基於 Navier-Stokes 氏微分方程式，研究粘滯性液體層的滑潤作用；此種流體動力學的理論，乃依據液體層之薄度進行續步漸近之解法而得。本文除證明初步近似解答與 Reynolds 氏方程式之結果相合外，更進一步將解答 Reynolds 方程式的問題，代以一實際相等的變值問題。其結果非但減少計算工作，且亦正確可靠。