RELATIVISTIC HAMILTONIAN SYSTEM OF A PARTICLE AND RELATIVISTIC HEISENBERG'S EQUATION

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(*Received November 16, 1946*)

ABSTRACT

In this paper, as suggested by the classical canonical equations, a new set of the corresponding relativistic equations is set up. Therefrom a relativistic form of Heisenberg's equation is deduced. The relativistic Hamiltonian system of a particle according to the formulation here established has been fully discussed and by following it, however, Dirac's equation appears naturally as a necessary form of relativistic wave equation for electron in quantum mechanics. The process of taking square root in Dirac's theory is seen to have its classical analogy. Finally, some applications of the relativistic Heisenberg's equation to Dirac's theory have been discussed and it has thereby been pointed out that this equation brings some quantities to being more symmetrical in the relativity sense and also some more general than those the non-relativistic equation can introduce.

1. THE RELATIVISTIC HAMILTONIAN SYSTEM OF A PARTICLE

It is well acquainted with, in the special theory of relativity, that the motion of a particle can be described by the canonical equations,

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial x_i} \qquad (i=1,\ 2,\ 3) \tag{1}$$

with the Hamiltonian function

$$H=c\sqrt{m_0^2c^2+p_1^2+p_2^2+p_3^2}-m_0c^2+W(x_1,\ x_2,\ x_3), \tag{2}$$

if the particle moves in a field of force with potential energy function $W(x_1, x_2, x_3)$.

In equations (1) the quantity H is generally understood as the energy of the conservative system (the particle in a conservative field) and thus constitutes only the fourth component of a 4-vector, namely, the energy-momentum vector according to the special theory of relativity, so also does the time t (in the form ict) occurring in these equations. Therefore the equations (1) are not themselves relativistically invariant. They are invariant only in the classical mechanics. Moreover, with eqs. (1) and (2), the relation corresponding to the law of conservation of energy cannot be obtained. With these relativistical asymmetries we are led to a more symmetrical relativistical form of the canonical equations analogous to eqs. (1),

$$\frac{dx_\mu}{d\tau}=\frac{\partial \mathcal{H}}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau}=-\frac{\partial \mathcal{H}}{\partial x_\mu} \qquad (\mu=1, 2, 3, 4), \tag{3}$$

where $d\tau$ (τ being called the proper time) is defined by

$$c^2 d\tau^2=-\sum_1^4 dx_\mu^2, \qquad x_\mu \equiv (x, y, z, ict),$$

and $\mathcal{H}$ is expected to be a relativistic scalar.

In the case of a particle, however, as will soon be shown, we have

$$\mathcal{H}=-c\sqrt{-\sum_1^4 p_\mu^2}, \tag{4}$$

if the particle is free, where p_μ is, according to the special theory of relativity, the energy-momentum-vector $(p_x, p_y, p_z, i\frac{W}{c})$, and

$$\mathcal{H}=-c\sqrt{-\sum_1^4 (p_\mu-\frac{e}{c}A_\mu)^2} \tag{5}$$

if it is charged (of charge e) and moves in an electromagnetic field defined by the 4-potential $A_\mu \equiv (A_x, A_y, A_z, i\varphi)$.

(*a*) *General constancy of* $\mathcal{H}$:

$\mathcal{H}$ is in general a function of x_μ and p_μ, thus by eqs. (3),

$$\frac{d\mathcal{H}}{d\tau}=\sum_1^4\left(\frac{\partial \mathcal{H}}{\partial x_\mu}\frac{dx_\mu}{d\tau}+\frac{\partial \mathcal{H}}{\partial p_\mu}\frac{dp_\mu}{d\tau}\right)=\sum_1^4\left(\frac{\partial \mathcal{H}}{\partial x_\mu}\frac{\partial \mathcal{H}}{\partial p_\mu}-\frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial \mathcal{H}}{\partial x_\mu}\right)=0,$$

or

$$\frac{d\mathcal{H}}{dt}=0.$$

Hence $\mathcal{H}$ is a constant of motion with respect to every system, and since $\mathcal{H}$ is a scalar, we have

$$\mathcal{H} = \text{constant},$$

this constant being independent of coordinate systems.

Therefore, if a dynamical system can be associated with such a function $\mathcal{H}$ and its motion described by eqs. (3), the function $\mathcal{H}$ must be a constant independent of coordinate systems.

In the case of a particle, we may put

$$\mathcal{H}= -m_0c^2,$$

completely irrespective of the condition of the particle, where m_0, as will later be shown, may be understood as the rest mass of the particle, and c is the light speed in vacuum.

(*b*) *The equations of motion of a particle* (proof of correctness of the above assumed Hamiltonian system):

For a charged particle moving in an electromagnetic field,

$$\mathcal{H}= -c\sqrt{-\sum_1^4 (p_\mu - \frac{e}{c}A_\mu)^2} = -m_0c^2, \tag{6}$$

as has already been discussed in (a); and from eqs. (3),

$$\frac{dx_\mu}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_\mu}=\frac{1}{m_0}(p_\mu - \frac{e}{c}A_\mu),$$

or

$$p_\mu = m_0\frac{dx_\mu}{d\tau}+\frac{e}{c}A_\mu \qquad (\mu=1,\,2,\,3,\,4), \tag{7}$$

and

$$\frac{dp_\mu}{d\tau}=-\frac{\partial\mathcal{H}}{\partial x_\mu}=\frac{e}{c}\frac{\partial A_\nu}{\partial x_\mu}\frac{dx_\nu}{d\tau}. \tag{8}$$

Combining (7) and (8), we have

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx_\mu}{dt}\right)=\frac{e}{c}\left(\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}\right)\frac{dx_\nu}{dt}\quad(\mu=1,2,3,4).\tag{9}$$

Putting
$$F_{\mu\nu}=\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu},$$

we have then

$$(F_{23},F_{31},F_{12})\equiv(H_x,H_y,H_z)$$
$$(F_{14},F_{24},F_{34})\equiv i(E_x\ E_y\ E_z),$$

Thus eqs. (9) become

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx_i}{dt}\right)=e(\vec{E}+\frac{1}{c}\vec{v}\times\vec{H})_i\quad(i=1,2,3)\tag{10}$$

and

$$\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-\beta^2}}\right)=e(\vec{E}\cdot\vec{v})\tag{11}$$

for $\mu=4$.

Here eqs. (10) represent the relativistic equations of motion of a particle in an electromagnetic field, and eq. (11) means, however, the law of conservation of energy.

m_0 is now obviously the rest mass of the particle.

(c) *The Einstein's energy-momentum relation:*

The equation (6) gives, when squared out,

$$m_0^2c^2=-\sum_1^4(p_\mu-\frac{e}{c}A_\mu)^2,$$

or, explicitly,

$$\frac{(W-e\phi)^2}{c^2}=(\vec{p}-\frac{e}{c}\vec{A})^2+m_0^2c^2.\tag{12}$$

This is Einstein's relativistic energy-momentum relation for a particle.

2. DIRAC'S EQUATION AS NATURALLY A NECESSARY FORM OF RELATIVISTIC WAVE EQUATION FOR ELECTRON

For the motion of an electron in an electromagnetic field, as classically described by eqs. (3) and (5), the wave equation is naturally of the form

$$\sqrt{-\sum_1^4 \left(p_\mu - \frac{e}{c} A_\mu\right)^2}\,\psi = m_0 c \psi. \tag{13}$$

It may here be quite reasonable to carry out the square root by putting

$$\sqrt{-\sum_1^4 \left(p_\mu - \frac{e}{c} A_\mu\right)^2} = i \sum_1^4 \beta_\mu \left(p_\mu - \frac{e}{c} A_\mu\right),$$

where

$$\beta_\mu \beta_\nu + \beta_\nu \beta_\mu = 2\,\delta_{\mu\nu} \qquad (\mu, \nu = 1, 2, 3, 4).$$

Thus, eq. (13) reduces to

$$\left\{\sum_1^4 \beta_\mu \left(p_\mu - \frac{e}{c} A_\mu\right) + i\, m_0 c\right\} \psi = 0, \tag{14}$$

and for a free particle,

$$\left(\sum_1^4 \beta_\mu\, p_\mu + i\, m_0 c\right) \psi = 0. \tag{15}$$

With the ordinary quantum-mechanical substitutions,

$$p_\mu = \frac{\hbar}{i} \frac{\partial}{\partial x_\mu} \qquad (\mu = 1, 2, 3, 4),$$

eqs. (14) and (15) are the Dirac's equations for the electron.

In its explicit form, eq. (15) reads

$$\sum_1^4 \beta_\mu \frac{\partial \psi}{\partial x_\mu} + \frac{m_0 c}{\hbar} \psi = 0,$$

with its conjugate equation[1],

$$\sum_1^4 \frac{\partial\phi}{\partial x_\mu}\beta_\mu - \frac{m_0 c}{\hbar}\phi = 0,$$

where

$$\phi = -\psi^+\beta_4.$$

The operator $\mathcal{K} = -ic\sum_1^4 \beta_\mu \left(p_\mu - \frac{e}{c}A_\mu\right)$ is obviously the rest-mass operator, which has arbitrary eigenvalue in flat space-time according to the special theory of relativity. The problem of its quantization with reference to various curved space-time has been discussed by Podolsky and Branson[2].

3. THE RELATIVISTIC HEISENBERG'S EQUATION

It is generally true, that, in non-commutative matrix algebra, if f is a function of x_l and p_l,

$$\frac{i}{\hbar}(x_l f - f x_l) = -\frac{\partial f}{\partial p_l},$$

$$\frac{i}{\hbar}(p_l f - f p_l) = \frac{\partial f}{\partial x_l}. \tag{16}$$

For classical Hamiltonian mechanics, eqs. (1) are valid, (with $l=1, 2, 3$ in the case of a particle); thus, by comparing them with eqs. (16) we have

$$\frac{dx_l}{dt} = \frac{i}{\hbar}(Hx_l - x_l H) = [H, x_l],$$

$$\frac{dp_l}{dt} = \frac{i}{\hbar}(Hp_l - p_l H) = [H, p_l].$$

It may easily be proved, that, if P is any dynamical variable and is a function of x_l and p_l, but not of time t explicitly,

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1. W. Pauli, ***Hand. d. Physik***, Band XXIV/I, 2te. Aufl. p. 220.
2. B. Podolsky and H. Branson, *Phys. Rev.*, **57**, (1940), 494.

$$\frac{dP}{dt}=[H, P].$$

This is the ordinary non-relativistic Heisenberg's equation in quantum mechanics.

In the case of a relativistic dynamical system, however, eqs. (3) are in general valid, thus, by comparing them with eqs. (16), we have

$$\frac{dx_\mu}{d\tau}=\frac{i}{\hbar}(\mathscr{H}\, x_\mu-x_\mu\,\mathscr{H})=[\mathscr{H},\, x_\mu],$$

$$\frac{dp_\mu}{d\tau}=\frac{i}{\hbar}(\mathscr{H}\, p_\mu-p_\mu\,\mathscr{H})=[\mathscr{H},\, p_\mu],$$

($\mu=1, 2, 3, 4$ in the case of a particle), and, similarly, if Q is a dynamical variable of the system considered and is a function of x_μ and p_μ,

$$\frac{dQ}{d\tau}=[\mathscr{H},\, Q], \tag{17}$$

which is the relativistic form of Heisenberg's equation.

It is obvious, by this equation, that the relativistic scalar $\mathscr{H}$ is a constant of motion of the dynamical system considered also in the sense of quantum mechanics.

4. SOME APPLICATIONS OF THE RELATIVISTIC HEISENBERG'S EQUATION TO DIRAC'S THEORY OF ELECTRON

Up to the present, Dirac's equation for electron serves as the only relativistic wave equation in quantum mechanics. It would thus be expected that some applications of the relativistic Heisenberg's equation may only be found in Dirac's theory. It will be seen in the following discussions that this equation will bring some quantities to being more symmetrical in the sense of relativity and also some more general than those the non-relativistic equation can give.

(a) *Velocity-vector operator*:

For an electron moving in an electromagnetic field

$$\mathscr{H}=-c\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}A_\mu\right)^2},$$

and in Dirac's theory,

$$\mathcal{H} = -ic\sum_{1}^{4}\beta_\mu\left(p_\mu - \frac{e}{c}A_\mu\right). \tag{18}$$

Thus, by eq. (17),

$$\frac{dx_\nu}{d\tau} = [\mathcal{H}, x_\nu] = -i\,c\,\beta_\nu \qquad (\nu = 1,\ 2,\ 3,\ 4). \tag{19}$$

The mean of it is

$$\phi\,\frac{dx_\nu}{d\tau}\,\psi = -i\,c\,\phi\,\beta_\nu\,\psi, \tag{20}$$

which is identical with that obtained ordinarily.

By considering the relativistic transformation properties of Dirac's equation, both (19) and (20) are 4-vectors, since β_ν and $\phi\psi$ transform respectively as a 4-vector and a scalar[3].

Eq. (19) may also be written,

$$\frac{1}{\sqrt{1-\beta^2}}\,\frac{dx_\nu}{dt} = -i\,c\,\beta_\nu \qquad (\nu = 1,\ 2,\ 3,\ 4),$$

the fourth component of which gives,

$$\frac{i\,c}{\sqrt{1-\beta^2}} = -ic\beta_4$$

or,

$$\beta_4 = -\frac{1}{\sqrt{1-\beta^2}}.$$

Thus,
$$\frac{dx_k}{dt} = i\,c\,\beta_4\,\beta_k = -c\,\alpha_k \qquad (k = 1,\ 2,\ 3),$$

where $\alpha_k = i\beta_k\beta_4$. This is just the velocity operator obtained directly from the non-relativistic Heisenberg's equation.

3. W. Pauli, *Hand. d. Physik*, Band XXIV/I, 2te. Aufl. p. 221.

The 4-velocity in Dirac's theory has already been discussed and is given by eq. (20), i.e.,

$$U^1_{av}=\left(\frac{dx_1}{d\tau}\right)_{av}=-ic\psi^\dagger\beta_1\psi=-c\psi^\dagger\alpha_1\psi,\ \text{etc.},$$

$$U^4_{av}=ic\left(\frac{dt}{d\tau}\right)_{av}=-ic\psi^\dagger\beta_4\psi=ic\psi^\dagger\psi,$$

we obtain therefrom the three-dimensional velocity components,

$$V_x=\frac{dx}{dt}=\left(\frac{dx}{d\tau}\right)_{av}\bigg/\left(\frac{dt}{d\tau}\right)_{av}=-c\frac{\psi^\dagger\alpha_1\psi}{\psi^\dagger\psi},\ \text{etc.}$$

These relations are identical with those obtained ordinarily[4].

(*b*) *The motion of a free electron. Oscillatory motion:*

The velocity operator is already obtained and is given by,

$$\frac{dx_\nu}{d\tau}=ic\beta_\nu \qquad (\nu=1,\ 2,\ 3,\ 4),$$

thus, the acceleration operator follows as

$$\frac{d^2x_\nu}{d^2\tau}=-ic\,\frac{d\beta_\nu}{d\tau}.$$

Now,

$$i\hbar\,\frac{d\beta_\nu}{d\tau}=\beta_\nu\mathcal{H}-\mathcal{H}\beta_\nu,$$

and

$$\beta_\nu\mathcal{H}+\mathcal{H}\beta_\nu=-2icp_\nu,$$

then,

$$i\hbar\,\frac{d\beta_\nu}{d\tau}=2\beta_\nu\mathcal{H}+2icp_\nu \qquad (\nu=1,\ 2,\ 3,\ 4): \tag{21}$$

4. E. L. Hill & R. Landshoff, *Rev. Mod. Phys.*, **10**, (1938) 115.

By a similar calculation, we have

$$i\hbar\frac{d^2\beta_\nu}{d\tau^2}=2\frac{d\beta_\nu}{d\tau}\mathcal{H},$$

hence,

$$\frac{d\beta_\nu}{d\tau}=\left(\frac{d\beta_\nu}{d\tau}\right)_0\exp\left(-\frac{4\pi i}{h}\mathcal{H}\tau\right). \tag{22}$$

Substituting (22) into (21), we have

$$\beta_\nu=-icp_\nu\mathcal{H}^{-1}+i\frac{\hbar}{2}\left(\frac{d\beta_\nu}{d\tau}\right)_0\exp\left(-\frac{2i}{\hbar}\mathcal{H}\tau\right)\mathcal{H}^{-1},$$

hence,

$$\frac{dx_\nu}{d\tau}=-ic\beta_\nu=-c^2p_\nu\mathcal{H}^{-1}+\frac{\hbar c}{2}\left(\frac{d\beta_\nu}{d\tau}\right)_0\exp\left(-\frac{2i}{\hbar}\mathcal{H}\tau\right)\mathcal{H}^{-1}. \tag{23}$$

The first term on the right of eq. (23) means the ordinarily observed velocity, since,

$$-c^2p_\nu\mathcal{H}^{-1}=-c^2p_\nu\frac{1}{-m_0c^2}=\frac{p_\nu}{m_0},$$

while the second term is called the oscillatory motion owing to the occurance of the exponential factor. The frequency of oscillation is given by,

$$\nu_z=\left|\frac{2\mathcal{H}}{h}\sqrt{1-\beta^2}\right|=\frac{2m_0c^2}{h}\sqrt{1-\beta^2},$$

hence,

$$2m_0c^2=\frac{h\nu_z}{\sqrt{1-\beta^2}}, \tag{24}$$

the energy difference between positive and negative energy levels ($2m_0c^2$) may therefore be measured by the frequency of the oscillatory motion according to the relation (24).

The results obtained with the help of the non-relativistic Heisenberg's equation have some deviations from those discussed here[5]. But no decision can here be made, since the motion is non-observable.

(*c*) *Law of conservation of total angular momentum. The spin tensor.*

For electrons moving in a 4-dimensional central field defined by

$$A_1=A_2=A_3=O,\quad A_4=A_4(s),\quad (ds^2=\sum_1^4 dx^2_\mu),$$

the relativistic Hamiltonian function is

$$\mathcal{H}=-ic\sum_1^4\beta_\mu p_\mu-ie\beta_4 A_4.$$

Now defining

$$M_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu\qquad(\mu,\nu=1,2,3,4),\tag{25}$$

we have, since, by a very simple calculation, $A_4(s)$ commutes with both x_μ and p_μ,

$$\frac{dM_{\mu\nu}}{d\tau}=[\mathcal{H},M_{\mu\nu}]=-ic\,(\beta_\mu p_\nu-\beta_\nu p_\mu).\tag{26}$$

The three spatial components of the tensor (i.e. M_{23}, M_{31}, M_{12}) are generally understood as the components of orbital angular momentum of the electron.

By eq. (26) the orbital angular momentum is not a constant of motion, we can thus expect that the total angular momentum should be composed of, apart from the orbital one, also another component.

Let us come to the tensor $\beta_\mu\beta_\nu$. The time variation of it is given by,

$$\frac{d}{d\tau}(\beta_\mu\beta_\nu)=[\mathcal{H},\beta_\mu\beta_\nu]=-\frac{2c}{\hbar}(\beta_\mu p_\nu-\beta_\nu p_\mu),\tag{27}$$

hence, by combining eqs. (26) and (27),

$$\frac{d}{d\tau}(M_{\mu\nu}-\frac{\hbar}{2}i\beta_\mu\beta_\nu)=O,$$

5. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 2nd. ed., p. 260.

or,

$$\frac{d}{dt}(M_{\mu\nu} - \frac{\hbar}{2} i\beta_\mu\beta_\nu) = 0. \qquad (\mu, \nu = 1, 2, 3, 4).$$

Thus the tensor $M_{\mu\nu} - \frac{\hbar}{2} i\beta_\mu\beta_\nu$ is a constant of motion, and is called the total angular momentum of the electron. The term $-\frac{\hbar}{2} i\beta_\mu\beta_\nu$ is generally called the spin angular momentum of it.

By means of the ordinary non-relativistic Heisenberg's equation, only the three spatial components of orbital and spin angular momentum are exhibited. This is obviously not symmetrical in the sense of relativity[6].

This law of conservation of total angular momentum is true, of course, also in the case of a free electron.

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6. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 2nd. ed., p. 264.