ON WEISS'S THEORY OF FIELDS

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ABSTRACT

Weiss's theory on the change of Schrödinger wave functional on a surface as the surface changes is given in a complete form, allowing the Lagrangian of the field to contain all derivatives of the field quantitiрs. The integrability of the resulting equation is proved by making use of the fact that the corresponding Hamilton-Jacobi equation is integrable. This gives at the same time a proof of the Lorentz invariancy of the commutation relations between the various conjugate variables, which so far remained obscure as soon as we allow derivatives higher than the second of the field quantities to appear in the Lagrangian.

In 1936, Weiss[1] discussed Schrödinger wave functionals on space-like surfaces and gave a formula for the change of the wave functional on a surface as the surface changes. His formulation is however restricted to what amounts to a one-parameter family of surfaces. In a previous paper by the author[2], extensions to general and arbitrary changes of the surfaces were made and it was shown that the wave equation in the extended form is integrable. Here we give the further extension to cases in which the Lagrangian of the field contains various derivatives of the field quantities. For clarity, we repeat part of the previous paper.

1. FORMULATION

Let x_μ be (x, y, z, ict) as usual and let q^α be the field quantities and let q^α_μ denote $\partial q^\alpha/\partial x_\mu$. Following Weiss, we let the Lagrangian L be a

1. Weiss, P. *Proc. Roy. Soc.*, **A 156** (1936), 192.
2. Chang. T. S. In course of publication.

function of q^α and q^α_μ and let q^α satisfy equations obtained from the variation of L, *i.e.*

$$\frac{\partial L}{\partial q^\alpha} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial q^\alpha{}_\mu}\right) = 0. \tag{1}$$

Let S be a space-like surface with its curvilinear coordinates (u_1, u_2, u_3) so that on it

$$x_\mu = b_\mu(u_1, u_2, u_3). \tag{2}$$

Let us introduce

$$N_\mu = \varepsilon_{\nu\varrho\theta\mu} \frac{\partial x_\nu}{\partial u_1} \frac{\partial x_\varrho}{\partial u_2} \frac{\partial x_\theta}{\partial u_3}, \tag{3}$$

where $\varepsilon_{\nu\varrho\theta\mu}$ is plus or minus one according to whether $\nu\varrho\theta\mu$ is an even or odd permutation of 1234 and is zero otherwise. Let us introduce p_α, G_μ, q^α_r by

$$p_\alpha = N_\mu (\partial L / \partial q^\alpha{}_\mu), \tag{4}$$

$$G_\mu = L N_\mu - \sum_\alpha p_\alpha q^\alpha{}_\mu, \tag{5}$$

$$q^\alpha{}_r = q^\alpha{}_\mu (\partial x_\mu / \partial u_r).^3 \qquad (r, s, \cdots = 1, 2, 3) \tag{6}$$

Let S' be a neighbouring surface with its curvilinear coordinates (u_1, u_2, u_3) and on it

$$x_\mu = b'_\mu(u) = b_\mu(u) + \varDelta x_\mu(u). \tag{7}$$

Denoting the Schrödinger wave functional Ψ on the parametrized surface $b_\mu(u)$ by $\langle m; b_\mu(u) | \rangle$ where m is an unspecified label for the coordinate of the wave functional and defining $\varDelta\Psi$ by

$$\langle m; b'_\mu(u) | \rangle - \langle m; b_\mu(u) | \rangle, \tag{8}$$

we introduce the wave equation

$$h i \varDelta \Psi = J \Psi, \tag{9}$$

$$J = i \int G_\mu \varDelta x_\mu \, du, \qquad (du = du_1\, du_2\, du_3) \tag{10}$$

3. For simplicity of writing, we drop the suffix α of p, q, q_μ, q_r, whenever this will not cause confusions.

where G_μ are functions of the symbols $q(u)$, $q_r(u)$, $p(u)$, $\partial b_\mu(u)/\partial u_r$ determined by (2)—(6) and q, q_r, p are now operators satisfying the familiar commutations laws

$$[p(u), q(u')] \equiv p(u)\, q(u') - q(u')\, p(u) = \hbar\, \delta(u-u'), \tag{11.1}$$

$$[p, p] = [q, q] = 0, \tag{11.2}$$

and

$$q_r = \partial q(u)/\partial u_r. \tag{12}$$

Obviously, we must prove that (9) as a total differential equation must be integrable, and that the expectation values of q at a point P constructed in the usual way in terms of $\langle m;\ b_\mu(u)|\rangle$ be independent of the choice of the functions $b_\mu(u)$ provided that $x_\mu = b_\mu(u)$ give a surface passing through P. The integrability will be proved in the next section (and also in the appendix) and the last point can be easily settled by making use of the fact that G_μ does not contain the derivatives of the operator p with respect to u_r. Let us assume that these two points are proved, then we may prove that the expectation values of q satisfy the field equations (1) by confining ourselves to surfaces x_4 = constant. In this special case, the Heisenberg's equation of motion are well known to be equivalent to (1).

2. INTEGRABILITY OF (9)

To prove that (9) is integrable, we calculate the change $\Delta\Psi$ of Ψ as the function $b_\mu(u)$ changes from the initial value $b_\mu^{(0)}(u)$ to the final value

$$b_\mu^{(f)}(u) = b_\mu^{(0)}(u) + \Delta_1 x_\mu(u) + \Delta_2 x_\mu(u) \tag{13}$$

in two ways, one by letting $b_\mu(u)$ pass through the intermediate function

$$b_\mu^{(1)}(u) = b_\mu^{(0)}(u) + \Delta_1 x_\mu(u) \tag{14}$$

and the other by letting $b_\mu(u)$ pass through the intermediate function

$$b_\mu^{(2)}(u) = b_\mu^{(0)}(u) + \Delta_2 x_\mu(u) \tag{15}$$

and compare the two results for $\Delta\Psi$ to the second order in $\Delta_1 x_\mu$, $\Delta_2 x_\mu$. If the two results are the same, the equation (9) is integrable.

In the first way, the change ot Ψ is given by

$$\frac{1}{\hbar}\left(\int_{(1)} G_\mu \Delta_2 x_\mu \, d u\right) \Psi_0 + \frac{1}{\hbar}\left(\int_{(0)} G_\mu \Delta_1 x_\mu \, d u\right) \Psi_0$$

$$+ \frac{1}{\hbar^2}\left(\int_{(1)} G_\mu \Delta_2 x_\mu \, d u\right)\left(\int_{(0)} G_\nu \Delta_1 x_\nu \, d u\right) \Psi_0 + [0(\Delta_1^2 x_\mu) + 0(\Delta_2^2 x_\mu)], \quad (16)$$

where Ψ_0 is the initial wave function, terms of the third order in Δx_μ are neglected, and the subscripts (1), (0) indicate what functions $b_\mu(u)$ are to be substituted for $x_\mu(u)$ in the G's in the integrand. The other $\Delta\Psi$ is given by (16) with the suffix 1,2 interchanged. On subtracting, the terms in the square bracket cancel out. If we neglect terms of the third order in Δx_μ, the difference is $1/\hbar$ times

$$\frac{1}{\hbar}\left(\int_{(0)} G_\mu \Delta_2 x_\mu \, d u\right)\left(\int_{(0)} G_\nu \Delta_1 x_\nu \, d u\right) \Psi_0$$

$$+ \left\{\left(\int_{(1)} - \int_{(0)}\right) G_\mu \Delta_2 x_\mu \, d u\right\} \Psi_0 - [1,2], \quad (17)$$

where [1,2] denotes all the expressions preceding it with the suffixes 1 and 2 interchanged.

(17) may be proved directly to be zero, and the direct proof is given in the appendix. However, the direct proof is complicated and is not easily extendable to cases in which the Lagrangian contains higher derivatives of the field quantities. For this reason, we give a simpler proof based on the integrability of the corresponding Hamilton-Jacobi equation.

The Hamilton-Jacobi equation from the Lagrangian L is simply

$$\Delta I = \int G_\mu \left(q, \frac{\partial q}{\partial u_r}, \frac{\delta I}{\delta q(u)}, \frac{\partial x_\mu}{\partial u_r}\right) \Delta x_\mu \, d u, \quad (18)$$

where G_μ is the same function of the arguments as that in (9), but the arguments are different. This is always integrable, provided that the field equations (1) are consistent and admit solutions, and the solution $I(q(u), b_\mu(u))$ of (18) is

$$\int L(q, q_\mu)\, d^4 x, \tag{19}$$

where the integral extends over a volume bounded by two space-like parametrized surfaces $b_\mu^{(00)}(u)$ and $b_\mu(u)$ and the surfaces at infinity (if the introduction of such to form a closed surface is necessary), q and q_μ in L satisfy the field equations and the whole integral is considered as a functional of $b_\mu^{(00)}$ and b_μ and the functions $q^{(00)}(u)$ and $q(u)$ which are the values $q(x)$ take on the surfaces. ΔI due to the change of the surface b_μ from $b^{(0)}$ to $b^{(1)}$ and from $b^{(1)}$ to $b^{(f)}$ is

$$\int_{(1)} G_\mu(I^{(1)})\, \Delta_2 x_\mu\, du + \int_{(0)} G_\mu(I^{(0)})\, \Delta_1 x_\mu\, du$$

$$+ \left(\int_{(1)} G_\mu(I^{(1)})\, \Delta_2 x_\mu\, du\right)\left(\int_{(0)} G_\nu(I^{(0)})\, \Delta_1 x_\nu\, du\right) + 0(\Delta_1^2 x_\mu) + 0(\Delta_2^2 x_\mu) \tag{20}$$

with obvious meanings for the notations $I^{(0)}$, $I^{(1)}$. Now, writing $p(u)$ for $\delta I/\delta q(u)$ we have

$$G_\mu(I^{(1)}, u') = G_\mu(I^{(0)}, u') + \int \frac{\delta G_\mu(u')}{\delta p(u'')}\, \frac{\delta(I^{(1)} - I^{(0)})}{\delta q(u'')}\, du''$$

$$= G_\mu(I^{(0)}, u') + \int \frac{\delta G_\mu(u')}{\delta p(u'')}\, \frac{\delta(\int G_\nu \Delta_1 x_\nu\, du)}{\delta q(u'')}\, du''. \tag{21}$$

Since (18) is integrable, (21) minus its [1, 2] is zero, or

$$\int_{(0)} \frac{\delta(\int G_\mu \Delta_2 x_\mu\, du)}{\delta p(u')}\, \frac{\delta(\int G_\nu \Delta_1 x_\nu\, du)}{\delta q(u')}\, du'$$

$$+ \left(\int_{(1)} - \int_{(0)}\right) G_\mu(I^{(0)})\, \Delta_2 x_\mu\, du - [1,2]$$

is zero. This is formally precisely the same operator in (17), if we replace $\delta I/\delta q(u)$ throughout by $p(u)$. Thus (17) is zero and the integrability of (9) established.

The proof also shows what changes in the formalism is necessary in case we have a Fermi Dirac field.

3. EXTENSION TO LAGRANGIANS WITH VARIOUS DERIVATIVES OF q

Let us extend the above theories to a Lagrangian $L(q, q_\mu, q_{\mu\nu}, \ldots)$ where $q_{\mu\nu}$ denotes $\partial^2 q/\partial x_\mu \partial x_\nu$ etc. (1) is replaced by

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial q_\mu} + (-)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial q_{\mu\nu}} + (-)^3 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\varrho}\frac{\partial L}{\partial q_{\mu\nu\varrho}} + \cdots = 0. \quad (22)$$

Let us form the complete variation of $I \equiv \int L d^4 x$. Let a part of the bounding surface be a plane extending to infinity and let us choose a system of reference (let us say, belonging to the observer $\underline{O}$) so that with respect to it, the plane is given by $\underline{x}_4 = \text{constant}$. ($\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, $\underline{x}_4$ are the coordinates of a point in this reference system.) Let $q^{(0)}$, $q^{(1)}$, $q^{(2)}, \ldots$ be defined by

$$q^{(0)} = q, \qquad q^{(1)} = \partial q/\partial \underline{x}_4 \qquad q^{(2)} = \partial^2 q/\partial \underline{x}_4^2, \qquad \cdots, \quad (23)$$

so that for a given $\underline{x}_4$, $L(x)$ may be considered as a functional of $q^{(0)}(\underline{\mathbf{x}})$, $q^{(1)}(\underline{\mathbf{x}}), \ldots$, $\underline{\mathbf{x}}$ denoting $(\underline{x}, \underline{y}, \underline{z})$. The complete variation of I due to the change of the boundary and that of $q(x)$ from a solution of (22) is a surface integral over the whole bounding surface, and the part of the surface integral over our plane $\underline{x}_4 = \text{constant}$ can be written as

$$\int d\underline{\mathbf{x}} \, \{G_\mu (D x_\mu) + \sum_{h=0}^{\varrho-1} p^{(h)} D q^{(h)}\}, \quad (24)$$

where Df is the change of f on the boundary and ϱ is the maximum number of suffixes 4 in $q_{\mu\nu}\ldots$ appearing in L. In (24), G_μ and $p^{(h)}$ are functionals of $q^{(0)}(\underline{\mathbf{x}})$, $q^{(1)}(\underline{\mathbf{x}}) \ldots$, $q^{(2\varrho-1)}(\underline{\mathbf{x}})$ determined by the procedure of taking the complete variation. By eliminating $q^{(\varrho)}(\underline{\mathbf{x}}) \ldots$, $q^{(2\varrho-1)}(\underline{\mathbf{x}})$ in a functional sense, we may express G_μ as functionals of $q^{(h)}(\underline{\mathbf{x}})$, $p^{(h)}(\underline{\mathbf{x}})$,

($h=0, 1, \ldots, \varrho-1$). Needless to say, the form of this functional depends on the position of the surface $x_4=$constant.

Restricting ourselves to planes only, we may now generalize (9) to our present case, i. e. choose a reference system O so that S is given by $x_4=$constant, determine G_μ in (24) as functionals of $q^{(h)}$, $p^{(h)}$, and let $\Delta\Psi$ in a change of S be given by

$$\hbar i \Delta \Psi = i \{ \int G_\mu \Delta x_\mu d \mathbf{x} \} \Psi, \tag{25}$$

in which $p(\mathbf{x})$ and $q(\mathbf{x})$ are operators satisfying the generalizations of (11), (12), *i.e.*

$$[p^{(h)}(\mathbf{x}), q^k(\mathbf{x}')] = \delta_{hk}\,\delta(\mathbf{x}-\mathbf{x}')\,\hbar, \tag{26}$$

etc. In the first place, (25) is integrable as will be shown in the next paragraph. In the second place, if L contains $q_{\mu\nu\cdots}$ in such a way that G_μ does not contain the space derivatives of $p^{(0)}$, the expectation value of q at a point P will be independent of what plane through P we have employed to find its Ψ and the corresponding expectation value of q at P. Finally we note that the expectation value satisfies (22) by specializing our surfaces to a family of parallel planes. That the Heisenberg's equation of motion for this special case reduces to (22) has been proved in a paper by the author[4] and in another by de Wet[5].

The proof of the integrability of (25) runs as follows. Let the domain of integration for I be bounded by a plane b, a plane $b^{(00)}$ and the surfaces at infinity. Consider I as a functional of the position of b, and $q^{(0)}, q^{(1)}, \ldots, q^{(\varrho-1)}$ on b, where $q^{(0)}, q^{(1)}, \ldots$ are defined by (23) after choosing the reference system O so that the plane b is given by $x_4=$constant, and similar quantities for $b^{(00)}$. Obviously

$$\delta I/\delta q^{(h)}(\mathbf{x}) = p^{(h)}(\mathbf{x}). \qquad (h=0,1,2,\cdots,\varrho-1) \tag{27}$$

Thus the change of I as b changes is given by

$$\Delta I = \int d\mathbf{x}\,\Delta x_\mu\,G_\mu\left(q^{(h)}, \frac{\partial q^{(h)}}{\partial \mathbf{x}_r}, \frac{\partial^2 q^{(h)}}{\partial \mathbf{x}_r\,\partial \mathbf{x}_s}, \cdots, \frac{\delta I}{\delta q^{(h)}(\mathbf{x})}, \frac{\partial}{\partial \mathbf{x}_r}\left(\frac{\delta I}{\delta q^{(h)}(\mathbf{x})}\right), \cdots\right). \tag{28}$$

4. Chang, T. S. *Proc. Cambridge Phil. Soc.*, 42 (1946), 132.
5. De Wet, J. S. *Proc. Cambridge Phil. Soc.*, 44 (1948), 546.

(If we write the above as

$$\Delta I = \int d\underline{\mathbf{x}}\, \Delta \underline{x}_\mu \underline{G}_\mu,$$

if q in (24) is chosen to be that q belonging to the observer $\underline{O}$, and if L is a scalar, the form of the functions $\underline{G}_\mu$ is independent of the surface b).
If we calculate $(20)-[1,2]$, we find

$$\sum_{h=0}^{\varrho-1}\int_{(0)} \frac{\delta(\int G_\mu \Delta_2 x_\mu d\underline{\mathbf{x}})}{\delta p^{(h)}(\underline{\mathbf{x}}')} \frac{\delta(\int G_\nu \Delta_1 x_\nu d\underline{\mathbf{x}})}{\delta q^{(h)}(\underline{\mathbf{x}}')} d\underline{\mathbf{x}}' + \left(\int_{(1)} - \int_{(0)}\right) G_\mu \Delta_2 x_\mu d\underline{\mathbf{x}} \\ -[1,2], \quad (29)$$

where $p^{(h)}(\underline{\mathbf{x}})$ stands for $\delta I/\delta q^{(h)}(\underline{\mathbf{x}})$. This is precisely the same operator in (17). Hence from the integrability of (28), we find $(20)-[1,2]=O$, and thus $(29)=O$, and thus $(17)=O$, and thus (25) is integrable. The above shows in fact that from (26) for one system of reference and the equations of motion (22), one arrives at (26) for another system of reference. If q in (24) is chosen to be that q belonging to the observer $\underline{O}$ and if L is a scalar, $p^{(h)}$ as a functional of $q^{(k)}$ $(k=0, 1, ..., 2\varrho-1)$ will be independent of the position of the plane $\underline{x}_4=$constant, and then the above shows that (26) is Lorentz-invariant[6].

The restriction of surfaces to be planes can now be removed. Given any surface, we may let q, $N_\mu q_\mu$, $N_\mu N_\nu q_{\mu\nu}$, ... play the role of $q^{(0)}$, $q^{(1)}$, ..., $q^{(\varrho-1)}$ defined by (23). We shall call them new $q^{(h)}$ and put the complete variation of $I \equiv \int L d^4x$ in the form (24), *i.e.*

$$\int du \left\{ (D x_\mu) G_{\mu,\, new} + \sum_{h=0}^{\varrho-1} p^{(h)}{}_{new} D q^{(h)}{}_{new} \right\}. \quad (30)$$

This is obviously possible and $G_{\mu\, new}$, $p^{(h)}_{new}$ are functions of $q^{(k)}$, $q^{(k)}_r$, $q^{(k)}_{rs}$,..., $(k=0, 1, ... 2\varrho-1)$, where

6. The Lorentz invariancy of (26) has been proved by de Wet (reference 5) for the case in which the Lagrangian contains the first and the second derivatives of the field quantities. The proof will soon become very cumbersome if one proceeds to include still higher derivatives.

$$\left.\begin{aligned} q_r^{(0)} &= q_\mu \frac{\partial x_\mu}{\partial u_r}, \qquad q_{rs}^{(0)} = q_{\mu\nu} \frac{\partial x_\mu}{\partial u_r} \frac{\partial x_\nu}{\partial u_s} + q_\mu \frac{\partial^2 x_\mu}{\partial u_r \partial u_s}, \\ q_r^{(1)} &= N_\mu q_{\mu\nu} \frac{\partial x_\nu}{\partial u_r} + \frac{\partial N_\mu}{\partial u_r} q_\mu, \quad \text{etc.} \end{aligned}\right\} \tag{31}$$

apart from being functions of $\partial x_\mu / \partial u_r$, $\partial^2 x_\mu / \partial u_r \partial u_s$, etc. This enables us to consider $G_\mu(u)$ as functionals of $x_\mu(u)$, $p^{(h)}(u)$, $q^{(h)}(u)$, $(h=0, 1, \ldots, \varrho-1)$ and to express G_μ as functions of $\partial x_\mu / \partial u_r$, $\partial^2 x_\mu / \partial u_r \partial u_s$, etc, $q^{(h)}$, $q_r^{(h)}$, $q_{rs}^{(h)}$,... etc. and $p^{(h)}$, $p_r^{(h)} \equiv \partial p^{(h)} / \partial u_r$, ... etc, $(h=0, 1, \ldots, \varrho-1)$. Then we can set up (25) with the operators q, q_r, q_{rs}, ..., p, p_r, ... etc. connected by $q_r = \partial q / \partial u_r$ etc. and satisfying

$$[p^{(h)}(u), q^{(k)}(u')] = \hbar\, \delta_{hk}\, \delta(u - u'). \tag{32}$$

Then (25) is again integrable and the proof proceeds exactly as before.

It may be pointed out that the set of the new $q^{(h)}$ may be replaced by some other set; one has essentially the same result as the above, but the new G_μ as functionals of $q^{(h)}$, $p^{(h)}$ may become complicated. A transformation from one set of $q^{(h)}$ to another with accompanying changes in $p^{(h)}$ and G_μ is essentially a contact transformation.

4. EXPLICIT FORM OF G_μ FOR WAVE FUNCTIONS ON PLANES

For completeness, let us add a word on the explicit form of G_μ in the cases in which S are always planes. In the proof for the integrability, we have not assumed that L is a scalar, but this assumption will be made now. Any two space-like neighbouring surfaces S and S' may be considered as the planes $x_4=0$ and $x'_4=0$ where x_μ and x'_μ are connected by an inhomogeneous Lorentz transformation

$$x_\mu' = x_\mu + \eta_\mu + \varepsilon_{\mu\nu} x_\nu. \qquad (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}) \tag{33}$$

Letting u_r in the plane S be x_r and u_r in S' be x'_r, we find for the case in which q are scalars and L contains the first derivatives of q only

$$\Delta \Psi = \eta_\mu D_\mu \Psi + \frac{1}{2} \varepsilon_{\mu\nu} D_{\mu\nu} \Psi, \qquad (D_{\mu\nu} = - D_{\nu\mu}) \tag{34}$$

$$\eta_\mu D_\mu = \frac{1}{\hbar}\int (H\eta_4 + p\,q_r\eta_r)\,du, \tag{34.1}$$

$$\frac{1}{2}\varepsilon_{\mu\nu}D_{\mu\nu} = \frac{1}{\hbar}\int (H\varepsilon_{4r}u_r + p\,q_s\varepsilon_{sr}u_r)\,du, \tag{34.2}$$

H being the Hamiltonian $pq_4 - L$. If q^α transforms under (33) according to

$$q^{\alpha\prime} = q^\alpha + \frac{1}{2}\varepsilon_{\mu\nu}(I_{\mu\nu})^\alpha_\beta q^\beta, \qquad (I_{\mu\nu} = -I_{\nu\mu}) \tag{35}$$

and if we use the $\langle q(u), b_\mu(u)|\rangle$ representation and let the q in $\langle q, b|\rangle$ to be that q belonging to the observer to whom the surface b is given by $x_4 = 0$, we include in the right-hand side of (34.2) an extra term

$$-\sum_{\alpha,\beta}\int p_\alpha \frac{1}{2}\varepsilon_{\mu\nu}(I_{\mu\nu})^\alpha_\beta q^\beta\,du. \tag{36}$$

Owing to the choice of u and the q's in the representation $\langle q, b|\rangle$ and the fact that L is a scalar, the forms of the different operators D_μ, $D_{\mu\nu}$ are independent of the plane S. It is easily seen that D_μ, $D_{\mu\nu}$ are the space integral of the $T_{\mu4}$ component of the energy momentum tensor $T_{\mu\nu}$ and the $M_{\mu\nu4}$ component of the angular momentum tensor $M_{\mu\nu\varrho}$, and this remains so when we extend the theory to include the various derivatives of q. Expressions for such tensors for cases in which the Lagrangian contains various derivatives of q were given in a paper by the author[7]. In all cases, it can be directly verified with the help of the commutation laws between q and p that they satisfy the well known commutation laws

$$\left.\begin{aligned} &[D_\mu, D_\nu] = 0, \qquad \left[\frac{1}{2}\varepsilon_{\mu\nu}D_{\mu\nu}, \eta_\varrho D_\varrho\right] = \varepsilon_{\mu\nu}\eta_\nu D_\mu, \\ &\left[\frac{1}{2}\varepsilon^{(1)}_{\mu\nu}D_{\mu\nu}, \frac{1}{2}\varepsilon^{(2)}_{\varrho\eta}D_{\varrho\eta}\right] = \varepsilon^{(1)}_{\nu\eta}\varepsilon^{(2)}_{\eta\varrho}D_{\nu\varrho}, \end{aligned}\right\} \tag{37}$$

where $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ are any two sets of ε. That they must satisfy (37) is obvious, since the general integrability of (9) or (25) is established.

7. Chang, T. S. *Proc. Cambridge Phil. Soc.* **44** (1948), 76.

APPENDIX

As mentioned in the text, it is possible to prove directly that (17) is zero. For this purpose, let us introduce arbitrary quantities with the symbols $\partial x_\mu/\partial w$, $\partial^2 x_\mu/\partial u_r\partial u_s$, $\partial^2 x_\mu/\partial w^2$, etc. and define the symbols $\partial w/\partial x_\mu$, $\partial u_r/\partial x_\mu$, $\partial^2 w/\partial x_\mu\partial x_\nu$ etc. as functions of $\partial x_\mu/\partial u_r$, $\partial^2 x_\mu/\partial u_r\partial u_s$, etc. and the arbitray quantities satisfying the relations

$$\frac{\partial x_\mu}{\partial u_\varrho}\frac{\partial u_\varrho}{\partial x_\nu}=\delta_{\mu\nu},\qquad \frac{\partial u_\mu}{\partial x_\varrho}\frac{\partial x_\varrho}{\partial u_\nu}=\delta_{\mu\nu},$$

$$(\mu,\nu,\varrho,\cdots=1,2,3,4,\qquad u_4\equiv w)$$

and the equations obtained from differentiating them formally. With this meaning of $\partial w/\partial x_\mu$, $\partial x_\mu/\partial w$, $\partial u_r/\partial x_\mu$, etc., and with q_w understood as $q_\mu\,(\partial x_\mu/\partial w)$, we have

$$\delta G_\mu=-q_\mu\,\delta p+N_\mu\frac{\partial L}{\partial q}\delta q+\Big(N_\mu\frac{\partial L}{\partial q_r}-p\frac{\partial u_r}{\partial x_\mu}\Big)\delta q_r+L\,\delta N_\mu$$

$$+\Big\{-N_\mu\frac{\partial L}{\partial q_\varrho}\frac{\partial u_r}{\partial x_\varrho}q_\nu+N_\varrho\frac{\partial L}{\partial q_\varrho}\frac{\partial u_r}{\partial x_\mu}q_\nu\Big\}\,\delta\Big(\frac{\partial x_\nu}{\partial u_r}\Big).\tag{38}$$

The term within the curly bracket in (17) is, due to (38),

$$\int\Big\{L\,\varDelta_1N_\mu\,\varDelta_2x_\mu+\Big[-N_\mu\frac{\partial L}{\partial q_\varrho}\frac{\partial u_r}{\partial x_\varrho}q_\nu+N_\varrho\frac{\partial L}{\partial q_\varrho}\frac{\partial u_r}{\partial x_\mu}q_\nu\Big]\frac{\partial(\varDelta_1x_\nu)}{\partial u_r}\varDelta_2x_\mu\Big\}du,$$

where $\varDelta_1N_\mu$ is the change of N_μ as b changes from $b^{(0)}$ to $b^{(1)}$. By partial integration,

$$\int L\,\varDelta_1N_\mu\,\varDelta_2x_\mu\,du-[1,2]=-\sum_{(per)}\int\frac{\partial L}{\partial u_1}\frac{\partial x_\tau}{\partial u_2}\frac{\partial x_\theta}{\partial u_3}\varepsilon_{\sigma\tau\theta\mu}\,\varDelta_1x_\sigma\,\varDelta_2x_\mu\,du,$$

where σ,τ,θ are suffixes of the same nature as $\mu\nu$ etc. and the summation (per) is taken over a simultaneous permutation of $u_1\,u_2\,u_3$ and $\sigma\,\tau\,\theta$. Inserting for $\partial L/\partial u_1$ the expression

$$\Big\{\frac{\partial L}{\partial q}\Big(q_r\frac{\partial u_r}{\partial x_\varrho}+X\frac{\partial w}{\partial x_\varrho}\Big)+\frac{\partial L}{\partial q_\gamma}\Big(\frac{\partial q_\gamma}{\partial u_r}\frac{\partial u_r}{\partial x_\varrho}+Y_\gamma\frac{\partial w}{\partial x_\varrho}\Big)\Big\}\frac{\partial x_\varrho}{\partial u_1},$$

where X and Y_γ are arbitrary quantities and rearranging, we get

$$-\int \Delta_1 x_\varrho \Delta_2 x_\mu N_\mu \left\{ \frac{\partial L}{\partial q} \left(q_r \frac{\partial u_r}{\partial x_\varrho} + X \frac{\partial w}{\partial x_\varrho} \right) \right.$$

$$\left. + \frac{\partial L}{\partial q_\gamma} \left(\frac{\partial q_\gamma}{\partial u_r} \frac{\partial u_r}{\partial x_\varrho} + Y_\gamma \frac{\partial w}{\partial x_\varrho} \right) \right\} du - [1,2]. \tag{39.1}$$

Similarly, we find

$$\int \left[-N_\mu \frac{\partial L}{\partial q_\varrho} \frac{\partial u_r}{\partial x_\varrho} q_\nu + N_\varrho \frac{\partial L}{\partial q_\varrho} \frac{\partial u_r}{\partial x_\mu} q_\nu \right] \frac{\partial (\Delta_1 x_\nu)}{\partial u_r} \Delta_2 x_\mu \, du - [1,2]$$

$$= \int \left[\Delta_1 x_\nu q_\nu \frac{\partial}{\partial u_r} \left\{ \left(N_\mu \frac{\partial L}{\partial q_r} - p \frac{\partial u_r}{\partial x_\mu} \right) \Delta_2 x_\mu \right\} \right. \tag{39.2}$$

$$\left. + \Delta_1 x_\nu \Delta_2 x_\mu N_\mu \frac{\partial L}{\partial q_\gamma} \left(\frac{\partial q_\nu}{\partial u_s} \frac{\partial u_s}{\partial x_\gamma} + Z_\nu \frac{\partial w}{\partial x_\gamma} \right) \right] du - [1,2],$$

where Z_ν is arbitrary, and

$$\frac{1}{\hbar} \left\{ \left(\int G_\mu \Delta_2 x_\mu \, du \right) \left(\int G_\nu \Delta_1 x_\nu \, du \right) - [1,2] \right\}'$$

$$= \int \left\{ \frac{\delta (\int G_\mu \Delta_2 x_\mu \, du)}{\delta p(u')} \frac{\delta (\int G_\nu \Delta_1 x_\nu \, du)}{\delta q(u')} - \dots \right\} du' \tag{39.3}$$

$$= -\int q_\mu \Delta_2 x_\mu \left[N_\nu \frac{\partial L}{\partial q} \Delta_1 x_\nu - \frac{\partial}{\partial u_r} \left\{ \left(N_\nu \frac{\partial L}{\partial q_r} - p \frac{\partial u_r}{\partial x_\nu} \right) \Delta_1 x_\nu \right\} \right] du - [1,2].$$

Noting that

$$\frac{\partial q_\nu}{\partial u_s} \frac{\partial u_s}{\partial x_\gamma} - \frac{\partial q_\gamma}{\partial u_s} \frac{\partial u_s}{\partial x_\nu}$$

is of the form

$$-Z_\nu \frac{\partial w}{\partial x_\gamma} + Y_\gamma \frac{\partial w}{\partial x_\nu}$$

we find that with a proper choice of X, Y, Z, the sum of (39.1), (39.2), (39.3) is zero. This completes the direct proof.

中 文 提 要

威士氏場論之討論

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對於一曲面上之施落亭格波函數在曲面變化時所發生之變化，威士氏有一理論。今將其理論補充，使其完整，使此理論，在場之蘭格倫日包含場量之各種微分時，依然可用。所獲得之方程式之可積分性，用與其相當之哈密爾頓—雅科俾方程式之可積分性証明之。此種討論，同時証明各種變數中之對換關係在羅蘭絲變化下之不變性。