橫觀各向同性體的彈性力學的空間問題*

胡 海 昌

（中國科學院數學研究所）

一、前言

近二十年來，各向異性體彈性力學在蘇聯獲得飛躍的發展[參 1]. 只涉及兩個變數（即坐標）的問題，如平面問題、柱體的扭轉和彎曲問題、廻轉體的軸對稱的形變問題，已經由許多學者加以研究，並且獲得了巨大的成就．在各向異性體彈性力學的領域中，要推 C. Г. 列赫尼茨基的貢獻最大（見[參 2]）．但是涉及三個變數的空間問題尚討論得很少．C. Г. 列赫尼茨基[參 3]在 1940 年求得了橫觀各向同性**的廻轉體的軸對稱形變問題的通解，可以說走上了研究空間問題的第一步並指出了此後努力的方向．本文的目的在求涉及三個變數的橫觀各向同性體的空間的問題的通解．我們放棄了 C. Г. 列赫尼茨基關於形變狀態的限制，僅僅保留他關於物性的假定．本文所得的結果為求解具體的空間問題創造了廣大的可能性．例如利用本文所求得的通解，即可解決橫觀各向同性的半無限體（假定其表面平行其物性的各向同性面）在其表面負担任意載荷的問題．

二、通解的推算

取一直角坐標系 (x, y, z)，使 z 軸跟物性的各向同性面垂直．這樣 xy 面上的彈性是各向同性的．在這坐標系下，橫觀各向同性體的廣義虎克定律是：

* 1952 年 11 月 29 日收到

** 意指在一系列互相平行的平面上彈性是各向同性的．

$$\frac{\partial u}{\partial x}=a_{11}\sigma_x+a_{12}\sigma_y+a_{13}\sigma_z,\quad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=a_{44}\tau_{yz},$$

$$\frac{\partial v}{\partial y}=a_{12}\sigma_x+a_{11}\sigma_y+a_{13}\sigma_z,\quad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=a_{44}\tau_{xz},\tag{1}$$

$$\frac{\partial w}{\partial z}=a_{13}\sigma_x+a_{13}\sigma_y+a_{33}\sigma_z,\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=a_{66}\tau_{xy},$$

其中

$$a_{66}=2(a_{11}-a_{12}),\tag{2}$$

或者亦可以寫成爲

$$\sigma_x=A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z},\quad \tau_{yz}=A_{44}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),$$

$$\sigma_y=A_{12}\frac{\partial u}{\partial x}+A_{11}\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z},\quad \tau_{xz}=A_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\tag{3}$$

$$\sigma_z=A_{13}\frac{\partial u}{\partial x}+A_{13}\frac{\partial v}{\partial y}+A_{33}\frac{\partial w}{\partial z},\quad \tau_{xy}=A_{66}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),$$

其中

$$A_{66}=\frac{1}{2}(A_{11}-A_{12}).\tag{4}$$

若將 (3) 式代入平衡方程式

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xy}}{\partial z}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0,$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0,\tag{5}$$

中，便得以位移表示的平衡方程式如下：

$$
\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2}\right)u+B_{12}\frac{\partial^2 v}{\partial x\partial y}+B_{13}\frac{\partial^2 w}{\partial x\partial z}=0,
$$

$$
B_{12}\frac{\partial^2 u}{\partial x\partial y}+\left(B_{66}\frac{\partial^2}{\partial x^2}+B_{11}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2}\right)v+B_{13}\frac{\partial^2 w}{\partial y\partial z}=0, \tag{6}
$$

$$
B_{13}\frac{\partial^2 u}{\partial x\partial z}+B_{13}\frac{\partial^2 v}{\partial y\partial z}+\left(B_{44}\frac{\partial^2}{\partial x^2}+B_{44}\frac{\partial^2}{\partial y^2}+B_{33}\frac{\partial^2}{\partial z^2}\right)w=0,
$$

其中

$$
B_{11}=A_{11},\quad B_{33}=A_{33},\quad B_{44}=A_{44},\quad B_{66}=A_{66},\quad B_{12}=A_{12}+A_{66},
$$

$$
B_{13}=A_{13}+A_{44}. \tag{7}
$$

B_{11}, B_{12}, B_{66} 有下面的關係

$$
B_{66}=B_{11}-B_{12}. \tag{8}
$$

這個關係在此後簡化算式時常常用到.

我們的問題是找尋 (6) 式的通解.

由 (6) 式消去任意兩個函數, 可知 u, v, w 都適合下面的方程式:

$$
\begin{vmatrix}
B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2} & B_{12}\frac{\partial^2}{\partial x\partial y} & B_{13}\frac{\partial^2}{\partial x\partial z} \\
B_{12}\frac{\partial^2}{\partial x\partial y} & B_{66}\frac{\partial^2}{\partial x^2}+B_{11}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2} & B_{12}\frac{\partial^2}{\partial y\partial z} \\
B_{13}\frac{\partial^2}{\partial x\partial z} & B_{13}\frac{\partial^2}{\partial y\partial z} & B_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+B_{33}\frac{\partial^2}{\partial z^2}
\end{vmatrix}\Phi=0,
$$

即

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2}{\partial z^2}\right)\left[B_{11}B_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\right.$$

$$\left.+\left(B_{44}^2+B_{11}B_{33}-B_{13}^2\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial z^2}+B_{33}B_{44}\frac{\partial^4}{\partial z^4}\right]\Phi=0. \tag{9}$$

爲了適合方程式 (6) 的前兩式，可以引進一個位移函數如下：

$$u=\begin{vmatrix} B_{12}\dfrac{\partial^2}{\partial x\partial y} & B_{13}\dfrac{\partial^2}{\partial x\partial z} \\ B_{66}\dfrac{\partial^2}{\partial x^2}+B_{11}\dfrac{\partial^2}{\partial y^2}+B_{44}\dfrac{\partial^2}{\partial z^2} & B_{13}\dfrac{\partial^2}{\partial y\partial z}\end{vmatrix}F,$$

$$v=\begin{vmatrix} B_{13}\dfrac{\partial^2}{\partial x\partial z} & B_{11}\dfrac{\partial^2}{\partial x^2}+B_{66}\dfrac{\partial^2}{\partial y^2}+B_{44}\dfrac{\partial^2}{\partial z^2} \\ B_{13}\dfrac{\partial^2}{\partial y\partial z} & B_{12}\dfrac{\partial^2}{\partial x\partial y}\end{vmatrix}F,$$

$$w=\begin{vmatrix} B_{11}\dfrac{\partial^2}{\partial x^2}+B_{66}\dfrac{\partial^2}{\partial y^2}+B_{44}\dfrac{\partial^2}{\partial z^2} & B_{12}\dfrac{\partial^2}{\partial x\partial y} \\ B_{12}\dfrac{\partial^2}{\partial x\partial y} & B_{66}\dfrac{\partial^2}{\partial x^2}+B_{11}\dfrac{\partial^2}{\partial y^2}+B_{44}\dfrac{\partial^2}{\partial z^2}\end{vmatrix}F,$$

即

$$u=-B_{13}B_{66}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2 F}{\partial x\partial z},$$

$$v=-B_{13}B_{66}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2 F}{\partial y\partial z},$$

$$w=B_{11}B_{66}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{B_{44}}{B_{11}}\frac{\partial^2 F}{\partial z^2}\right).$$

約去 u, v, w 算式中的公共微分算子，可知爲了適合 (6) 式的前兩式可以取

$$u=-\frac{\partial^2 F}{\partial x \partial z}, \quad v=-\frac{\partial^2 F}{\partial y \partial z}, \quad w=\frac{B_{11}}{B_{13}}\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{B_{44}}{B_{11}} \frac{\partial^2 F}{\partial z^2}\right). \tag{10}$$

將此公式入 (6) 式的第三式，便得 F 所適合的方程式如下:

$$\left[B_{11} B_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2+\left(B_{44}^2+B_{11} B_{33}-B_{13}^2\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \frac{\partial^2}{\partial z^2}+\right.$$
$$\left.+B_{33} B_{44} \frac{\partial^4}{\partial z^4}\right] F=0. \tag{11}$$

這裏的微分算子正是 (9) 式中的一個因子．這個方程式和 C. Γ. 列赫尼茨基[參 3] 研究廻轉體的軸對稱形變問題時所引進的應力函數所適合的方程式基本上相同．唯一的差別只在他所得到的是軸對稱的一個特殊情形，而我們的則是一般的形式．依照 C. Γ. 列赫尼茨基的記號，命

$$a=\frac{a_{13}\left(a_{11}-a_{12}\right)}{a_{11} a_{33}-a_{13}^2}, \quad c=\frac{a_{13}\left(a_{11}-a_{12}\right)+a_{11} a_{14}}{a_{11} a_{33}-a_{13}^2}, \quad d=\frac{a_{11}^2-a_{12}^2}{a_{11} a_{33}-a_{13}^2}. \tag{12}$$

則經過一個繁長的計算後可以證明

$$\frac{B_{33}}{B_{11}}=d, \quad \frac{B_{44}^2+B_{11} B_{33}-B_{13}^2}{B_{11} B_{44}}=a+c, \tag{13}$$

所以 F 所適合的方程式是

$$\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2+(a+c)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \frac{\partial^2}{\partial z^2}+d \frac{\partial^4}{\partial z^4}\right] F=0. \tag{14}$$

若再應用 C. Γ. 列赫尼茨基的記號

$$s_1=\sqrt{\frac{a+c+\sqrt{(a+c)^2-4 d}}{2 d}}, \quad s_2=\sqrt{\frac{a+c-\sqrt{(a+c)^2-4 d}}{2 d}} \tag{15}$$

則 F 所適合的方程式可以寫成爲

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)F=0. \tag{16}$$

C. Г. 列赫尼茨基應用應變能必須是正數這一原理,證明了 s_1 和 s_2 不可能爲純虛數.

F 適合一個四級方程式, 所以由 (10) 和 (16) 所規定的解答只能適合兩個邊界條件. 但是 u, v, w 的通解應能適合三個邊界條件, 所以上面所求得的解答尚不是通解.

不論 u, v, w 是什麼函數, 我們總能用另外三個函數* $F, \varphi, \bar{w}$ 表示如下:

$$u=-\frac{\partial^2 F}{\partial x\partial z}-\frac{\partial\varphi}{\partial y},\quad v=-\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial\varphi}{\partial x},$$

$$w=\frac{B_{11}}{B_{13}}\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{B_{44}}{B_{11}}\frac{\partial^2 F}{\partial z^2}\right)+\bar{w}. \tag{17}$$

這個表示法不是唯一的, 因爲下面這個齊次方程式有解答存在:

$$\frac{\partial^2 F_0}{\partial x\partial z}+\frac{\partial\varphi_0}{\partial y}=0,\quad \frac{\partial^2 F_0}{\partial y\partial z}-\frac{\partial\varphi_0}{\partial x}=0.$$

$$\frac{B_{11}}{B_{13}}\left(\frac{\partial^2 F_0}{\partial x^2}+\frac{\partial^2 F_0}{\partial y^2}+\frac{B_{44}}{B_{11}}\frac{\partial^2 F_0}{\partial z^2}\right)+\bar{w}_0=0. \tag{18}$$

事實上只要適當地選擇 F_0, 和 $\varphi_0, \bar{w}_0$ 可以是任意一個不含 z 的函數, 再加上任意一個 x, y 的調和函數 (可以包含 z, 如 $f(x+iy, z)$ 的形式). 所以在 (17) 式的表示法中, $\bar{w}$ 可以加減一個不含 z 的函數, 和加減一個 x, y 的調和函數.

將算式 (17) 代入平衡方程式 (16) 中, 得 $F, \varphi, \bar{w}$ 所適合的方程式如下:

$$-\frac{\partial}{\partial y}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2\varphi}{\partial z^2}\right)+\frac{B_{13}}{B_{66}}\frac{\partial^2\bar{w}}{\partial x\partial z}=0,$$

---

* 暫時假定這裏的 F 和上面的 F 不同. 後面將要證明兩者是相同的.

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2 \varphi}{\partial z^2}\right)+\frac{B_{13}}{B_{66}}\frac{\partial^2 \bar{w}}{\partial y\,\partial z}=0, \tag{19}$$

$$\left[\frac{B_{11}}{B_{13}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2+\frac{B_{44}^2+B_{11}B_{13}-B_{13}^2}{B_{13}B_{44}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial z^2}+\frac{B_{33}}{B_{13}}\frac{\partial^4}{\partial z^4}\right]F$$

$$+\frac{\partial^2 \bar{w}}{\partial x^2}+\frac{\partial^2 \bar{w}}{\partial y^2}+\frac{B_{33}}{B_{44}}\frac{\partial^2 \bar{w}}{\partial z^2}=0.$$

從 (19) 式的前兩式可知

$$\frac{B_{13}}{B_{66}}\frac{\partial \bar{w}}{\partial z}+i\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2 \varphi}{\partial z^2}\right)=f(x+iy,z),$$

但是 $\bar{w}$ 可以加減一個 x,y 的調和函數, 所以我們可以命 f 爲零, 這樣得

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2 \varphi}{\partial z^2}=0 \tag{20}$$

和

$$\frac{\partial \bar{w}}{\partial z}=0, \quad 即 \quad \bar{w}=\bar{w}(x,y).$$

但是 $\bar{w}$ 可以加減一個不含 z 的函數, 所以我們可以命

$$\bar{w}=0. \tag{21}$$

這樣 (19) 式的第三式就簡化爲 (11) 式.

爲了此後書寫方便起見, 我們命

$$\frac{B_{11}}{B_{13}}=\alpha, \quad \frac{B_{33}}{B_{13}}=\beta, \quad \frac{B_{44}}{B_{13}}=\gamma, \quad \frac{B_{66}}{B_{44}}=s_0^2, \quad \frac{B_{12}}{B_{13}}=\frac{B_{11}-B_{66}}{B_{13}}=\alpha-\gamma s_0^2. \tag{22}$$

於是可以總結起來說, 方程式 (16) 的通解是

$$u=-\frac{\partial^2 F}{\partial x\,\partial z}-\frac{\partial \varphi}{\partial y}, \quad v=-\frac{\partial^2 F}{\partial y\,\partial z}+\frac{\partial \varphi}{\partial x}, \tag{23}$$

$$w=\alpha\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\gamma\frac{\partial^2 F}{\partial z^2}\right),$$

其中 F 和 φ 分別適合下列兩方程式:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)F=0, \tag{24}$$

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{1}{s_0^2}\frac{\partial^2 \varphi}{\partial z^2}=0. \tag{25}$$

三、圓柱坐標系下的通解

取一圓柱坐標系 (r, θ, z), 使 z 軸跟物性的各向同性面垂直. 於是在 (r,θ) 平面上的彈性是各向同性的. 設 u_r, u_θ, w 爲各點沿 r, θ, z 三方向上的位移. 這樣的物體在這樣的坐標系下的廣義虎克定律是:

$$\begin{aligned}
\sigma_r &= A_{11}\frac{\partial u_r}{\partial r}+A_{12}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+A_{13}\frac{\partial w}{\partial z}, & \tau_{\theta z} &= A_{44}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right),\\
\sigma_\theta &= A_{12}\frac{\partial u_r}{\partial r}+A_{11}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+A_{13}\frac{\partial w}{\partial z}, & \tau_{rz} &= A_{44}\left(\frac{\partial w}{\partial r}+\frac{\partial u_r}{\partial z}\right), \\
\sigma_z &= A_{13}\frac{\partial u_r}{\partial r}+A_{13}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+A_{33}\frac{\partial w}{\partial z}, & \tau_{r\theta} &= A_{66}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right).
\end{aligned} \tag{26}$$

這裏的 $A_{11}, A_{12}, \cdots A_{66}$ 就是 (3) 式中的 $A_{11}, A_{12}, \cdots A_{66}$. 物體的平衡方程式是:

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0,\\
&\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0,\\
&\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0.
\end{aligned} \tag{27}$$

欲求適合於 (26) (27) 兩式的 u_r, u_θ, w 的通解,可由 (23) (24) (25) 三式經一坐標變換而得. 設想本節圓柱坐標系的 z 軸跟上節直角坐標系的 z 軸相重,並使 (r, θ) 跟 (x, y) 的關係如下:

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{28}$$

則利用向量變換的公式容易證明 u_r, u_θ, w 的通解是

$$\begin{gathered} u_r = -\frac{\partial^2 F}{\partial r\,\partial z} - \frac{1}{r}\frac{\partial \varphi}{\partial \theta}, \quad u_\theta = -\frac{1}{r}\frac{\partial^2 F}{\partial \theta\,\partial z} + \frac{\partial \varphi}{\partial r}, \\ w = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \gamma\frac{\partial^2 F}{\partial z^2}\right). \end{gathered} \tag{29}$$

其中 F 和 φ 分别適合下列兩方程式:

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right) F = 0, \tag{30}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{s_0^2}\frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{31}$$

在這裏有兩類重要的軸對稱的場合特別值得指出. 第一類是橫觀各向同性的廻轉體 (假定物形的對稱軸跟物性的各向同性面垂直) 的扭轉問題. 這時我們命

$$F = 0, \quad \varphi = \varphi(r, z). \tag{32}$$

於是得

$$\begin{gathered} u_r = 0, \quad w = 0, \quad u_\theta = \frac{\partial \varphi}{\partial r}, \\ \frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial \varphi}{\partial r} + \frac{1}{s_0^2}\frac{\partial^2 \varphi}{\partial z^2} = 0. \end{gathered} \tag{33}$$

第二類是同上的廻轉體的軸對稱形變的問題. 這時我們命

$$\varphi = 0, \quad F = F(r, z). \tag{34}$$

於是得

$$u_\theta = 0,\quad u_r = -\frac{\partial^2 F}{\partial r\,\partial z},\quad w = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial F}{\partial r} + \gamma\frac{\partial^2 F}{\partial z^2}\right),$$
$$\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)F = 0. \tag{35}$$

這兩個軸對稱的特殊場合，C. Γ. 列赫尼茨基[參 3]在 1940 年由應力函數出發而解決. 他的結果和我們的結果形式上雖有分別. 本質上是完全一致的.

四、横觀各向同性的圓錐的彎曲問題

在本節裏我們欲利用上節求得的通解來解決横觀各向同性的圓錐（假定圓錐的軸跟物性的各向同性面垂直）在角頂負担一個横向載荷* 的問題. 同上的圓錐在角頂負担軸向載荷所產生的軸對稱形變問題，已由 C. Γ. 列赫尼茨基[參 3 第 62 節]解決.

設有一圓錐，其頂角爲 2α,（當 $\alpha=\frac{\pi}{2}$ 時，圓錐變爲半無限體），在其角頂 0 負担一横向載荷 P，如圖 (1) 所示. 圓錐的側面不受外力作用. 在圓錐的底面，我們只要求應力符合經聖維那原理簡化後的邊界條件，即只要求應力適合平衡條件，

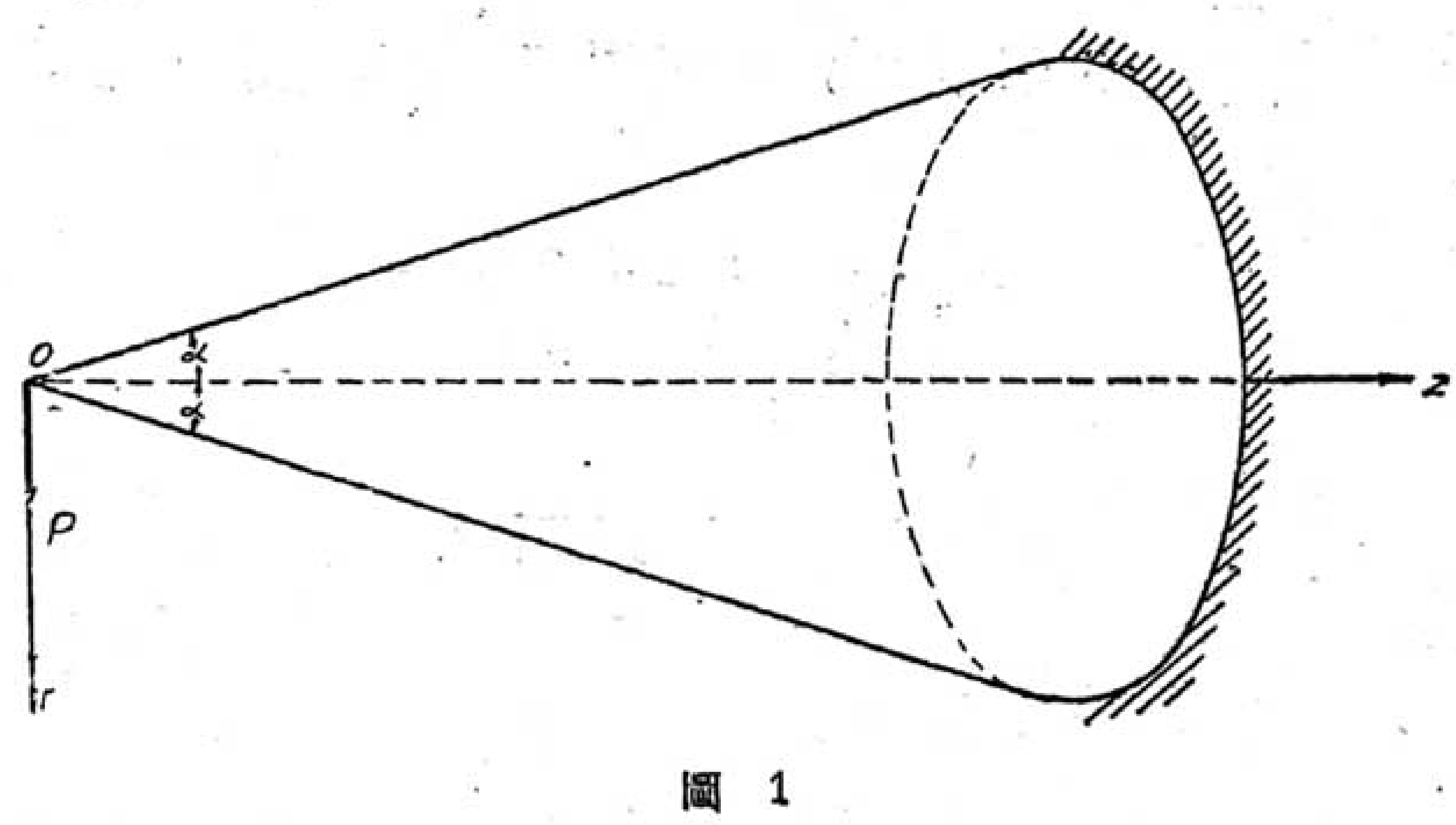

圖 1

* 係指載荷的方向跟圓錐的軸垂直.

或者假定圓錐係一直延伸到無窮遠處. 取一圓柱坐標系 (r, θ, z), 命其原點即爲圓錐的角頂, 命 z 軸跟圓錐的軸相重並指向圓錐的內部; 命 $\theta=0$ 時 r 的方向跟載荷 P 的方向相重.

在根據 (30) (31) 兩式求解 F 和 φ 之前, 我們先將 (29) 式再簡化一下. 假定 $s_1 \neq s_2$.** 設 $f_i(r, \theta, s_i z)\ (i=1,2)$ 爲 $(r, \theta, s_i z)$ 的調和函數, 則顯而易見

$$F=f_1(r, \theta, s_1 z)+f_2(r, \theta, s_2 z) \tag{36}$$

即爲 (30) 的解. 但是這個解答並不是方程式 (30) 的通解, 因爲除此之外, 尙有一類解答不能包括在 (36) 中. 這類解答的形式如下:

$$F=B_0(r, \theta)+z B_1(r, \theta), \tag{37}$$

其中 B_0 和 B_1 是只包含 r, θ 的雙調和函數. 然而像 (37) 式的解答在本節的問題中是不需要的, 因此我們設 F 的形式是 (36). 如果命

$$\frac{\partial f_1}{\partial z}=\varphi_1(r, \theta, s_1 z), \quad \frac{\partial f_2}{\partial z}=\varphi_2(r, \theta, s_2 z), \tag{38}$$

而把公式 (29) 中的 $\varphi(r, \theta\, s_0 z)$ 叫做 φ_0, 則 (29) 式便簡化爲

$$u_r=-\sum_{i=1,2} \frac{\partial \varphi_i}{\partial r}-\frac{1}{r} \frac{\partial \varphi_0}{\partial \theta}, \quad u_\theta=-\sum_{i=1,2} \frac{1}{r} \frac{\partial \varphi_i}{\partial \theta}+\frac{\partial \varphi_0}{\partial r},$$

$$w=\sum_{i=1,2} \frac{\gamma s_i^2-\alpha}{s_i^2} \frac{\partial \varphi_i}{\partial z}, \tag{39}$$

其中 $\varphi_i(r, \theta, s_i z)\ (i=0,1,2)$ 是 $(r, \theta, s_i z)$ 的調和函數.

相當於這組位移的應力是

** 如果 $s_1=s_2$, 則方程式 (24) 和 (30) 變爲雙調和方程式, 因此本問題的解法和相當的各向同性問題的解法沒有很大的區別.

$$
\frac{\sigma_r}{B_{66}} = \sum_{i=1,2} \frac{q_i}{s_i^2} \frac{\partial^2 \varphi_i}{\partial z^2} - 2 \left\{ \sum_{i=1,2} \frac{\partial^2 \varphi_i}{\partial r^2} + \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \varphi_0}{\partial \theta} \right\},
$$

$$
\frac{\sigma_\theta}{B_{66}} = \sum_{i=1,2} \frac{q_i}{s_i^2} \frac{\partial^2 \varphi_i}{\partial z^2} - 2 \left\{ \sum_{i=1,2} \left(\frac{1}{r} \frac{\partial \varphi_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi_i}{\partial \theta^2} \right) - \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \varphi_0}{\partial \theta} \right\},
$$

$$
\frac{\sigma_z}{B_{66}} = \sum_{i=1,2} \frac{k_i}{s_i^2} \frac{\partial^2 \varphi_i}{\partial z^2},
$$

$$
\frac{\tau_{\theta z}}{B_{66}} = \frac{1}{s_0^2} \sum_{i=1,2} \frac{(\gamma - 1) s_i^2 - \alpha}{s_i^2} \frac{1}{r} \frac{\partial^2 \varphi_i}{\partial \theta \, \partial z} + \frac{\partial^2 \varphi_0}{\partial r \, \partial z}, \tag{40}
$$

$$
\frac{\tau_{rz}}{B_{66}} = \frac{1}{s_0^2} \sum_{i=1,2} \frac{(\gamma - 1) s_i^2 - \alpha}{s_i^2} \frac{\partial^2 \varphi_i}{\partial r \, \partial z} - \frac{1}{r} \frac{\partial^2 \varphi_0}{\partial \theta \, \partial z},
$$

$$
\frac{\tau_{r\theta}}{B_{66}} = -2 \sum_{i=1,2} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \varphi_i}{\partial \theta} + \frac{\partial^2 \varphi_0}{\partial r^2} - \frac{1}{r} \frac{\partial \varphi_0}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \varphi_0}{\partial \theta^2},
$$

其中

$$
q_i = \frac{\alpha - 2\gamma s_0^2 + (\gamma s_i^2 - \alpha)(1 - \gamma)}{\gamma s_0^2}, \quad k_i = \frac{1 - \gamma + \beta(\gamma s_i^2 - \alpha)}{\gamma s_0^2}. \tag{41}
$$

根據維分析的原理，本問題中的應力、位移以及位移函數 $\varphi_j\,(j = 0, 1, 2)$ 都必須是 (r, z) 的齊次函數，因爲在所設問題中找不到一個長度標準能够把坐標 (r, θ, z) 變成無維坐標．平衡條件要求應力必須是 (r, z) 的負二次齊次函數，因此 $\varphi_j\,(j = 0, 1, 2)$ 必爲 (r, z) 的零次齊次函數．最後照顧到對稱的情形，我們假定

$$
\varphi_0 = \sin\theta\, \psi_0(r, z), \quad \varphi_1 = \cos\theta\, \psi_1(r, z), \quad \varphi_2 = \cos\theta\, \psi_2(r, z),
$$

其中 $\psi_j\,(j = 0, 1, 2)$ 是 (r, z) 的零次齊次函數，並且當 $r = 0$ 而 $z \neq 0$ 時是有限的．這樣的調和函數是

$$
\varphi_0 = \frac{A_0 r \sin\theta}{\sqrt{r^2 + s_0^2 z^2} + s_0 z}, \quad \varphi_1 = \frac{A_1 r \cos\theta}{\sqrt{r^2 + s_1^2 z^2} + s_1 z},
$$

$$
\varphi_2 = \frac{A_2 r \cos\theta}{\sqrt{r^2 + s_2^2 z^2} + s_2 z}. \tag{42}
$$

其中 $A_j\,(j=0,1,2)$ 是常數. 將 (42) 式代入 (40), 我們得本問題的應力算式如下:

$$\frac{\sigma_r}{B_{66}}=\cos\theta\left\{\sum_{i=1,2}A_i\frac{q_i r(r^2+2s_i^2z^2+2s_iz\sqrt{r^2+s_i^2z^2})+2rs_iz(s_iz+2\sqrt{r^2+s_i^2z^2})}{(r^2+s_i^2z^2+s_iz\sqrt{r^2+s_i^2z^2})^2\sqrt{r^2+s_i^2z^2}}\right.$$

$$\left.-2A_0\frac{s_0z-\sqrt{r^2+s_0^2z^2}}{r(r^2+s_0^2z^2+s_0z\sqrt{r^2+s_0^2z^2})}\right\},$$

$$\frac{\sigma_\theta}{B_{66}}=\cos\theta\left\{\sum_{i=1,2}A_i\left[\frac{q_ir}{(r^2+s_i^2z^2)\sqrt{r^2+s_i^2z^2}}-\frac{2(s_iz-\sqrt{r^2+s_i^2z^2})}{r(r^2+s_i^2z^2+s_iz\sqrt{r^2+s_i^2z^2})}\right]\right.$$

$$\left.+2A_0\frac{s_0z-\sqrt{r^2+s_0^2z^2}}{r(r^2+s_0^2z^2+s_0z\sqrt{r^2+s_0^2z^2})}\right\}, \tag{43}$$

$$\frac{\sigma_z}{B_{66}}=\cos\theta\sum_{i=1,2}\frac{A_ik_ir}{(r^2+s_i^2z^2)\sqrt{r^2+s_i^2z^2}},$$

$$\frac{\tau_{\theta z}}{B_{66}}=\sin\theta\left\{\frac{1}{s_0^2}\sum_{i=1,2}\frac{A_i[(\gamma-1)s_i^2-\alpha]}{s_i(r^2+s_i^2z^2+s_iz\sqrt{r^2+s_i^2z^2})}\right.$$

$$\left.+\frac{A_0[(r^2-s_0^2z^2)\sqrt{r^2+s_0^2z^2}-s_0^3z^3]}{(r^2+s_0^2z^2+s_0z\sqrt{r^2+s_0^2z^2})^2\sqrt{r^2+s_0^2z^2}}\right\},$$

$$\frac{\tau_{rz}}{B_{66}}=\cos\theta\left\{\frac{1}{s_0^2}\sum_{i=1,2}\frac{A_i[(\gamma-1)s_i^2-\alpha][(r^2-s_i^2z^2)\sqrt{r^2+s_i^2z^2}-s_i^3z^3]}{s_i(r^2+s_i^2z^2+s_iz\sqrt{r^2+s_i^2z^2})^2\sqrt{r^2+s_i^2z^2}}\right.$$

$$\left.+\frac{A_0}{r^2+s_0^2z^2+s_0z\sqrt{r^2+s_0^2z^2}}\right\},$$

$$\frac{\tau_{r\theta}}{B_{66}}=\sin\theta\left\{2\sum_{i=1,2}\frac{A_i(s_iz-\sqrt{r^2+s_i^2z^2})}{r(r^2+s_i^2z^2+s_iz\sqrt{r^2+s_i^2z^2})}\right.$$

$$\left.+\frac{A_0(r^3-2rs_0z\sqrt{r^2+s_0^2z^2})}{(r^2+s_0^2z^2+s_0z\sqrt{r^2+s_0^2z^2})^2\sqrt{r^2+s_0^2z^2}}\right\}.$$

在圓錐的側面應無外力作用, 所以 $A_j\,(j=0,1,2)$ 應適合下面的邊界條件:

當 $r:z=1:\cot\alpha$ 時,

$$\tau_{rz}-\sigma_r\cot\alpha=0,\quad \tau_{\theta z}-\tau_{r\theta}\cot\alpha=0,\quad \sigma_z-\tau_{rz}\cot\alpha=0. \tag{44}$$

在圓錐的任意一截面 $z=c$ 上，應力的合力應和作用在角頂的外力 P 維持平衡；所以 $A_j\,(j=0,1,2)$ 應適合方程式

$$\int_0^{z\tan\alpha}\int_0^{2\pi}(\tau_{rz}\cos\theta-\tau_{\theta z}\sin\theta)\,r\,dr\,d\theta=-P, \tag{45-a}$$

$$\int_0^{z\tan\alpha}\int_0^{2\pi}(\tau_{rz}\sin\theta+\tau_{\theta z}\cos\theta)\,r\,dr\,d\theta=0,\quad \int_0^{z\tan\alpha}\int_0^{2\pi}\sigma_z\,r\,dr\,d\theta=0. \tag{45-b}$$

極易證明，不論 $A_j\,(j=0,1,2)$ 等於何值，(45-b) 就已經適合．同時經過一個繁長的演算之後，亦可以證明 (44) 的三個方程式並不是獨立無關的，實際上只有兩個是獨立的．所以 (44) 會同 (45-a) 後有而且只有一個解答．把這解得的 $A_j\,(j=0,1,2)$ 代入 (43) 式，便得橫觀各向同性的圓錐（假定其軸垂直其物性的各向同性面）在角頂負担一個橫向載荷所產生的應力．

當 $\alpha=\dfrac{\pi}{2}$ 時，這時候我們的問題變為橫觀各向同性的半無限體（假定其表面平行其物性的各向同性面）在其表面負担一個平行其表面的集中載荷，(44) 和 (45-a) 簡化為

$$\begin{gathered}
\frac{(\gamma-1)s_1^2-\alpha}{s_1}A_1+\frac{(\gamma-1)s_2^2-\alpha}{s_2}A_2+s_0^2A_0=0,\\
\frac{(\gamma-1)s_1^2-\alpha}{s_1}A_1+\frac{(\gamma-1)s_2^2-\alpha}{s_2}A_2+s_0^2A_0=0,\\
k_1A_1+k_2A_2=0,\\
-\frac{(\gamma-1)s_1^2-\alpha}{s_1}A_1-\frac{(\gamma-1)s_2^2-\alpha}{s_2}A_2+s_0^2A_0=-\frac{s_0^2}{\pi}P.
\end{gathered} \tag{46}$$

解之得

$$A_0=\frac{-P}{2\pi},\quad A_1=\frac{-P}{2\pi}\cdot\frac{s_0^2}{k_1}\cdot\frac{1}{\dfrac{(\gamma-1)s_2^2-\alpha}{s_2k_2}-\dfrac{(\gamma-1)s_1^2-\alpha}{s_1k_1}},$$

$$A_2=\frac{-P}{2\pi}\frac{s_0^2}{k_2}\frac{1}{\dfrac{(\gamma-1)s_1^2-\alpha}{s_1k_1}-\dfrac{(\gamma-1)s_2^2-\alpha}{s_2k_2}}. \tag{47}$$

將此組 $A_j(j=0,1,2)$ 代入 (43) 式, 便得橫觀各向同性的半無限體 (假定其表面平行其物性的各向同性面) 在其表面負担一個平行其表面的集中載荷 P 所產生的應力. С. Г. 列赫尼茨基[參 3] 已經解决了同上的半無限體在其表面負担一個垂直其表面的集中載荷的問題. 解决了上面的兩個問題之後, 則同上的半無限體在其表面負担任意載荷的問題即可根據疊加原理用一積分求得.

作者在研究本問題時得到清華大學錢偉長教授熱心的支持與鼓勵, 謹表示深切的謝意.

參 考 文 獻

1.* М. М. Фридман: Математическая теория упругости анизотропных сред (обзор). Прикладная математика и механика, Том 14, Стр. 321.

2. С. Г. Лехницкий: Теория упругости анизотропного тела. Гостехиздат. 1950. М.—Л.

3. С. Г. Лехницкий: Симметричная деформация и кручение тела вращения с анизотропией частного вида. Прикладная математика и механика. Том 4, Вып. 3, 1940.

* 已由本文作者譯成中文, 不久即將出版.

ON THE THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY OF A TRANSVERSELY ISOTROPIC BODY*

HU HAI-CHANG (胡海昌)

Institute of Mathematics, Academia Sinica

ABSTRACT

In this paper, we follow S. G. Lehnitzky[3] in the study of the three-dimensional problems of a transversely isotropic body depending on three variables. We give up the restriction of S. G. Lehnitzky about deformation and reserve only the assumption about the elastic property of the body. The results of this paper give a wide possibility for the solution of special problems.

Consider a transversely isotropic elastic body in equilibrium. Take a rectangular coordinate system (x, y, z) with the axis z perpendicular to the plane of isotropy of the body. In this system of coordinates, the equation of equilibrium in terms of displacement (u, v, w) is

$$\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2}\right)u+B_{12}\frac{\partial^2 v}{\partial x\partial y}+B_{13}\frac{\partial^2 w}{\partial x\partial z}=0,$$

$$B_{12}\frac{\partial^2 u}{\partial x\partial y}+\left(B_{66}\frac{\partial^2}{\partial x^2}+B_{11}\frac{\partial^2}{\partial y^2}+B_{44}\frac{\partial^2}{\partial z^2}\right)v+B_{13}\frac{\partial^2 w}{\partial y\partial z}=0, \tag{6}$$

$$B_{13}\frac{\partial^2 u}{\partial x\partial z}+B_{13}\frac{\partial^2 v}{\partial y\partial z}+\left(B_{44}\frac{\partial^2}{\partial x^2}+B_{44}\frac{\partial^2}{\partial y^2}+B_{33}\frac{\partial^2}{\partial z^2}\right)w=0,$$

* Received November 29, 1952.

where $B_{66}=B_{11}-B_{12}$ and $B_{11},\cdots,B_{66}$ are constants depending on the elastic property of the body. The problem is to find the general solution of (6). A particular solution of (6) may be found by expressing u, v, w in terms of a function F as follows:

$$u=-\frac{\partial^2 F}{\partial x\partial z},\quad v=-\frac{\partial^2 F}{\partial y\partial z},\quad w=\frac{B_{11}}{B_{13}}\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{B_{44}}{B_{11}}\frac{\partial^2 F}{\partial z^2}\right), \tag{10}$$

where F satisfies the equation

$$\left[B_{11}B_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2+(B_{44}^2+B_{11}B_{33}-B_{13}^2)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial z^2}\right.$$
$$\left.+B_{33}B_{44}\frac{\partial^4}{\partial z^4}\right]F=0. \tag{11}$$

It can be verified that Eq. (11) is the three-dimensional form of the equation satisfied by the stress function introduced by S. G. Lehnitzky for the solution of the symmetrical deformation of a body of revolution. The general solution of Eq. (6) may be obtained by expressing (u, v, w) in terms of two functions F and φ as follows:

$$u=-\frac{\partial^2 F}{\partial x\partial z}-\frac{\partial\varphi}{\partial y},\quad v=-\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial\varphi}{\partial x},$$
$$w=\frac{B_{11}}{B_{13}}\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{B_{44}}{B_{11}}\frac{\partial^2 F}{\partial z^2}\right), \tag{23}$$

where F satisfies Eq. (11), and φ satisfies the equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{B_{44}}{B_{66}}\frac{\partial^2\varphi}{\partial z^2}=0. \tag{25}$$

In cylindrical coordinate system (r, θ, z) the general solution of the displacement (u_r, u_θ, w) may be obtained from formula (23) through a vectorial transformation. Two particular cases may be noted. The first is the torsion of a transversely isotropic body of revolution. Here we take

$$F = 0, \quad \varphi = \varphi(r, z). \tag{32}$$

The second is the symmetrical deformation of the same body. Here we take

$$F = F(r, z), \quad \varphi = 0. \tag{33}$$

Finally, we apply the general formula to the case of a transversely iotropic circular cone with axis perpendicular to the plane of isotropy loaded by a transverse force at its vertex.