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容變粘滯性與聲之速變及吸收

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作者前[1]曾就一個弛緩過程之假設建立流體之容變粘滯性之唯象理論。近來一小部分試驗結果指出同一種流體，例如 CO_2 及 CH_3CHO 等，得有數個弛緩過程，即數個弛緩時間。本文即將原有之單弛緩過程所致容變粘滯性之理論推廣至多個弛緩過程所致者，並由所導出之方程式推究流體中聲音之速變及吸收諸現象。

設 n_i/n 爲流體中 i 種弛緩過程之部分，即 $\sum_i n_i = n$，則

$$s = \sum_i \frac{n_i}{n} s_i, \quad 及 \quad \frac{ds}{dt} = \sum_i \frac{n_i}{n} \frac{ds_i}{dt}.$$

及由之得屬於 i 種弛緩部分之方程式爲

$$\frac{ds_i}{dt} = \beta_\infty \frac{dp}{dt} + \frac{s_o - s_i}{\beta_o \eta_{2i}}, \tag{1}$$

式中 η_{2i} 乃 i 種過程所致之容變粘滯性係數，其定義爲

$$\bar{p} - p_i' = \eta_{2i}\left(\frac{ds_i}{dt}\right)_{\text{vis}}.$$

由方程式 (1) 知 i 種過程之弛緩時間爲

$$\tau_{2i} = \beta_o \eta_{2i}.$$

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1. 盧鶴紱，中國物理學報 **7** (1950)，362。

一氣體或似氣體之液體中之聲速與頻率之關係經由式 (1) 推得爲

$$v_i = \left[\frac{1}{\rho}\frac{\beta_o + \beta_\infty\,\omega^2\,\tau_{2i}^2}{\beta_0^2 + \beta_\infty^2\,\omega^2\,\tau_{2i}^2}\right]^{\frac{1}{2}},$$

卽不同過程所成之聲波其傳播之速度略異，蓋

$$\text{當 } \omega\,\tau_{2i} \longrightarrow 0, \text{ 則} \qquad v \longrightarrow v_o = 1/(\rho\,\beta_o)^{\frac{1}{2}};$$

$$\text{當 } \omega\,\tau_{2i} \longrightarrow \infty, \text{ 則} \qquad v \longrightarrow v_\infty = 1/(\rho\,\beta_\infty)^{\frac{1}{2}};$$

卽在兩極限頻率時皆合而爲一也。 每波長之聲强度吸收係數 $\gamma = 2\lambda\alpha$ 爲

$$\gamma = \sum_i \frac{n_i}{n}\gamma_i,$$

式中

$$\gamma_i = 2\pi\,\omega\,\beta_o\left[\frac{4}{3}\eta_1\frac{\beta_o^2 + \beta_\infty^2\,\omega^2\,\tau_{2i}^2}{\beta_o(\beta_o + \beta_\infty\,\omega^2\,\tau_{2i}^2)} + \eta_{2i}\frac{\beta_r}{\beta_o + \beta_\infty\,\omega^2\,\tau_{2i}^2}\right]$$

爲 i 種過程所致之每波長吸收係數。

$$\text{當 } \omega\,\tau_{2i} \longrightarrow 0, \text{ 則} \qquad \gamma \longrightarrow \gamma_o = 2\pi\,\omega\,\beta_o\left(\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_o}\sum_i\frac{n_i}{n}\eta_{2i}\right);$$

$$\text{當 } \omega\,\tau_{2i} \longrightarrow \infty, \text{ 則} \qquad \gamma \longrightarrow \gamma_\infty = 2\pi\,\omega\,\beta_\infty\left(\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_o\,\beta_\infty^2\,\omega^2}\sum_i\frac{n_i}{n}\frac{1}{\eta_{2i}}\right).$$

故知在低頻率時，流體動力學中之 η_2 爲

$$\eta_2 = \sum_i \frac{n_i}{n}\eta_{2i}.$$

容變粘滯性 η_{2i} 所致之 γ 部分，在頻率爲

$$\omega_{mi} = (\beta_o/\beta_\infty)^2/\tau_{2i} = v_\infty/v_o\,\tau_{2i}$$

時爲最大，其值爲

$$\frac{n_i}{n}\pi\frac{\beta_r}{(\beta_o\,\beta_\infty)^{\frac{1}{2}}} = \frac{n_i}{n}\pi\frac{v_\infty^2 - v_o^2}{v_\infty\,v_o}.$$

一似固體之液體之聲速經推得有下式之關係：

$$v_i^2 = \frac{1}{\rho}\left(\frac{4}{3}\,\frac{\mu\,\omega^2\,\tau_1^2}{1+\omega^2\,\tau_1^2} + \frac{\beta_o + \beta_\infty\,\omega^2\,\tau_{2i}^2}{\beta_0^2 + \beta_\infty^2\,\omega^2\,\tau_{2i}^2}\right).$$

當 $\omega\,\tau_1 \longrightarrow 0,\quad \omega\,\tau_{2i} \longrightarrow 0,$ 則 $\quad v \longrightarrow v_o = 1\,/\,(\rho\,\beta_o)^{\frac{1}{2}};$

當 $\omega\,\tau_1 \longrightarrow \infty,\quad \omega\,\tau_{2i} \longrightarrow \infty,$ 則 $\quad v \longrightarrow v_\infty = [(k_\infty + \frac{4}{3}\mu)\,/\,\rho]^{\frac{1}{2}}.$

其 i 種過程所致之振幅吸收係數爲

$$\alpha_i = \frac{\omega^2}{2\,\rho\,v_i^3}\left[\frac{4}{3}\,\frac{1}{1+\omega^2\,\tau_1^2}\,\eta_1 + \frac{\beta_o\,\beta_r}{\beta_0^2 + \beta_\infty^2\,\omega^2\,\tau_{2i}^2}\,\eta_{2i}\right].$$

由之得：

當 $\omega\,\tau_1 \longrightarrow 0,\quad \omega\,\tau_{2i} \longrightarrow 0,$ 則強度吸收係數

$$\gamma \longrightarrow \gamma_o = 2\,\pi\,\omega\,\beta_o\left(\frac{4}{3}\,\eta_1 + \frac{\beta_r}{\beta_o}\sum_i \frac{n_i}{n}\,\eta_{2i}\right).$$

當 $\omega\,\tau_1 \longrightarrow \infty,\quad \omega\,\tau_{2i} \longrightarrow \infty,$ 則

$$\gamma \longrightarrow \gamma_\infty = \frac{2\,\pi\,\beta_\infty}{\omega\,(1+\frac{4}{3}\,\mu\,\beta_\infty)}\left(\frac{4}{3}\,\frac{\mu^2}{\eta_1} + \frac{\beta_r}{\beta_o\,\beta_\infty^2}\sum_i \frac{n_i}{n}\,\frac{1}{\eta_{2i}}\right).$$

是在低頻率時得與前相同之結果。

依此理論經由試驗數據計算知普通氣體之 η_2 皆遠大於 η_1. 例如，純氫在平常溫度及壓力時之 η_2 爲其 η_1 之 270 倍。 多原子分子氣體之 η_2 且千倍之，萬倍之。 其 $\beta_r \approx \beta_\infty\,/\,10.$

傅瑞克曾由試驗測得 CO_2 在 $23°C$ 之聲吸收係數與頻率關係之曲線。 經皮來邁之分析發見兩個高峯，卽爲兩個弛緩過程部分所組成。 依作者之理論，由此分析所給之數據，經算知 CO_2 之 $\beta_o = 7.69 \times 10^{-7}$ 及 $\beta_\infty = 7.09 \times 10^{-7}$ 厘米 $^2/$ 達因，其兩容變粘滯性係數爲 5.82 及 12.7 波愛斯，其兩過程之弛緩時間分別爲 4.48×10^{-6} 及 9.76×10^{-6} 秒，其低頻有效容變粘滯性係數 $\eta_2 =$ 11.5 波愛斯。

普通液體最大吸收時之頻率太高，即其 τ_{2i} 皆太小（$\sim 10^{-10}$ 秒），故其速變現象難於實驗示明。 惟就所知之吸收係數得計算適於普通提剎式流體動力學之有效容變粘滯係數 $\frac{\beta_r}{\beta_o}\eta_2 = \frac{\beta_r}{\beta_o}\sum_i \frac{n_i}{n}\eta_{2i}$ 之值。 其值亦均大於 η_1 數倍，數十倍，或逾百倍。

黎伯曼曾量度聲源所致導聲媒液之流動，以計算各種液體之第二粘滯性係數之值。 其所用算式係伊卡就提剎式之流體動力學方程式所推得者。 此提剎式方程式與前由作者之弛緩壓縮理論所示者，除前者少去後者之最末項 $\eta_2\,\beta_\infty\,\nabla\frac{dp}{dt}$ 外，完全相同。 提剎式方程式所給之低頻率時振幅吸收係數爲

$$\alpha = \frac{\omega^2}{2\rho v^3}\left(\frac{4}{3}\eta_1 + \eta_2\right).$$

故在低頻率時，提剎之 η_2 實即吾人之 $\eta_2\,\beta_r/\beta_o$. 明乎此，得由黎伯曼之實驗結果計算 $\eta_2\,\beta_r/\beta_o$. 其所得之數值與前由吸收係數算得者大體吻合。 故對容變粘滯性之存在益增信念。

能觀察其速變區域之流體，其計算之結果恆給 $\beta_r < \beta_\infty$，則提剎式方程式不能適用。 然常見流體之速變區域多遠非試驗所能達到，則提剎式方程式適用，而據吾人之弛緩理論所示應得 $\beta_r \gtrsim \beta_\infty$ 或竟 $\beta_\infty \approx 0$.

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ON VOLUME VISCOSITY AND ACOUSTIC DISPERSION PHENOMENA

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ABSTRACT

Our bulk-visco-elastic theory is extended to the case, as recently discovered in ultrasonic absorption measurements, in which there are present more than one relaxation times. This multi-relaxational theory is applied to the study of acoustic dispersion phenomena. The bearing of our relaxational theory in general on classical hydrodynamics is further examined. Illustrative calculations from certain available experimental data are given and their indications discussed.

It seems now well accepted that the excessive absorption of sound in fluids is due to relaxational molecular processes. A relaxational bulk-visco-elastic theory, assuming a single relaxational process, has been given by the writer[1]. Certain ultrasonic experimental data on sound absorption[2, 3] have shown the existence of more than one relaxation times for a single fluid-substance. More than one relaxational molecular processes are, indeed, to be expected in view of the existence of several vibration modes in the case of a gas and of the possible existence of several ways of molecular rearrangement in the case of a liquid. This fact necessitates an extension of our relaxational theory previously given.

1. MULTI-RELAXATIONAL VISCO-ELASTIC THEORY

Assume the simultaneous existence of several components or alternative molecular processes with different relaxation times $\tau_1, \tau_2, \ldots, \tau_i, \ldots$. Let n_i/n be the fractional occurrence of the ith relaxational process. Then $\sum_i n_i = n$. Recollecting that the compressibility, β_o, of a fluid element due to a constant change of the applied pressure is composed of two parts, one, β_∞,

1. Hoff Lu, *Chinese J. Phys.* **7** (1950), 365.
2. W. H. Pielemeier, *Jour. Acous. Soc. Am.* **15** (1943), 25.
3. E. A. Alexander and J. D. Lambert, *Proc. Roy. Soc.* A, **179** (1942), 499.

representing an instantaneous compression and another, β_r representing a relaxational compression due to volume viscosity, the time variation of relative compression, $s = -\Delta V/V$, may be represented by

$$\frac{ds}{dt} = \frac{1}{n} \sum_i n_i \frac{ds_i}{dt} , \tag{1}$$

with

$$\frac{ds_i}{dt} = \beta_\infty \frac{dp}{dt} + \frac{s_0 - s_i}{\beta_0 \eta_{2i}} , \tag{2}$$

where $s_0 = \beta_0 (p - p_0)$ is the equilibrium value of s ultimately produced by a constant pressure $p = p(\rho)$. As before, we may formally arrive at (2) if we define the ith coefficient of volume viscosity η_{2i} by

$$\bar{p} - p_i' = \eta_{2i} \left(\frac{ds_i}{dt} \right)_{\text{vis}} , \tag{3}$$

where

$$\left(\frac{ds_i}{dt} \right)_{\text{vis}} = \frac{ds_i}{dt} - \frac{ds_\infty}{dt} = \frac{ds_i}{dt} - \beta_\infty \frac{dp}{dt} ,$$

$\bar{p}$ is the mean dynamic pressure of hydrodynamics, being equivalent to a static pressure p as far as the steady value of s is concerned, and $p_i' = p_i' (\rho)$ is an effective dynamic pressure that determines the actual s_i according to

$$s_i = \beta_0 (p_i' - p_0) . \tag{4}$$

Equation (2) shows a relaxation time

$$\tau_{2i} = \beta_0 \eta_{2i} \tag{5}$$

for the part of compressional strain depending on η_{2i}.

Since obviously

$$s = \frac{1}{n} \sum_i n_i s_i = \frac{1}{n} \beta_0 \sum_i n_i (p_i' - p_0)$$

$$= \beta_0 \left(\frac{1}{n} \sum n_i p_i' - p_0\right) = \beta_0 (p' - p_0) ,$$

the effective dynamic pressure is

$$p' = \frac{1}{n} \sum_i n_i \, p_i' \tag{6}$$

The overall volume viscosity η_2 as defined by

$$\bar{p} - p' = \eta_2 \left(\frac{ds}{dt} \right)_{\text{vis}} \tag{7}$$

is, by (3) and (6),

$$\eta_2 = \frac{1}{n} \sum_i n_i \, \eta_{2i} \left(\frac{ds_i}{dt} \right)_{\text{vis}} \Big/ \left(\frac{ds}{dt} \right)_{\text{vis}} . \tag{8}$$

2. DISPERSION AND ABSORPTION OF SOUND

Writing $$s_i = C_i \, e^{-\alpha_i x} \, e^{i\omega(t - x/v_i)} = C_i \, e^{i\omega(t - x/v_i^*)} ,$$

we have, for the practical case $\omega \gg \alpha_i v_i$, approximately.

$$v_i^{*2} \simeq v_i^2 + 2i\alpha_i \, v_i^3 / \omega , \tag{9}$$

and

$$v_i^{*2} = (k_i^* + \frac{4}{3} \mu^*)/\rho , \tag{10}$$

where $k_i^* = 1/\beta_i^*$ is the complex bulk modulus and μ^* the complex shear modulus. (2) yields

$$\beta_i^* = \beta_\infty + \frac{\beta_r}{1 + i\omega\tau_{2i}} , \tag{11}$$

while μ^* has been found[1] for two cases.

In the case of a gas or a more gas-like liquid, $\mu^* = i\omega\eta_1$, and we find, by comparing the real parts of (9) and (10),

$$v_i^2 = \frac{1}{\rho} \frac{\beta_o + \beta_\infty \omega^2 \tau_{2i}^2}{\beta_0^2 + \beta_\infty^2 \omega^2 \tau_{2i}^2} . \tag{12}$$

Thus, there are as many sound velocities as there are relaxational processes causing the volume viscosity. However, in the following limiting cases, these velocities all merge into one, viz.,

$$\text{as} \quad \omega \tau_{2i}\text{'s} \longrightarrow 0 , \qquad v \longrightarrow v_o = 1/(\rho \beta_o)^{\frac{1}{2}} ; \tag{13}$$

$$\text{as} \quad \omega \tau_{2i}\text{'s} \longrightarrow \infty , \qquad v \longrightarrow v_\infty = 1/(\rho \beta_\infty)^{\frac{1}{2}} ; \tag{14}$$

giving

$$\rho v_0^2 \beta_o = \rho v_\infty^2 \beta_\infty = 1 . \tag{15}$$

The inflectional point of each dispersion curve, v_i^2 versus log ω, is found to be at the frequency $(\omega_i)_i = \beta_o / \beta_\infty \tau_{2i} = v_\infty^2 / v_o^2 \tau_{2i}$.

Comparing the imaginary parts of (9) and (10), we find

$$\alpha_i = \frac{\omega^2}{2 \rho v_i^3} \left[\frac{4}{3} \eta_1 + \frac{\beta_o \beta_r}{\beta_0^2 + \beta_\infty^2 \omega^2 \tau_{2i}^2} \eta_{2i} \right] . \tag{16}$$

The experimentally measured coefficient of absorption is usually the intensity coefficient per wave length, i.e., γ as defined in $I = I_o e^{-\gamma x/\lambda} = I_o e^{-2\alpha x}$ so that $\gamma = 2\lambda\alpha$. Thus, we have

$$\gamma_i = \gamma_{1i} + \gamma_{2i} = 2\pi \omega \beta_o \left[\frac{4}{3} \eta_1 \frac{\beta_0^2 + \beta_\infty^2 \omega^2 \tau_{2i}^2}{\beta_o \{ \beta_o + \beta_\infty \omega^2 \tau_{2i}^2 \}} + \eta_{2i} \frac{\beta_r}{\beta_o + \beta_\infty \omega^2 \tau_{2i}^2} \right] \tag{17}$$

The observed absorption coefficient per wave length is

$$\gamma = \frac{1}{n} \sum_i n_i \gamma_i , \tag{18}$$

$$\text{As} \quad \omega \tau_{2i}\text{'s} \longrightarrow 0, \qquad \gamma \longrightarrow \gamma_o = 2\pi \omega \beta_o \left[\frac{4}{3} \eta_1 + \frac{\beta_r}{\beta_o} \sum_i \frac{n_i}{n} \eta_{2i} \right] ; \tag{19}$$

$$\text{as} \quad \omega \tau_{2i}\text{'s} \longrightarrow \infty , \quad \gamma \longrightarrow \gamma_\infty = 2\pi \omega \beta_\infty \left[\frac{4}{3} \eta_1 + \frac{\beta_r}{\beta_o \beta_\infty^2 \omega^2} \sum_i \frac{n_i}{n} \frac{1}{\eta_{2i}} \right] . \tag{20}$$

The part of γ due to η_{2i} has its maximum value equal to

$$\frac{n_i}{n}(\gamma_2)_{max} = \frac{n_i}{n}\pi\frac{\beta_r}{(\beta_o\,\beta_\infty)^{1/2}} = \frac{n_i}{n}\pi\frac{v_\infty^2 - v_0^2}{v_\infty\, v_o}, \tag{21}$$

and at the frequency

$$(\omega_m)_i = (\beta_o/\beta_\infty)^{\frac{1}{2}}/\tau_{2i} = v_\infty/v_o\,\tau_{2i}\ . \tag{22}$$

These equations are found to be satisfactory in representing the observed dispersion phenomena. An illustrative calculation of η_{2i}, β_o, and β_∞, in the case of dry CO_2, is given in § 4.

For a more solid-like liquid, we had[1] $\mu^* = \mu i\omega\tau_1/(1+i\omega\tau_1)$, where τ_1 is Maxwell's relaxation time for the shearing process, i.e. $\tau_1=\eta_1/\mu$. Likewise, we may find the corresponding expressions for v_i and γ, the results being as follows:

$$v^2 = \frac{1}{\rho}\left(\frac{4}{3}\frac{\mu\,\omega^2\tau_1^2}{1+\omega^2\tau_1^2} + \frac{\beta_o + \beta_\infty\,\omega^2\tau_{2i}^2}{\beta_0^2 + \beta_\infty^2\,\omega^2\tau_{2i}^2}\right). \tag{23}$$

As $\omega\tau_1 \longrightarrow 0$, and $\omega\tau_{2i}$'s $\longrightarrow 0$, $\quad v \longrightarrow v_o = 1/(\rho\,\beta_o)^{\frac{1}{2}}$; (24)

as $\omega\tau_1 \longrightarrow \infty$, and $\omega\tau_{2i}$'s $\longrightarrow \infty$, $\quad v \longrightarrow v_\infty = \left[(k_\infty + \frac{4}{3}\mu)/\rho\right]^{\frac{1}{2}}$. (25)

$$\alpha_i = \frac{\omega^2}{2\rho\,v_i^3}\left[\frac{4}{3}\frac{1}{1+\omega^2\tau_1^2}\eta_1 + \frac{\beta_o\,\beta_r}{\beta_0^2+\beta_\infty^2\,\omega^2\tau_{2i}^2}\eta_{2i}\right]. \tag{26}$$

As $\omega\tau_1 \longrightarrow 0$, and $\omega\tau_{2i}$'s $\longrightarrow 0$,

$$\gamma \longrightarrow \gamma_o = 2\pi\,\omega\,\beta_o\left[\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_o}\sum_i\frac{n_i}{n}\eta_{2i}\right]; \tag{27}$$

as $\omega\tau_1 \longrightarrow \infty$, and $\omega\tau_{2i}$'s $\longrightarrow \infty$,

$$\gamma \longrightarrow \gamma_\infty = \frac{2\pi\,\beta_\infty}{\omega\,(1+\frac{4}{3}\mu\,\beta_\infty)}\left[\frac{4}{3}\frac{\mu^2}{\eta_1} + \frac{\beta_r}{\beta_o\,\beta_\infty^2}\sum_i\frac{n_i}{n}\frac{1}{\eta_{2i}}\right]. \tag{28}$$

There does not exist absorption data of solid-like liquids at frequencies near the dispersion region to allow a test of these equations.

3. ON THE GENERALIZED CLASSICAL HYDRODYNAMICS

Classical hydrodynamics has been generalized[1] to embody the effect of η_2 by assuming

$$\bar{p} - p' = (\lambda + \frac{2}{3}\eta_1)\left(\frac{ds}{dt}\right)_{\text{vis}} = (\lambda + \frac{2}{3}\eta_1)\left(\frac{ds}{dt} - \frac{ds_\infty}{dt}\right)$$

$$= -(\lambda + \frac{2}{3}\eta_1)\left(\nabla \cdot \boldsymbol{v} + \beta_\infty \frac{dp}{dt}\right),$$

where λ and η_1 are Lamé's type coefficients. The equation of motion then becomes

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{f} - \nabla p' + (\frac{1}{3}\eta_1 + \eta_2)\nabla\nabla \cdot \boldsymbol{v} + \eta_1 \nabla^2 \boldsymbol{v} + \eta_2 \beta_\infty \nabla \frac{dp}{dt}. \tag{29}$$

This equation differs in form from the classical equation first explicitly given by Tisza[4] and also used by Eckart[5] merely in the presence of the extra term $\eta_2 \beta_\infty \nabla \frac{dp}{dt}$. It is seen that Tisza's hydrodynamics neglects the instantaneous dilatational strain and corresponds to the approximation $\nabla \cdot \boldsymbol{v} \approx -(ds/dt)_{\text{vis}}$, implying that $ds_\infty/dt << (ds/dt)_{\text{vis}}$. We note that for an incompressible fluid, $\nabla \cdot \boldsymbol{v} = 0$ and $dp/dt = 0$, we have $p' = \bar{p}$ and (29) reduces to

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{f} - \nabla \bar{p} + \eta_1 \nabla^2 \boldsymbol{v},$$

so that the effect of η_2 vanishes. Since hydrodynamics is ordinarily applied to fluids that may be regarded as incompressible, we see why η_2 remains unnoticed for so long. We further note that in (29) the effect of any shear modulus that may exist has been neglected. How this may be included has been indicated by Frenkel[6].

Classical hydrodynamics as here generalized according to (29) leads to[1] the approximate expression

$$\alpha = \frac{\omega^2}{2\rho v^3}\left(\frac{4}{3}\eta_1 + \frac{\beta_r}{\beta_o}\eta_2\right) \tag{30}$$

4. Tisza, L., *Phys. Rev.* **61** (1942), 531.
5. Eckart, C., *Phys. Rev.* **73** (1948), 68.
6. Frenkel, J., *Kinetic Theory of Liquids* (Clarendon Press, Oxford, 1946), IV, 10, pp. 248-249.

for the amplitude coefficient of absorption at sufficiently low frequencies. This is seen to be the same as the limiting form of our visco-elastic theory for $\omega\tau_1 \to 0$ and $\omega\tau_{2i}$'s $\to 0$ with

$$\eta_2 = \frac{1}{n} \sum_i n_i \, \eta_{2i} \, . \tag{31}$$

It is to be noted that the hydrodynamical equation involving η_2 employed by Tisza[4], Eckart[5], and others is the one obtained from (29) by dropping the last term which represents the effect of instantaneous strain. Such an equation leads to the approximate expression

$$\alpha = \frac{\omega^2}{2\,\rho\, v^3} \left(\frac{4}{3}\,\eta_1 + \eta_2 \right) \tag{32}$$

for sufficiently low frequencies. Comparing this with (16) and (26), we see that

$$\beta_o\,\beta_r \sum_i \frac{n_i}{n} \frac{\eta_{2i}}{\beta_0^2 + \beta_\infty^2\,\omega^2\,\tau_{2i}^2} \, , \tag{33}$$

or $\frac{\beta_r}{\beta_o} \sum_i \frac{n_i}{n} \eta_{2i}$ for sufficiently low frequencies, plays the role of an effective volume viscosity for such a hydrodynamical theory.

4. ILLUSTRATIVE CALCULATIONS

Results of calculations of the volume viscosity and the static and instantaneous compressibilities from the observed maxima of sound absorption in the case of a single relaxation time have been given elsewhere[7] for a number of polyatomic gases. It turns out that η_2 is several thousand times greater than η_1 and that $\beta_r \approx \beta_\infty / 10$.

Pielemeier[2] has shown that Fricke's absorption curve for dry CO_2 at 23°C may be resolved into two components, one having its peak of 0.215 at 17 kc while the other having its peak of 0.043 at about 37 kc. Assuming these values, let us calculate the two relaxation times and hence the two coefficients of volume viscosity of dry CO_2. Since, practically, we have $(n_1/n)\ (\gamma_2)_{max} = 0.215$ and $(n_2/n)\ (\gamma_2)_{max} = 0.043$, we get $(\gamma_2)_{max} = 0.258$, and $n_1/n = 5/6$ and $n_2/n = 1/6$. By (21), we find $v_\infty / v_o = 1.042$. Since

7. Hoff Lu, *Jour. Acous. Soc. Am.* **23** (1951), 12.

$(\omega_m)_1 = 2\pi \times 17\times10^3$ and $(\omega_m)_2 = 2\pi \times 37 \times 10^3$, we have, by (22), $\tau_{21} = 9.76\times10^{-6}$ second and $\tau_{22} = 4.48\times10^{-6}$ second. Taken from Handbook of Chemistry and Physics, $\rho = 1.81\times10^{-3}$ gm/cm^3, and $v_o = 268$ m/sec, so that, by (15), we get $\beta_o = 7.69\times10^{-7}$ cm^2/dyne and $\beta_\infty = 7.09\times10^{-7}$ cm^2/dyne, and, by (5), $(\eta_2)_1 = 12.7$ poises and $(\eta_2)_2 = 5.82$ poises. At frequencies much below those of the dispersion region (i.e., for $\omega \ll (\omega_m)_i$), we have, by (31), $\eta_2 = 11.5$ poises.

For liquids, τ_2 is, in general, so small ($\sim 10^{-10}$ sec) that the dispersion region is usually at inaccessibly high frequencies. As a result, only the effective viscosity $\eta_2\, \beta_r/\beta_o$ can be calculated from the observed attenuation coefficient. The results[7] of such calculations for a number of liquids have shown that the value of $\eta_2\, \beta_r/\beta_o$ ranges from several to several ten times greater than η_1. Sensibly, the same values for $\eta_2\, \beta_r/\beta_o$ are obtained[7] from the results[8] of Liebermann's experiment on a hydrodynamical effect of the volume viscosity.

One case of Liebermann's data is of interest. Ethyl formate ($HCOOC_2H_5$) is the only liquid that has its dispersion region fall in the experimental range of frequencies in Liebermann's experiment. Hence, for this case, Liebermann's n'/n is

$$\frac{n'}{n} = \frac{\beta_o\, \beta_r}{\beta_0^2 + \beta_\infty^2\, \omega^2 \tau_2^2} \frac{\eta_2}{\eta_1} - \frac{2}{3} = \frac{1}{1 + \omega^2\, \beta_\infty^2\, \eta_2^2} \frac{\beta_r}{\beta_o} \frac{\eta_2}{\eta_1} - \frac{2}{3}\,.$$

Using the first and last data, this formula yields $\eta_2\, \beta_r/\beta_o = 2.3$ poises and $\beta_\infty\, \eta_2 = 1.9\times10^{-7}$ poise-cm^2/dyne. The theoretical curve thus given is drawn in Fig. 1, the experimental points being also given (crossed) for comparison.

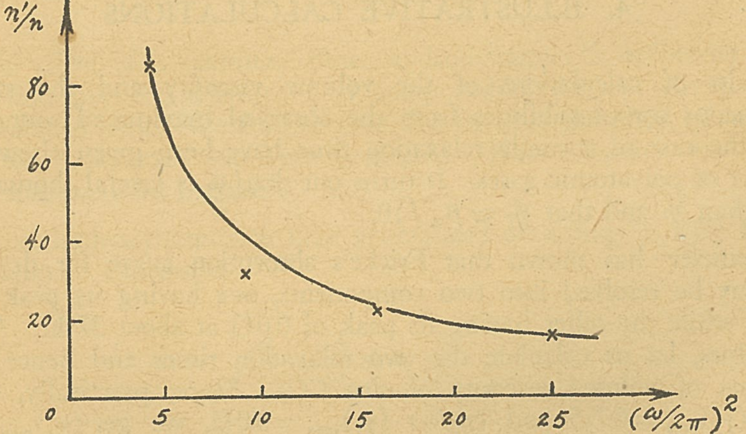

Fig. 1

8. L. N. Liebermann, *Phys. Rev.* **75** (1949), 1415.

Taking $\beta_0 \approx 1 \times 10^{-10}$ cm^2/dyne, we have, for ethyl formate at 17.4°C, $\eta_2 \approx 2000$ poises, $\tau \approx 2 \times 10^{-7}$ sec, $\beta_\infty \approx \beta_0$, and $\beta_r \approx 1 \times 10^{-13}$ cm^2/dyne. Unfortunately, there are too few experimental points to enable us say anything as to whether there exists an extra relaxation time that causes one of the point go off the theoretical curve considerably.

5. CONCLUDING REMARKS

We note that in all cases in which we have been able to calculate β_r and β_∞, it turns out that $\beta_r < \beta_\infty$, which means that $s_r < s_\infty$ so the we expect $(ds/dt)_{\text{vis}} < ds_\infty/dt$ and, hence, Tisza's simplified type of generalized hydrodynamics will not be expected to apply in these cases, as is already known from the observed dependence of sound attenuation on frequency. However, for the more common fluids whose dispersion region is so remote that no appreciable effect of frequency is observable, Tisza's type of equation will be applicable, and for this case our theory would indicate that $\beta_r > \beta_\infty$, or $\beta_\infty \approx 0$.