圓弧二邊形柱體的扭轉問題*

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一. 前 言

圓弧二邊形在數學上不算一個很複雜的圖形，而在實用上却可近似地描寫許多有用的截面，因此圓弧二邊形柱體的扭轉問題，受到許多學者注意. 最簡單的圓弧二邊形顯然要推半圓形，因此這種柱體的扭轉問題解決得最早. 早在 1881 年，A. G. 格林希耳[1]便求得形式緊縮的解答. 較繁一些的圖形是正交圓弧二邊形. 這類形狀的柱體的扭轉問題，最先由 T. H. 格郎瓦耳[2] 加以研究. 他利用保角映象和複格林函數，求得了共軛函數和最大剪應力的形式緊縮的算式. 但他沒有充份簡化他的算式，以致錯誤地認爲抗扭剛度不可能用初等函數的有限算式來表示.

最近 Я. С. 烏弗良特[3,4]全面地研究了圓弧二邊形柱體的扭轉問題. 他利用雙極坐標和福里哀積分，求得了任意圓弧二邊形柱體的應力函數，最大剪應力和抗扭剛度的算式. 這是重要的貢獻. 但是他的算式還未充份簡化，以致在計算實用上重要的數據時，需化費很大的人力.

本文重新全面地討論了圓弧二邊形柱體的扭轉問題. 我們雖然還是採用雙極坐標和福里哀積分，但是充份簡化了算式，並且指出，當兩圓弧的交角與 π 之比爲一有理數時，不但應力函數（因而最大剪應力），並且抗扭剛度，都可以用初等函數（三角函數和雙曲線函數）的有限算式來表示.

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[1] Greenhill, A. G., On the motion of a frictionless liquid in a rotating sector, *Messenger of Math.* **10** (1881), 83.

[2] Gronwall, T. H., On the influence of keyways on the stress distribution in cylindrical shafts, *Trans. Amer. Math. Soc.* **20** (1919), 234.

[3] Уфлянд, Я. С., Кручение призматического стержня с профилем ограниченным дугами двух пересекающихся окружностей, ДАН СССР, Том **68** (1949), Вып. 1.

[4] ————, Биполярные координаты в теории упругости, Гостехиздат 1950.

本文最後特別討論了正交圓弧二邊形柱體的扭轉問題．我們將最大剪應力和抗扭剛度都用三角函數的簡單有限算式表示出來．

二．扭轉問題的建立

今設圓弧二邊形柱體的截面如圖 1 所示，二圓弧的半徑各爲 r_1 和 r_2，公共弦的長度爲 $2a$．取一直角坐標系 (x, y) 如圖 1 所示．再根據這個坐標系選用雙極坐標系 (ξ, η) 如下：

$$x + iy = a \tanh \frac{\xi + i\eta}{2}, \quad \xi + i\eta = \log \frac{a + x + iy}{a - x - iy}. \tag{1}$$

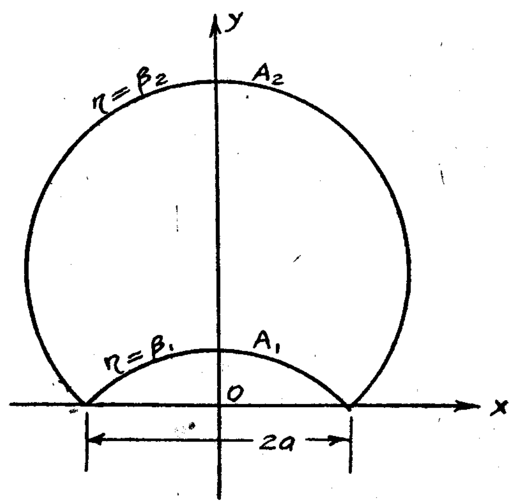

圖 1.

採用這樣的坐標系，二邊界圓弧的方程便可表示爲

$$\eta = \beta_1 \quad 和 \quad \eta = \beta_2, \tag{2}$$

其中 β_1，β_2 與 r_1，r_2 及 a 的關係爲

$$|\sin \beta_1| = \frac{a}{r_1}, \qquad |\sin \beta_2| = \frac{a}{r_2}. \tag{3}$$

β 爲正爲負，視該圓弧在 x 軸上或 x 軸下而定．截面所佔的區域爲 $-\infty < \xi < \infty$，$\beta_1 \leqslant \eta \leqslant \beta_2$．

對於這種坐標系，容易求得

$$h = \left| \frac{d\,(x+iy)}{d\,(\xi+i\eta)} \right| = \frac{a}{\cosh\xi+\cos\eta}, \qquad (4)$$

$$x^2+y^2 = \frac{a\,(\cosh\xi-\cos\eta)}{\cosh\xi+\cos\eta}.$$

根據柱體的自由扭轉理論，應力函數 Ψ 應適合下面的泊松方程

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2} = -2, \qquad (5)$$

這在雙極坐標系裏變爲

$$\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2} = -\frac{2a^2}{(\cosh\xi+\cos\eta)^2}, \qquad (6)$$

和適合下面的邊界條件：

$$當\ \eta=\beta_1\ 和\ \eta=\beta_2\ 時,\quad \Psi=0. \qquad (7)$$

Ψ 的一個特解可取

$$\Psi_0 = \frac{a^2}{\sin\beta_2}\cdot\frac{\sin(\beta_2-\eta)}{\cosh\xi+\cos\eta}.$$

這相當於在直角坐標系裏取

$$\Psi_0 = -\frac{1}{2}(x^2+y^2) - ay\cot\beta_2 + \frac{a^2}{2}.$$

因此我們設

$$\Psi = \frac{a^2}{\sin\beta_2}\,\frac{\sin(\beta_2-\eta)}{\cosh\xi+\cos\eta}+\psi, \qquad (8)$$

其中共軛函數 ψ 應爲調和函數，並適合下面的邊界條件：

$$當\ \eta=\beta_2\ 時\quad \psi=0; \qquad (9\text{-a})$$

$$當\ \eta=\beta_1\ 時\quad \psi=-\frac{a^2}{\sin\beta_2}\cdot\frac{\sin(\beta_2-\beta_1)}{\cosh\xi+\cos\beta_1}. \qquad (9\text{-b})$$

因爲 ξ 變化的範圍爲 $-\infty<\xi<\infty$，所以 ψ 適宜於展成 ξ 的福里哀積分．照顧到關於 y 軸的對稱關係和邊界條件（9-a），我們設

$$\psi=\int_0^\infty A(m)\sinh m(\beta_2-\eta)\cos m\xi\,dm\,, \tag{10}$$

其中 $A(m)$ 爲待定函數．將此式代入邊界條件（9-b），得

$$\int_0^\infty A(m)\sinh m(\beta_2-\beta_1)\cos m\xi\,dm=-\frac{a^2}{\sin\beta_2}\cdot\frac{\sin(\beta_2-\beta_1)}{\cosh\xi+\cos\beta_1}. \tag{11}$$

根據福里哀積分理論可知

$$A(m)=-\frac{2}{\pi}\,\frac{a^2\sin(\beta_2-\beta_1)}{\sin\beta_2}\int_0^\infty\frac{\cos m\xi\,d\xi}{\cosh\xi+\cos\beta_1}.$$

注意到

$$\int_0^\infty\frac{\cos m\xi\,d\xi}{\cosh\xi+\cos\beta_1}=\frac{\pi}{\sin\beta_1}\cdot\frac{\sinh m\beta_1}{\sinh m\pi}, \tag{A}$$

便得

$$A(m)=-2a^2(\cot\beta_1-\cot\beta_2)\frac{\sinh m\beta_1}{\sinh m\pi\sinh m(\beta_2-\beta_1)}.$$

因此共軛函數爲

$$\psi=-2a^2(\cot\beta_1-\cot\beta_2)\int_0^\infty\frac{\sinh m\beta_1\sinh m(\beta_2-\eta)}{\sinh m\pi\sinh m(\beta_2-\beta_1)}\cos m\xi\,dm. \tag{12}$$

於是最後得應力函數 Ψ 的算式如下：

$$\Psi=\frac{a^2}{\sin\beta_2}\cdot\frac{\sin(\beta_2-\eta)}{\cosh\xi+\cos\eta}-$$

$$-2a^2(\cot\beta_1-\cot\beta_2)\int_0^\infty\frac{\sinh m\beta_1\sinh m(\beta_2-\eta)}{\sinh m\pi\sinh m(\beta_2-\beta_1)}\cos m\xi\,dm. \tag{13}$$

這個公式比 Я. С. 烏弗良特求得的同一公式少了一個積分式．

從應力函數計算剪應力的公式是

$$\frac{\tau_{\xi z}}{\alpha G}=\frac{1}{h}\frac{\partial\Psi}{\partial\eta},\qquad\frac{\tau_{\eta z}}{\alpha G}=-\frac{1}{h}\frac{\partial\Psi}{\partial\xi}, \tag{14}$$

式中 α 是柱體的扭轉率（卽單位長度的扭轉角），G 是剪彈性係數．將（13）和（14）代入此式，得

$$\frac{\tau_{\xi z}}{\alpha Ga} = (\cosh\xi + \cos\eta)\left\{-\frac{1}{\sin\beta_2}\cdot\frac{\cosh\xi\cos(\beta_2-\eta)+\cos\beta_2}{(\cosh\xi+\cos\eta)^2} + \right.$$

$$\left. + 2(\cot\beta_1 - \cot\beta_2)\int_0^\infty \frac{m\sinh m\beta_1\cosh m(\beta_2-\eta)}{\sinh m\pi\sinh m(\beta_2-\beta_1)}\cos m\xi\, dm\right\}, \quad (15\text{-a})$$

$$\frac{\tau_{\eta z}}{\alpha Ga} = (\cosh\xi + \cos\eta)\left\{\frac{1}{\sin\beta_2}\cdot\frac{\sinh\xi\sin(\beta_2-\eta)}{(\cosh\xi+\cos\eta)^2} - \right.$$

$$\left. - 2(\cot\beta_1 - \cot\beta_2)\int_0^\infty \frac{m\sinh m\beta_1\sinh m(\beta_2-\eta)}{\sinh m\pi\sinh m(\beta_2-\beta_1)}\sin m\xi\, dm\right\}. \quad (15\text{-b})$$

從對稱關係可知最大剪應力必在圖 1 中的 $A_1(\xi=0,\eta=\beta_1)$ 和 $A_2(\xi=0,\eta=\beta_2)$ 兩點．利用薄膜相似性，可以看出當 $|\beta_2|\geqslant|\beta_1|$ 時，最大剪應力在 A_1 點．今後我們便假定 $|\beta_2|\geqslant|\beta_1|$．這個假定顯然無損於問題的普遍性．因此在公式（15-a）中命 $\xi=0$，$\eta=\beta_1$，得最大剪應力的算式如下：

$$\frac{\tau_{\max}}{\alpha Ga} = (1+\cos\beta_1)\left\{-\frac{1}{\sin\beta_2}\cdot\frac{\cos(\beta_2-\beta_1)+\cos\beta_2}{(1+\cos\beta_1)^2} + \right.$$

$$\left. + 2(\cot\beta_1 - \cot\beta_2)\int_0^\infty \frac{m\sinh m\beta_1\cosh m(\beta_2-\beta_1)}{\sinh m\pi\sinh m(\beta_2-\beta_1)}\, dm\right\}. \quad (16)$$

有了應力函數，便可計算柱體的抗扭剛度 $D=M/\alpha G$，其中 M 是扭力矩．計算 D 的公式是

$$D = 2\iint \Psi\, dx\, dy = 2\iint h^2\Psi\, d\xi\, d\eta. \quad (17)$$

將（13）式和（4）式代入（17）式，並經過一個繁長的簡化手續後，可以證明我們這樣求得的 D 的算式和 Я. С. 烏弗良特所求得的完全相同，卽

$$\frac{D}{a^4} = \frac{\eta - \sin\eta\cos\eta - 2\sin^3\eta\cos\eta}{2\sin^4\eta}\Bigg|_{\beta_1}^{\beta_2} -$$

$$- 4\pi(\cot\beta_1 - \cot\beta_2)^2\int_0^\infty \frac{m\sinh m\beta_1\sinh m\beta_2}{\sinh^2 m\pi\sinh m(\beta_2-\beta_1)}\, dm. \quad (18)$$

這個公式形式上雖然簡單整齊,但是用於數字計算有時却有困難,因爲當 β_1 很小時,第一項和第二項都是很大的數目,而兩者之差却是一個頗小的數目. 所以在這種場合,數字結果的準確性毫無保證. 爲了使數字計算方便起見,可將原式中的積分稍加變換如下:

$$\int_0^\infty \frac{m \sinh m\beta_1 \sinh m\beta_2}{\sinh^2 m\pi \sinh m(\beta_2-\beta_1)}\, dm = \int_0^\infty \frac{m \sinh^2 m\beta_1 \cosh m(\beta_2-\beta_1)}{\sinh^2 m\pi \sinh m(\beta_2-\beta_1)}\, dm + \\ + \int_0^\infty \frac{m \sinh m\beta_1 \cosh m\beta_1}{\sinh^2 m\pi}\, dm, \qquad (19)$$

式中第二個積分可以積出

$$\int_0^\infty \frac{m \sinh m\beta_1 \cosh m\beta_1}{\sinh^2 m\pi}\, dm = \frac{1}{2}\int_0^\infty \frac{m \sinh 2m\beta_1}{\sinh^2 m\pi}\, dm = \\ = \frac{1}{4\pi}\left(\frac{\beta_1}{\sin^2 \beta_1} - \cot \beta_1\right). \qquad (B)$$

這樣 (18) 式便變爲

$$\frac{D}{a^4} = \frac{\beta_2 - \sin\beta_2\cos\beta_2 - 2\sin^3\beta_2\cos\beta_2}{2\sin^4\beta_2} - \frac{1}{2\sin^4\beta_1}\Big\{3(\beta_1 - \sin\beta_1\cos\beta_1) - \\ - 2\beta_1\sin^2\beta_1 + 2\cot\beta_2\sin\beta_1(\cot\beta_2\sin\beta_1 - 2\cos\beta_1)\times(\beta_1 - \sin\beta_1\cos\beta_1)\Big\} - \\ - 4\pi(\cot\beta_1 - \cot\beta_2)^2 \int_0^\infty \frac{m \sinh^2 m\beta_1 \cosh m(\beta_2-\beta_1)}{\sinh^2 m\pi \sinh m(\beta_2-\beta_1)}\, dm. \qquad (20)$$

在本文中, (18) 和 (20) 兩式同時並用. 當 β_1 很小時, 求 D 用 (20) 式, 當 β_1 比較大時,則用 (18) 式.

當 $\beta_1=0$ 時,最大剪應力和抗扭剛度的算式 (16) 和 (20) 簡化爲:

$$\frac{\tau_{\max}}{aG\alpha} = -\cot\beta_2 + 4\int_0^\infty \frac{m^2 \coth m\beta_2}{\sinh m\pi}\, dm, \qquad (21\text{-a})$$

$$\frac{D}{a^4} = \frac{\beta_2 - \sin\beta_2\cos\beta_2 - 2\sin^3\beta_2\cos\beta_2}{2\sin^4\beta_2} + \\ + \frac{4}{3}\cot\beta_2 - 4\pi\int_0^\infty \frac{m^3 \coth m\beta_2}{\sinh^2 m\pi}\, dm. \qquad (21\text{-b})$$

三. 解答的簡化

上節已求得了扭轉問題中的應力函數的算式(13)、最大剪應力的算式 (16) 和抗扭剛度的算式 (18) 和 (20). 在每一個公式中都有一個無窮積分. 當 β_1 和 β_2 爲任意數時,這些積分不易或竟不可能用初等函數的有限算式來表示. 但是,正如本節將欲指出,當 $(\beta_2-\beta_1)/\pi$ 爲一有理數時,這些積分可以用初等函數(三角函數和雙曲線函數)的有限算式來表示.

當 $(\beta_2-\beta_1)/\pi$ 爲一有理數時, 它必可用兩個不可約的正整數 s, r 表示如下:

$$\frac{\beta_2-\beta_1}{\pi}=\frac{r}{s}. \tag{22}$$

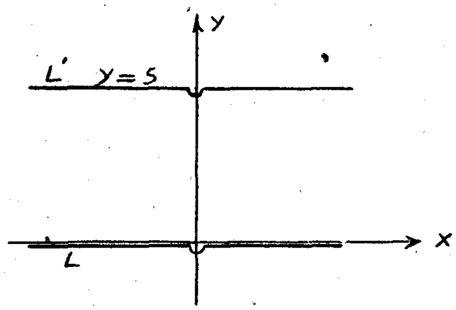

圖 2.

爲了簡明起見,玆先討論下列複變數 $z=x+iy$ (這裏的 x, y 與上節中的直角坐標完全無關)的圍線積分:

$$I\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L\frac{e^{\alpha z}\,dz}{\sinh^2\pi z\,\sinh\frac{r\pi}{s}z},\quad \left|Rl\cdot\alpha\right|<\frac{r+2s}{s}\pi, \tag{23}$$

式中的積分線 L 係由實軸加上在原點的極小的向下圓弧構成,如圖 2 所示. 玆再取一由直線 $y=s$ 加上橫跨 y 軸的極小的向下圓弧構成的圍線 L'. 因爲

$$\frac{1}{2}\int_{L'}\frac{e^{\alpha z}\,dz}{\sinh^2\pi z\,\sinh\frac{r\pi}{s}z}=\frac{1}{2}\int_L\frac{e^{\alpha(z+is)}\,dz}{\sinh^2\pi(z+is)\,\sinh\frac{r\pi}{s}(z+is)}=$$

$$=(-1)^r e^{is\alpha}\frac{1}{2}\int_L\frac{e^{\alpha z}\,dz}{\sinh^2\pi z\,\sinh\frac{r\pi}{s}z}=(-1)^r e^{is\alpha}I\left(\frac{r}{s},\alpha\right), \tag{24}$$

所以根據古西餘值定理得

$$\left[1-(-1)^r e^{is\alpha}\right] I = \pi i \left\{ \text{被積函數在 } LL' \text{ 內的餘值} \right\}. \tag{25}$$

被積函數在 LL' 內的異點有下列三類:

1) 在 $z=0$ 處,

$$\frac{e^{\alpha z}}{\sinh^2 \pi z \sinh \frac{r\pi}{s} z} = \frac{1+\alpha z+\frac{1}{2}\alpha^2 z^2}{\left(\pi z+\frac{1}{6}\pi^3 z^3\right)^2 \left(\frac{r\pi}{s} z+\frac{1}{6}\frac{r^3\pi^3}{s^3} z^3\right)} =$$

$$= \frac{s}{r\pi^3}\cdot\frac{1}{z^3}\left(1+\alpha z+\frac{1}{2}\alpha^2 z^2\right)\left(1+\frac{1}{6}\pi^2 z^2\right)^{-2}\left(1+\frac{1}{6}\frac{r^2\pi^2}{s^2} z^2\right)^{-1},$$

$$\therefore r_1 = \frac{s}{r\pi^3}\left(\frac{\alpha^2}{2}-\frac{\pi^2}{3}-\frac{r^2\pi^2}{6s^2}\right); \tag{26-a}$$

2) 在 $z=ik$ 處 $(k=1,2,\cdots,s-1)$. 命 $z=\zeta+ik$, 則得

$$\frac{e^{\alpha(\zeta+ik)}}{\sinh^2 \pi(\zeta+ik) \sinh \frac{r\pi}{s}(\zeta+ik)} = \frac{e^{ik\alpha}(1+\alpha\zeta)}{\sinh^2 \pi\zeta\left(i\sin\frac{kr\pi}{s}+\zeta\frac{r\pi}{s}\cos\frac{kr\pi}{s}\right)} =$$

$$= \frac{e^{ik\alpha}}{i\pi^2\sin\frac{kr\pi}{s}}\cdot\frac{1}{\zeta^2}\cdot\frac{1+\alpha\zeta}{1-i\zeta\frac{r\pi}{s}\cot\frac{kr\pi}{s}},$$

$$\therefore r_2 = \sum_{k=1}^{s-1}\frac{e^{ik\alpha}}{i\pi^2}\csc\frac{kr\pi}{s}\left(\alpha+i\frac{r\pi}{s}\cot\frac{kr\pi}{s}\right); \tag{26-b}$$

3) 在 $z=i\frac{ks}{r}$ 處 $(k=1,2,\cdots,r-1)$. 命 $z=\zeta+i\frac{ks}{r}$, 則得

$$\frac{e^{\alpha\left(\zeta+i\frac{ks}{r}\right)}}{\sinh^2 \pi\left(\zeta+i\frac{ks}{r}\right)\sinh\frac{r\pi}{s}\left(\zeta+i\frac{ks}{r}\right)} = \frac{(-1)^{k+1} e^{i\frac{ks\alpha}{r}}}{\sin^2\frac{ks\pi}{r}\sinh\frac{r\pi}{s}\zeta},$$

$$\therefore r_3 = \sum_{k=1}^{r-1}(-1)^{k+1}\frac{s}{r\pi}e^{i\frac{ks\alpha}{r}}\csc^2\frac{ks\pi}{r}. \tag{26-c}$$

將 (26) 式的三個結果代入 (25) 式,得

$$\left[1-(-1)^r e^{is\alpha}\right] I = \frac{is}{r\pi^2}\left(\frac{\alpha^2}{2}-\frac{\pi^2}{3}-\frac{r^2\pi^2}{6s^2}\right)+$$
$$+\frac{1}{\pi}\sum_{k=1}^{s-1} e^{ik\alpha}\csc\frac{kr\pi}{s}\left(\alpha+i\frac{r\pi}{s}\cot\frac{kr\pi}{s}\right)-$$
$$-\frac{is}{r}\sum_{k=1}^{r-1}(-1)^k e^{i\frac{ks\alpha}{r}}\csc^2\frac{ks\pi}{r}. \tag{27}$$

這個算式表明 $I(\frac{r}{s},\alpha)$ 可以用初等函數的有限算式來表示. I 是一個重要的積分,許多積分可以用 I 及其對於 α 的導數來表示. 但是 I 是一個複數,實用上不甚方便. 爲了避免這個缺點,可採用下面兩個積分作爲基本公式:

$$J\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L\frac{\sinh\alpha z\,dz}{\sinh^2\pi z\sinh\frac{r\pi}{s}z}=\frac{1}{2}\left\{I\left(\frac{r}{s},\alpha\right)-I\left(\frac{r}{s},-\alpha\right)\right\}, \tag{28-a}$$

$$K\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L\frac{\cosh\alpha z\,dz}{\sinh\pi z\sinh\frac{r\pi}{s}z}=\frac{1}{2}\left\{J\left(\frac{r}{s},\pi+\alpha\right)+J\left(\frac{r}{s},\pi-\alpha\right)\right\}. \tag{28-b}$$

經過一個繁長的計算後不難求得

$$2\left[1-(-1)^r\cos\alpha s\right] J=(-1)^{r+1}\frac{s}{r\pi^2}\sin\alpha s\left(\frac{\alpha^2}{2}-\frac{\pi^2}{3}-\frac{r^2\pi^2}{6s^2}\right)+$$
$$+\frac{1}{\pi}\sum_{k=1}^{s-1}\csc\frac{kr\pi}{s}\left\{\alpha\left[\cos k\alpha-(-1)^r\cos(s-k)\alpha\right]-\right.$$
$$\left.-\frac{r\pi}{s}\cot\frac{kr\pi}{s}\left[\sin k\alpha+(-1)^r\sin(s-k)\alpha\right]\right\}+$$
$$+\frac{s}{r}\sum_{k=1}^{r-1}(-1)^k\csc^2\frac{ks\pi}{r}\left\{\sin\frac{ks\alpha}{r}-(-1)^r\sin\frac{(r-k)s\alpha}{r}\right\}, \tag{29-a}$$

$$2\left[1-(-1)^{r+s}\cos\alpha s\right] K=(-1)^{r+s+1}\frac{\alpha s}{r\pi}\sin\alpha s+$$
$$+\sum_{k=1}^{s-1}(-1)^k\csc\frac{kr\pi}{s}\left[\cos k\alpha-(-1)^{r+s}\cos(s-k)\alpha\right]+$$
$$+\frac{r}{s}\sum_{k=1}^{r-1}(-1)^k\csc\frac{ks\pi}{r}\left[\cos\frac{ks\alpha}{r}-(-1)^{r+s}\cos\frac{(r-k)s\alpha}{r}\right]. \tag{29-b}$$

上節公式中遇到的無窮積分，都可以用 J 和 K 以及它們對於 α 的導數來表示．首先把那些公式中的 m 看作是複變數 z，其次用 $\frac{1}{2}\int_L$ 代替原來的 $\int_0^\infty$，最後將雙曲線函數的乘積化爲若干個雙曲線函數之和便得．這樣我們求得應力函數算式（13）中的積分

$$\int_0^\infty \frac{\sinh m\beta_1 \sinh m(\beta_2-\eta)}{\sinh m\pi \sinh \frac{r\pi}{s} m} \cos m\xi \, dm = \frac{1}{4}\Big\{K(\beta_1+\beta_2-\eta-i\xi)+ +K(\beta_1+\beta_2-\eta+i\xi)-K(\beta_2-\beta_1-\eta+i\xi)-K(\beta_1-\beta_2+\eta+i\xi)\Big\}; \quad (30)$$

最大剪應力算式（16）中的積分〔$K'(\alpha)=\partial K/\partial\alpha$〕

$$\int_0^\infty \frac{m \sinh m\beta_1 \cosh m(\beta_2-\beta_1)}{\sinh m\pi \sinh \frac{r\pi}{s} m} dm = \frac{1}{2}\Big\{K'(\beta_2)+K'(2\beta_1-\beta_2)\Big\}; \quad (31)$$

和抗扭剛度算式（18）中的積分〔$J'(\alpha)=\partial J/\partial\alpha$〕

$$\int_0^\infty \frac{m \sinh m\beta_1 \sinh m\beta_2}{\sinh^2 m\pi \sinh \frac{r\pi}{s} m} dm = \frac{1}{2}\Big\{J'(\beta_1+\beta_2)-J'(\beta_2-\beta_1)\Big\}. \quad (32)$$

四. 正交圓弧二邊形柱體的扭轉問題

正交圓弧二邊形是上節討論的情形中最簡單的一種．因爲這時 $(\beta_2-\beta_1)/\pi=\frac{1}{2}$，所以在（29）式中命 $r=1$，$s=2$，然後稍加簡化，得兩個基本積分公式如下：

$$J(\alpha)=\frac{1}{2}\int_L \frac{\sinh \alpha z \, dz}{\sinh^2 \pi z \sinh \frac{\pi}{2} z} = \left(\frac{\alpha^2}{2\pi^2}-\frac{3}{8}\right)\tan\alpha + \frac{\alpha}{2\pi\cos\alpha}, \quad (33\text{-a})$$

$$K(\alpha)=\frac{1}{2}\int_L \frac{\cosh \alpha z \, dz}{\sinh \pi z \sinh \frac{\pi}{2} z} = \frac{\alpha}{\pi}\tan\alpha - \frac{1}{2\cos\alpha}. \quad (33\text{-b})$$

根據這兩個基本積分，不難導得下列三個積分公式：

$$J'(\alpha) = \frac{1}{2}\int_L \frac{z\cosh\alpha z\, dz}{\sinh^2 \pi z \sinh\frac{\pi}{2}z} =$$

$$= \frac{\alpha}{\pi^2}\tan\alpha + \left(\frac{\alpha^2}{2\pi^2} - \frac{3}{8}\right)\frac{1}{\cos^2\alpha} + \frac{\cos\alpha + \alpha\sin\alpha}{2\pi\cos^2\alpha}, \qquad (34\text{-a})$$

$$J'\left(\frac{\pi}{2}\right) = \frac{1}{2}\int_L \frac{z\cosh\frac{\pi}{2}z\, dz}{\sinh^2 \pi z \sinh\frac{\pi}{z}z} = -\frac{1}{2\pi^2} - \frac{1}{8}, \qquad (34\text{-b})$$

$$K'(\alpha) = \int_0^\infty \frac{x\sinh\alpha x\, dx}{\sinh\pi x \sinh\frac{\pi}{2}x} = \frac{1}{\pi}\tan\alpha + \frac{2\alpha - \pi\sin\alpha}{2\pi\cos^2\alpha}. \qquad (34\text{-c})$$

為了書寫方便起見，在正交圓弧二邊形的場合，我們設 $\beta_2 = \beta$, $\beta_1 = \beta - \frac{\pi}{2}$. 於是最大剪應力和抗扭剛度的算式（16）和（18）變為

$$\frac{\tau_{\max}}{\alpha G a} = -\frac{2(1-\sin\beta)}{\sin 2\beta} - $$

$$-\frac{4(1+\sin\beta)}{\sin 2\beta}\int_0^\infty \frac{m\sinh m\left(\beta - \frac{\pi}{2}\right)\cosh\frac{\pi}{2}m}{\sinh m\pi \sinh\frac{\pi}{2}m}\, dm, \qquad (35)$$

$$\frac{D}{a^4} = \frac{\pi}{4\cos^4\beta} + \frac{8\beta\cos 2\beta}{\sin^4 2\beta} - \frac{4}{\sin^3 2\beta} - $$

$$-\frac{16\pi}{\sin^2 2\beta}\int_0^\infty \frac{m\sinh m\left(\beta - \frac{\pi}{2}\right)\sinh m\beta}{\sinh^2 m\pi \sinh\frac{\pi}{2}m}\, dm. \qquad (36)$$

利用（34）式不難求得

$$\int_0^\infty \frac{m\sinh m\left(\beta - \frac{\pi}{2}\right)\cosh\frac{\pi}{2}m}{\sinh m\pi \sinh\frac{m\pi}{2}}\, dm = \frac{1}{\pi}\tan\beta + \frac{2\beta - \pi}{2\pi\cos^2\beta}, \qquad (37)$$

$$\int_0^\infty \frac{m \sinh m\left(\beta-\frac{\pi}{2}\right) \sinh m\beta}{\sinh^2 m\pi \sinh \frac{m\pi}{2}} dm = -\frac{4\beta-\pi}{4\pi^2} \cot 2\beta + \frac{8\beta^2-4\pi\beta-\pi^2}{8\pi^2 \sin^2 2\beta} +$$

$$+\frac{\sin 2\beta-\left(2\beta-\frac{\pi}{2}\right)\cos 2\beta}{4\pi \sin^2 2\beta} + \frac{1}{4\pi^2} + \frac{1}{16}. \tag{38}$$

因此最後得

$$\frac{\tau_{\max}}{\alpha G a} = -\frac{2\beta-2\pi \sin \beta+\pi \sin^2 \beta+\sin 2\beta}{\pi \sin \beta \cos \beta (1-\sin \beta)}, \tag{39}$$

$$\frac{D}{a^4} = \frac{1}{\sin^4 2\beta}\left\{4\pi \sin^4 \beta+2(8\beta-\pi) \cos 2\beta-8 \sin 2\beta + \frac{2(4\beta-\pi) \sin 4\beta}{\pi} -\right.$$

$$\left.-\left(\frac{4}{\pi}+\pi\right)\sin^2 2\beta - \frac{2(8\beta^2-4\pi\beta-\pi^2)}{\pi}\right\}. \tag{40}$$

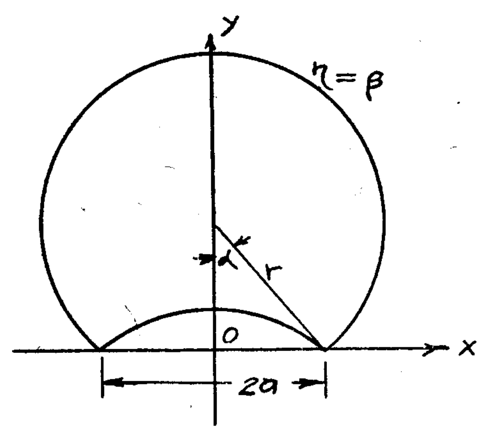

圖 3.

不難證明本節中最大剪應力的算式（39）和 T. H. 格郎瓦耳[2]所求得的相同. 參照圖 3，命

$$r = a \csc \beta, \qquad \alpha = \pi - \beta, \tag{41}$$

則（39）式變為

$$\frac{\tau_{\max}}{\alpha G r} = \frac{2(\pi-\alpha)-2\pi \sin \alpha+\pi \sin^2 \alpha-\sin 2\alpha}{\pi \cos \alpha (1-\sin \alpha)}. \tag{42}$$

這就是 T. H. 格郎瓦耳的算式.

當 β 趨近於 $\frac{\pi}{2}$ 時, (40) 式的極限是

$$\frac{D}{a^4} = \frac{\pi}{2} - \frac{4}{\pi}, \tag{43}$$

這個結果和 Sheng P. L.[5]最近求得的結果相同.

TORSION OF PRISMS BOUNDED BY TWO INTERSECTING CIRCULAR CYLINDERS*

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ABSTRACT

In this paper, the problem of torsion of prisms bounded by two intersecting circular cylinders is solved by means of Fourier's integrals. It is found that when the angle of intersection of these two circular cylinders is commensurable with π the stress function and the torsional rigidity of the prism can be expressed in closed form in terms of circular and hyperbolic sines and cosines.

Let the cross section of the prism be refered to bipolar coordinates (ξ, η) which connects with rectangular coordinates (x, y) as follows

$$x + iy = a \tanh \frac{\xi + i\eta}{2}, \quad \xi + i\eta = \log \frac{a + x + iy}{a - x - iy}. \tag{1}$$

Let the region occupied by the cross section is $-\infty < \xi < \infty$, $\beta_1 \leqslant \eta \leqslant \beta_2$, as shown in Fig. 1.

The stress function Ψ satisfies both Poisson's equation which in bipolar coordinates becomes

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} = -\frac{2a^2}{(\cosh \xi + \cos \eta)^2}, \tag{6}$$

[5] Sheng, P. L., Note on the torsional rigidity of semicircular bars, *Quart. Appl. Math.* **9** (1951), 309.

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and the following boundary condition:

$$\Psi = 0 \qquad \text{when} \qquad \eta = \beta_1 \quad \text{or} \quad \eta = \beta_2 . \tag{7}$$

As a special solution of Eq. (6), we take

$$\Psi_0 = \frac{a^2}{\sin \beta_2} \cdot \frac{\sin(\beta_2 - \eta)}{\cosh \xi + \cos \eta},$$

which satisfies the boundary condition at $\eta = \beta_2$. Therefore we write

$$\Psi = \frac{a^2}{\sin \beta_2} \cdot \frac{\sin(\beta_2 - \eta)}{\cosh \xi + \cos \eta} + \psi, \tag{8}$$

where ψ is a harmonic function. By expressing ψ as a Fourier's integral

$$\psi = \int_0^\infty A(m) \sinh m(\beta_2 - \eta) \cos m\xi \, dm, \tag{10}$$

it may be found that

$$A(m) = -2a^2 (\cot \beta_1 - \cot \beta_2) \frac{\sinh m\beta_1}{\sinh m\pi \sinh m(\beta_2 - \beta_1)} .$$

Hence we get finally

$$\Psi = \frac{a^2}{\sin \beta_2} \cdot \frac{\sin(\beta_2 - \eta)}{\cosh \xi + \cos \eta} - $$
$$- 2a^2 (\cot \beta_1 - \cot \beta_2) \int_0^\infty \frac{\sinh m\beta_1 \sinh m(\beta_2 - \eta)}{\sinh m\pi \sinh m(\beta_2 - \beta_1)} \cos m\xi \, dm . \tag{13}$$

The expression of the torsional rigidity D calculated from (13) coinsides with Ufliand's[3, 4] result

$$\frac{D}{a^4} = \frac{\eta - \sin\eta \cos\eta - 2\sin^3\eta \cos\eta}{2\sin^4\eta} \Bigg|_{\beta_1}^{\beta_2} - $$
$$- 4\pi (\cot \beta_1 - \cot \beta_2)^2 \int_0^\infty \frac{m \sinh m\beta_1 \sinh m\beta_2}{\sinh^2 m\pi \sinh m(\beta_2 - \beta_1)} dm . \tag{18}$$

When β_1 is a small number, this formula is not convenient for numerical calculation, for in this case, the first and the second terms are large numbers while their difference is a small number. For the convenience of numerical calculation, formula (18) may be transformed to the following form:

$$\frac{D}{a^4} \doteq \frac{\beta_2-\sin\beta_2\cos\beta_2-2\sin^3\beta_2\cos\beta_2}{2\sin^4\beta_2} - \frac{1}{2\sin^4\beta_1}\Big\{3(\beta_1-\sin\beta_1\cos\beta_1) -$$

$$-2\beta_1\sin^2\beta_1+2\cot\beta_2\sin\beta_1(\cot\beta_2\sin\beta_1-2\cos\beta_1)\times(\beta_1-\sin\beta_1\cos\beta_1)\Big\} -$$

$$-4\pi(\cot\beta_1-\cot\beta_2)^2\int_0^\infty \frac{m\sinh^2 m\beta_1\cosh m(\beta_2-\beta_1)}{\sinh^2 m\pi\sinh m(\beta_2-\beta_1)}\,dm\,. \tag{20}$$

In formulas (13) and (18), there are infinite integrals. For arbitrary values of β_1 and β_2 these integrals may not expressible in closed form in terms of elementary functions. When $\beta_2-\beta_1$ is commensurable with π, then it can be shown that they are expressible in closed form in terms of circular and hyperbolic sines and cosines.

When $\beta_2-\beta_1$ is commensurable with π, we can find two relative prime positive integers r and s such that

$$\frac{\beta_2-\beta_1}{\pi}=\frac{r}{s}\,. \tag{22}$$

Let us calculate first the following complex contour integral

$$I\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L \frac{e^{\alpha z}\,dz}{\sinh^2\pi z\sinh\frac{r\pi}{s}z}\,,\quad |Rl\cdot\alpha|<\frac{r+2s}{s}\pi\,, \tag{23}$$

where the contour L consists of the real axis with a small downward semi-circular arc at the origion as shown in Fig. 2. Take another contour L' which consists of a straight line $y=s$ with a small downward semi-circular arc at the point $(0,s)$. It is evident that

$$\frac{1}{2}\int_{L'} \frac{e^{\alpha z}\,dz}{\sinh^2\pi z\sinh\frac{r\pi}{s}z}=(-1)^r e^{is\alpha} I\left(\frac{r}{s},\alpha\right). \tag{24}$$

Therefore by taking $L+L'$ as a closed contour and using Cauchy's theorem of residues, we can find that

$$[1-(-1)^r e^{is\alpha}]\, I = \frac{is}{r\pi^2}\left(\frac{\alpha^2}{2}-\frac{\pi^2}{3}-\frac{r^2\pi^2}{6s^2}\right)+$$

$$+\frac{1}{\pi}\sum_{k=1}^{s-1} e^{ik\alpha}\csc\frac{kr\pi}{s}\left(\alpha+i\,\frac{r\pi}{s}\cot\frac{kr\pi}{s}\right)-$$

$$-\frac{is}{r}\sum_{k=1}^{r-1}(-1)^k e^{i\frac{ks\alpha}{r}}\csc^2\frac{ks\pi}{r}. \tag{27}$$

From this formula we can further derive the following two integrals:

$$J\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L \frac{\sinh \alpha z\, dz}{\sinh^2 \pi z \sinh\frac{r\pi}{s} z}=$$

$$=\frac{1}{2}\left\{I\left(\frac{r}{s},\alpha\right)-I\left(\frac{r}{s},-\alpha\right)\right\}, \tag{28-a}$$

$$K\left(\frac{r}{s},\alpha\right)=\frac{1}{2}\int_L \frac{\cosh \alpha z\, dz}{\sinh \pi z \sinh\frac{r\pi}{s} z}=$$

$$=\frac{1}{2}\left\{J\left(\frac{r}{s},\pi+\alpha\right)+J\left(\frac{r}{s},\pi-\alpha\right)\right\}. \tag{28-b}$$

Now the integrals appeared in formulas (13) and (18) can be expressed in terms of K, J and their derivatives with respect to α as follows:

$$\int_0^\infty \frac{\sinh m\beta_1 \sinh m(\beta_2-\eta)}{\sinh m\pi \sinh\frac{r\pi}{s} m}\cos m\xi\, dm=\frac{1}{4}\Big\{K(\beta_1+\beta_2-\eta-i\xi)+$$

$$+K(\beta_1+\beta_2-\eta+i\xi)-K(\beta_2-\beta_1-\eta+i\xi)-K(\beta_1-\beta_2+\eta+i\xi)\Big\}, \tag{30}$$

$$\int_0^\infty \frac{m\sinh m\beta_1 \sinh m\beta_2}{\sinh^2 m\pi \sinh\frac{r\pi}{s} m}\, dm=\frac{1}{2}\Big\{J'(\beta_1+\beta_2)-J'(\beta_2-\beta_1)\Big\}. \tag{32}$$

For prisms bounded by two orthogonal circular cylinders, we get the following results:

$$\frac{\tau_{\max}}{\alpha G a}=-\frac{2\beta-2\pi\sin\beta+\pi\sin^2\beta+\sin 2\beta}{\pi\sin\beta\cos\beta\,(1-\sin\beta)}, \tag{39}$$

$$\frac{D}{a^4} = \frac{1}{\sin^4 2\beta}\left\{4\pi \sin^4 \beta + 2\,(8\beta - \pi) \cos 2\beta - 8 \sin 2\beta - \left(\frac{4}{\pi} + \pi\right) \sin^2 2\beta + \right.$$
$$\left. + \frac{2\,(4\beta - \pi)}{\pi} \sin 4\beta - \frac{2\,(8\beta^2 - 4\,\pi\beta - \pi^2)}{\pi}\right\}. \tag{40}$$

Eq. (39) coinsides with Gronwall's[2] result. But Gronwall failed to obtain the closed expression for the torsional rigidity.