THE LAMINAR MIXING MOTION OF TWO INCOMPRESSIBLE GASES

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ABSTRACT

Equations determining the velocity and density distributions within the mixing region of two incompressible gases with different densities are set up, their temperatures being assumed to be the same. For incompressible mixing the total number of gas molecules per unit volume is constant, although the density of the gaseous mixture varies from point to point due to diffusion of matter. As an illustration we consider the plane jet and steady motion. The boundary layer method of approximation can still be applied. The boundary of the jet is shown to be the same as that for one fluid. The solution of the problem then depends upon the numerical value of the coefficient of viscosity of the mixture which is a function of the number of molecules of each constituent gas in the unit volume. The present method of investigation is applicable to the cylindrical and half jets and also to the case where the two gases are at different temperatures.

1. EQUATIONS OF MOTION

Let m_1 and m_2 be the molecular masses of the two gases whose mixing motion is under investigation. Denote their numbers of molecules per unit volume by n_1 and n_2 respectively. If the motion of the two gases were incompressible, the total number n of the gas molecules in a unit volume which is defined by

$$n = n_1 + n_2 \tag{1.1}$$

is a constant.

The density ϱ of the gaseous mixture is the sum

$$\varrho = n_1 m_1 + n_2 m_2. \tag{1.2}$$

For their mean mass velocity v_i we have the definition

$$\varrho v_i = n_1 m_1 v_{1i} + n_2 m_2 v_{2i}, \quad (i = 1, 2, 3) \tag{1.3}$$

where v_{1i} and v_{2i} are the velocities of the two component gases. We use cartesian tensors and the summation convention for the Latin letters throughout the paper.

The equations of motion determining v_i are given by

$$\varrho\frac{\partial v_i}{\partial t}+\varrho v_j\frac{\partial v_i}{\partial x_j}=\frac{\partial}{\partial x_j}\sigma_{ij} \tag{1.4}$$

in which the stress tensor σ_{ij} for laminar motion is known to be

$$\sigma_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\,. \tag{1.5}$$

Substituting expression (1.5) into Eq. (1.4), we find

$$\varrho\frac{\partial v_i}{\partial t}+\varrho v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right]-\frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu\frac{\partial v_k}{\partial x_k}\right). \tag{1.6}$$

For the present discussion we disregard the body force acting upon the fluid and assume the temperature of the two gases to be the same.

2. EQUATIONS OF DIFFUSION OF MATTER

The equation of continuity of the two constituent gases can be summarized in the form,

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha)+\frac{\partial}{\partial x_j}(n_\alpha m_\alpha v_{\alpha j})=0 \tag{2.1}$$

$$(\alpha=1,\ 2;\ \alpha \text{ not summed})$$

We introduce the velocity difference $u_{\alpha i}$ between the velocity of one component gas and the mean mass velocity v_i by the relation

$$u_{\alpha i}=v_{\alpha i}-v_i\,. \tag{2.2}$$

Obviously on account of the definition of v_i according to Eq. (1.3), we have

$$n_1m_1u_{1i}+n_2m_2u_{2i}=0\,. \tag{2.3}$$

Then in terms of v_i and u_i, the equation of continuity (2.1) becomes

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha)+\frac{\partial}{\partial x_j}(n_\alpha m_\alpha v_j)+\frac{\partial}{\partial x_j}(n_\alpha m_\alpha u_{\alpha j})=0. \tag{2.4}$$

The sum of the above two equations yields the equations of continuity for the mixture

$$\frac{\partial \varrho}{\partial t}+\frac{\partial}{\partial x_j}(\varrho v_j)=0. \tag{2.5}$$

The velocity of diffusion, $u_{1i}-u_{2i}$, of the two gases has been shown to be[1]

$$u_{1i}-u_{2i}=-\frac{n^2}{n_1 n_2}\{D_{12}d_{1i}+D_T\frac{\partial}{\partial x_i}\log T\}, \tag{2.6}$$

where D_{12} is the coefficient of diffusion, D_T the coefficient of thermal diffusion and T is the temperature. The vector d_{1i} stands for the expression,

$$d_{1i}=\frac{\partial}{\partial x_i}\frac{n_1}{n}+\frac{n_1 n_2}{n\varrho p}(m_2-m_1)\frac{\partial p}{\partial x_i}-\frac{n_1 n_2}{\varrho p}m_1 m_2(F_{1i}-F_{2i}). \tag{2.7}$$

Here p is the pressure of the gaseous mixture as before and F_{1i} and F_{2i} are the body forces acting upon the two gases. In the application to follow we shall ignore both the second and third terms on the right hand side of the above equaoion. Due to the relations (2.3) and (2.6) and neglecting thermal diffusion, we tbtain

$$n_1 m_1 u_{1i}=-n_2 m_2 u_{2i}=-\frac{n^2}{\varrho}m_1 m_2 D_{12}\, d_{1i}. \tag{2.8}$$

The equations of continuity (2.4) then become

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha)+\frac{\partial}{\partial x_j}(n_\alpha m_\alpha v_j)=\frac{\partial}{\partial x_j}\left\{\frac{n^2}{\varrho}m_1 m_2\, D_{12}\frac{\partial}{\partial x_j}\frac{n_\alpha}{n}\right\}. \tag{2.9}$$

The coefficient of viscosity μ for a gaseous mixture is also known[2]. Then we have five scalar equations (1.6) and (2.9) for the six functions v_i, n_α and p. The sixth equation is the adiabatic equation of state for gases. The condition of incompressibility is only consistent with this equation, if the velocity of the gaseous mixture were small.

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1. S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge (1939), 143.

2. Chapman and Cowling, loc. cit., 167, 230; J.H. Jeans, *An introduction to the kinetic theory of gases*, Cambridge (1940), 183.

3. STEADY TWO-DIMENSIONAL MOTION: THE PLANE JET

Choose the positive x-axis to be the axis of the jet and the z-axis coinciding with the slit of the jet out of which the gas with density ϱ_1 flows into the gas of density ϱ_2 at rest; the y-axis is taken perpendicular to the jet. Let u and v represent the x and y components of the fluid velocity respectively. From symmetry consideration the z-component of the velocity is zero. Let the motion be steady. In order to satisfy the equation of continuity for the gaseous mixture (2.5) which now becomes

$$\frac{\partial}{\partial x}(\varrho u)+\frac{\partial}{\partial y}(\varrho v)=0, \tag{3.1}$$

we introduce a stream function ψ and put u and v equal to the expressions,

$$\varrho u=\frac{\partial \psi}{\partial y}, \quad \varrho v=-\frac{\partial \psi}{\partial x}. \tag{3.2}$$

Next apply the condition of dynamical similarity to the flow in the mixing region downstream by assuming that ψ is a function of the type,

$$\psi=x^r F(\eta); \quad \eta=y/x^s, \tag{3.3}$$

where r and s are two constants to be determined and that all the other scalar functions p, ϱ, n_1 and n_2 are functions of η only. Since we are investigating the region at great distances x from the orifice of the jet, we only have to use the x-component of the equations of motion (1.6) and one equation of continuity (2.9) in the following form respectively:

$$\varrho u\frac{\partial u}{\partial x}+\varrho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{3.4}$$

$$\frac{\partial}{\partial x}(n_1 m_1 u)+\frac{\partial}{\partial y}(n_1 m_1 v)=\frac{\partial}{\partial y}\left(\frac{n}{\varrho}m_1 m_2 D_{12}\frac{\partial n_1}{\partial y}\right). \tag{3.5}$$

In Eq. (3.5) we have adopted the condition of incompressibility n=constant.

From Eq. (3.4) we obtain, by integration, the rate M at which the momentum of fluid flows across a section of the jet per unit height must be the same for all sections downstream, namely,

$$M=2\int_0^\infty u^2 dy=\text{constant}. \tag{3.6}$$

Substituting ψ from Eq. (3.3) into (3.2), we find

$$u=x^{r-s}F'/\varrho, \quad v=x^{r-1}(-rF+s\eta F')/\varrho. \tag{3.7}$$

Condition (3.6) then yields the relation,

$$2r-s=0. \tag{3.8}$$

If u and v from Eq. (3.7) are put into the equation of motion (3.4), we obtain another relation betwee r and s,

$$r+s=1. \tag{3.9}$$

The two equations (3.8) and (3.9) determine r and s to be

$$r=\frac{1}{3}, \qquad s=\frac{2}{3}. \tag{3.10}$$

This result remains the same as in the ordinary theory of spread of jet of one fluid. The equation of motion itself becomes

$$-\frac{1}{3}\frac{d}{d\eta}\frac{FF'}{\varrho}=\frac{d}{d\eta}\left[\mu\frac{d}{d\eta}\frac{F'}{\varrho}\right]. \tag{3.11}$$

Eq. (3.9) for the exponents r and s can also be obtained from the equation of continuity (3.5) which after cancelling out the common factor then becomes

$$-\frac{1}{3}F\frac{d}{d\eta}\frac{n_1}{\varrho}=n\,m_2\frac{d}{d\eta}\frac{D_{12}n'_1}{\varrho}. \tag{3.12}$$

The boundary conditions for the equations (3.11) and (3.12) are given below:

$$F=\frac{d}{d\eta}(F'/\varrho)=0,\ n_1=\varrho_1/m_1=n,\ n_2=0, \text{ when } \eta=0;$$

$$F'=0,\quad n_1=0,\ n_2=\varrho_2/m_2=n, \quad \text{when } \eta=\infty. \tag{3.13}$$

Under the above conditions Eq. (3.11) can be integrated once, yielding the result,

$$-\frac{1}{3}\frac{FF'}{\varrho}=\mu\frac{d}{d\eta}\frac{F'}{\varrho}. \tag{3.14}$$

The above method and the condition of dynamical similarity can be applied to cylindrical and half jets. If one of the gases is at higher temperature, we have to take the equation of heat transfer[3] into consideration, For steady motion the temperature will be a function of the variable η alone. Here further complication arises in the calculation due to the fact that both the coefficients of viscosity of the component gases will depend upon the temperature.

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3. S. Goldstein, *Modern developments in fluid dynamics*, Oxford (1938), II, 606, Eq. 12.