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應用忽魯登變分法決定核子與核子散射的週相*

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我們應用忽魯登低決定週相的變分法於包含二核子的系統，假定核子勢爲不同力程之湯川勢低重疊；同時在需要的時候推廣忽魯登法以包括庫侖勢。所得冪級數形狀的公式可用在能量小於 40 Mev. 時。

一，忽魯登底變分法[1]

本節將略述忽魯底決定週相的變分法，並指出如何可使實際計算簡化。

我們的數學問題是：對於方程式

$$L\,u \equiv \left\{ \frac{d^2}{dr^2} - \frac{l\,(l+1)}{r^2} + k^2 - U \right\} u = 0, \tag{1}$$

（此中 $U(r)$ 在無窮遠處下落較 r^{-1} 爲快）

決定其解 u 底漸近展開式中的週相 δ。此處解 u 應適合：

$$u = 0 \qquad (\text{在 } r = 0),$$

$$u \sim \sin\left(kr - \frac{1}{2}\,l\pi + \delta\right) \qquad (\text{當 } r \longrightarrow \infty).$$

假定 u' 是和 u 滿足同樣邊界條件及具有相似漸近展開式的嘗試函數。u' 中的週相 δ 可看成一參變數，另外 u' 可能還包含其他參變數 c_ν，$\nu = \pm 1, \pm 2, \cdots$。忽魯登證明：

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[1] 忽魯登 (Hulthén), *K. Fysiograf. Saellsk. Lund Foerhandl.* **14** (1944), No. 8 及 No. 21.

若 $I' = \int_o^\infty u' L u' dr$，則 δ 及 c_ν 可從下列條件決定，

$$\frac{\partial I'}{\partial c_\nu} = 0, \ \nu = \pm 1, \pm 2, \cdots; \text{ 及 } I' = 0. \tag{2}$$

設 $U(r)$ 當 $\lambda r > 1$ 時，下降很快。忽魯登認爲下列的嘗試函數相當合理，

$$u' = (1 + c_1 e^{-2\lambda r} + c_2 e^{-2\lambda r} + \cdots) F(r) \cos\delta + (1 + c_{-1} e^{-\lambda r} + \cdots)(1 - e^{-\lambda r})^{2l+1} G(r) \sin\delta,$$

此處 $F(r)$，及 $G(r)$ 爲方程式 (1) 中略去 $U(r)$ 後之解。當 $r \to \infty$ 時，

$$F(r) \sim \sin\left(kr - \frac{1}{2} l\pi\right), \quad G(r) \sim \cos\left(kr - \frac{1}{2} l\pi\right).$$

很容易看出：在條件 (2) 中，我們若將 c_ν 寫成 C_ν / C_o，$\nu = \pm 1, \pm 2, \cdots$，並令

$$u = C_o u' = fF + gG = \sum_{n=o}^{\infty} C_n e^{-n\lambda r} F \cos\delta + \sum_{m=o}^{\infty} C_{-m} e^{m\lambda r} (1 - e^{-\lambda r})^{2l+1} G \sin\delta, \tag{3}$$

而 $$I = C_0^2 I' = \int_o^\infty u L u \, dr = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m I_{mn} C_m, \tag{4}$$

則 (2) 變成一組齊次線方程式，

$$\frac{\partial I}{\partial C_m} = 0, \quad m = 0, \pm 1, \pm 2, \cdots,$$

或 $$\sum_{n=-\infty}^{\infty} I_{mn} C_n = 0, \quad m = 0, \pm 1, \pm 2, \cdots. \tag{5}$$

C_n 不全爲零之必要條件爲，

$$\det \left| I_{mn} \right| = 0. \tag{6}$$

此條件近似地決定週相 δ 之值。爲了免除含混及看出此近似值底精確度，我們可以利用下面兩恆等式，

$$\left.\begin{aligned} C_o \cos\delta &= \lim_{r\to o} \frac{u}{F} + \frac{1}{k}\int_o^\infty G\,U\,u\,dr, \\ C_o \sin\delta &= -\frac{1}{k}\int_o^\infty F\,U\,u\,dr. \end{aligned}\right\} \tag{7}$$

這兩恆等式爲方程式 (1) 底精確解所滿足。我們算出 δ 之近似值好壞如何可以看此兩式滿足底程度如何來決定。

把 (3) 式代入積分 I 中，稍加計算，可得

$$I = -I_1 - I_2, \quad \text{其中} \tag{8}$$

$$I_1 = \int_o^\infty \left(f\frac{dg}{dr} - g\frac{df}{dr} \right) k\,dr$$

$$= k\sin\delta\cos\delta \left\{ C_0^2 + \sum_{m=o}^\infty \sum_{n=o}^\infty 2n \frac{\Gamma(m+n)\,\Gamma(2l+2)}{\Gamma(m+n+2l+2)} C_{-m}\,C_n \right\}, \tag{9}$$

$$I_2 = \int_o^\infty \left[F^2\left\{ \left(\frac{df}{dr}\right)^2 + Uf^2 \right\} + G^2\left\{ \left(\frac{dg}{dr}\right)^2 + Ug^2 \right\} + 2FG\left\{ \frac{df}{dr}\frac{dg}{dr} + Ufg \right\} \right] dr.$$

若設

$$U = \sum_\alpha U_\alpha \frac{e^{-\lambda_\alpha r}}{r} = \sum_\alpha \lambda\, w_\alpha \frac{e^{-\xi_\alpha \lambda r}}{r}, \quad (\lambda \text{ 爲諸 } \lambda_\alpha \text{ 中最小的})$$

且令

$$\left.\begin{aligned} R(s,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s (\lambda r)^{t-1} e^{-x\lambda r} F^2 \lambda\,dr. \\ S(s.\,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s (\lambda r)^{t-1} e^{-x\lambda r} FG\,\lambda\,dr, \\ T(s,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s (\lambda r)^{t-1} e^{-x\lambda r} G^2 \lambda\,dr, \end{aligned}\right\} \tag{10}$$

則可算出 I_2 如下：

$$I_2 = \lambda \cos^2\delta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_m C_n \Big\{ mn\, R(0,1,m+n) + \sum_{\alpha} w_\alpha R(0,0,m+n+\xi_\alpha) \Big\}$$

$$+ \lambda \sin^2\delta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{-m} C_{-n} \Big\{ mn\, T(4l+2,1,m+n)$$

$$-(2l+1)(m+n)\, T(4l+1,1,m+n+1) + (2l+1)^2\, T(4l,1,m+n+2)$$

$$+ \sum_{\alpha} w_\alpha T(4l+2,0,m+n+\xi_\alpha) \Big\}$$

$$+2\lambda \cos\delta \sin\delta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{-m} C_n \Big\{ mn\, S(2l+1,1,m+n)$$

$$-(2l+1)\, n\, S(2l,1,m+n+1) + \sum_{\alpha} w_\alpha S(2l+1,0,m+n+\xi_\alpha) \Big\}. \tag{9a}$$

二，質子對中子散射底情形

在質子對中子散射底情形，我們有

$$\left. \begin{aligned} F(r) &= \left(\frac{1}{2}\pi kr\right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr), \\ G(r) &= \left(\frac{1}{2}\pi kr\right)^{\frac{1}{2}} (-)^l J_{-l-\frac{1}{2}}(kr). \end{aligned} \right\} \tag{11}$$

由此容易算出積分 $R(s,t,x)$, $S(s,t,x)$ 及 $T(s,t,x)$，再代入 (8), (9) 及 (9a) 式中加以計算後，可將積分 I 寫成如下形式：

$$-I = k \sin\delta \cos\delta \cdot y^{-1} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m (\alpha_{mn} y^2 + \beta_{mn} y + \gamma_{mm}) C_n, \tag{12}$$

其中我們用了下面一些記號，

$$y = \frac{\pi}{2^{2l+2}\Gamma^2(l+\frac{3}{2})} \left(\frac{k}{\lambda}\right)^{2l+1} \cot\delta; \tag{13}$$

$$\left.\begin{aligned}
\alpha_{mn} &= \sum_{\rho=0}^{\infty} \frac{(-)^{\rho}\,(2l+\rho+2)_{\rho}\,\Gamma^2(l+\tfrac{3}{2})}{2^{2\rho}\,\rho!\,\Gamma^2(l+\rho+\tfrac{3}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} a\,(2\rho)\,, \\
\beta_{-m,n} &= \beta_{n,-m} = n\,\frac{\Gamma(m+n)\,\Gamma(2l+2)}{\Gamma(m+n+2l+2)} + \delta_{-m,n} \\
&\quad + \sum_{\rho=0}^{\infty} \frac{(-)^{\rho+1}\,(\rho+1)_{\rho}\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\tfrac{3}{2})\,\Gamma(-l+\rho+\tfrac{1}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} b(2\rho)\,, \\
\gamma_{-m,-n} &= \sum_{\rho=0}^{\infty} \frac{(-)^{\rho}\,(-2l+\rho)_{\rho}\,\pi^2}{2^{2\rho+2}\,\rho!\,\Gamma^2(-l+\rho+\tfrac{1}{2})\,\Gamma^2(l+\tfrac{3}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} c(2\rho)\,;
\end{aligned}\right\} \quad (14)$$

對於 $m \geqslant 0$, $n \geqslant 0$. 若 m 或 $n < 0$, 則 α_{mn}, $\beta_{-m,n}$, $\beta_{n,-m}$, $\gamma_{-m,-n}$ 均$=0$. (14) 式中

$$\left.\begin{aligned}
a(\nu) &= mn\,I(0,\nu+2l+3,m+n) + \sum_{\alpha} w_{\alpha}\,I(0,\nu+2l+2,m+n+\xi_{\alpha})\,, \\
b(\nu) &= mn\,I(2l+1,\nu+2,m+n) - (2l+1)\,n\,I(2l,\nu+2,m+n+1) \\
&\quad + \sum_{\alpha} w_{\alpha}\,I(2l+1,\nu+1,m+n+\xi_{\alpha})\,, \\
c(\nu) &= mn\,I(4l+2,\nu-2l+1,m+n) - (2l+1)(m+n)\,I(4l+1,\nu-2l+1,m+n+1) \\
&\quad + (2l+1)^2\,I(4l,\nu-2l+1,m+n+2) + \sum_{\alpha} w_{\alpha}\,I(4l+2,\nu-2l,m+n+\xi_{\alpha})\,.
\end{aligned}\right\} \quad (15)$$

而 $I\,(s,t,x) = \int_0^{\infty} (1-e^{-\lambda r})^s\,(\lambda r)^{t-1}\,e^{-x\lambda r}\,\lambda\,dr$. 此積分可用解析函數之理論算出，結果如下

$$I\,(s,t,x) = \sum_{j=0}^{s} (-)^j \binom{s}{j} (j+x)^{-t}\,\Gamma\,(t)$$

(對於 $s+t>0$, $t \neq$ 零或負整數)， (16)

$$I\,(s,t,x) = \sum_{j=0}^{s} (-)^j \binom{s}{j} (j+x)^{-t}\,\frac{(-)^{1-t}}{\Gamma\,(1-t)} \log\,(j+x)$$

(對於 $s+t>0$, $t=$ 零或負整數). (16a)

現在可以看出：決定週相 δ 的條件 (6) 爲一 y 之二次方程式，其係數可表成 $(k/\lambda)^2$ 之冪級數。因此，當這些冪級收斂時，也就是說，當 $(k/\lambda)^2<1$ 或質子底能量 E 約小於 40 Mev. 時，y 也就可表成 $(k/\lambda)^2$ 之冪級數。

對免除含混和看出精確度有用的恆等式 (7) 對於我們所採取的嘗試函數當然不會剛好滿足。若令恆等式右邊與左邊之比爲 $1+\varepsilon_{\cos}$ 或 $1+\varepsilon_{\sin}$，則 $\varepsilon_{\cos}$ 及 $\varepsilon_{\sin}$ 應該很小。(7) 式可以寫成下列形狀

$$\sum_{m=-\infty}^{\infty}(\beta_m y+\gamma_m)\,C_m=0, \tag{17}$$

其中，當 $m\geq 0$ 時，

$$\beta_{m,\cos}=\sum_{\rho=0}^{\infty}\frac{(-)^{\rho+l}(\rho+1)_\rho\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\frac{3}{2})\,\Gamma(-l+\rho+\frac{1}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}\sum_\alpha w_\alpha I(0,2\rho+1,m+\xi_\alpha)$$
$$+1-(1+\varepsilon_{\cos})\,\delta_{m,0}\,,$$

$$\gamma_{-m,\cos}=\sum_{\rho=0}^{\infty}\frac{(-)^{\rho}(-2l+\rho)_\rho\,\pi^2}{2^{2\rho+2}\,\rho!\,\Gamma^2(-l+\rho+\frac{1}{2})\,\Gamma^2(l+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}\sum_\alpha w_\alpha I(2l+1,2\rho-2l,m+\xi_\alpha)$$
$$+\frac{1}{2l+1}, \tag{18}$$

$$\beta_{m,\sin}=\sum_{\rho=0}^{\infty}\frac{(-)^{\rho}(2l+\rho+2)_\rho\,\Gamma^2(l+\frac{3}{2})}{2^{2\rho}\,\rho!\,\Gamma^2(l+\rho+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}\sum_\alpha w_\alpha I(0,2\rho+2l+2,m+\xi_\alpha)\,,$$

$$\gamma_{-m,\sin}=\sum_{\rho=0}^{\infty}\frac{(-)^{\rho+l}(\rho+1)_\rho\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\frac{3}{2})\,\Gamma(-l+\rho+\frac{1}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}\sum_\alpha w_\alpha I(2l+1,2\rho-2l,m+\xi_\alpha)$$
$$+(1+\varepsilon_{\sin})\,\delta_{m,0}\,,$$

若 $m<0$，則 β_m 及 γ_{-m} 均等於零。

藉方程組 (5) 之助，可將參變數 C_m 從 (17) 式消去。

三，質子對質子散射底情形

在質子對質子散射底情形，我們要考慮方程式，

$$Lu\equiv\left\{\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2-\frac{1}{Rr}-U\right\}u=0,\ \left(\frac{1}{R}=\frac{Me^2}{\hbar^2}=2k\eta\right) \tag{19}$$

決定由於短程勢 U 所引起的週相 δ。δ 出現在 (19) 式之解 u 底漸近展開式中，如下式所示，($u=0$ 在 $r=0$ 處)

$$u \sim \sin\left(kr - \eta \log 2\,kr - \frac{1}{2}\,l\pi + \sigma + \delta\right), \tag{20}$$

此處 $\sigma = \arg \Gamma\,(l+1+i\,\eta)$ 為庫侖週相。

如果我們把 F 及 G 取作具有下列漸近展開式的庫侖波函數，(滿足略去 U 後之 (19) 式)

$$F \sim \sin\left(kr - \eta \log 2\,kr - \frac{1}{2}\,l\pi + \sigma\right),$$

$$G \sim \cos\left(kr - \eta \log 2\,kr - \frac{1}{2}\,l\pi + \sigma\right),$$

則第一節中所說的仍然可以應用。

根據約斯特，灰勒和布來特[2]，我們有

$$\left.\begin{aligned} F(r) &= \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} C\cdot(\lambda r)^{l+1} \sum_{j=l+1}^{\infty} A_j\cdot(\lambda r)^{j-l-1}, \\ G(r) &= \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} D\cdot(\lambda r)^{-l} \sum_{j=-l}^{\infty} a_j\cdot(\lambda r)^{j+l} \\ &\quad + \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} D\,(Q+p\log\lambda r)\,(\lambda r)^{l+1} \sum_{j=l+1}^{\infty} A_j\,(\lambda r)^{j-l-1}. \end{aligned}\right\} \tag{21}$$

此處我們用了下面一些符號：

$$\left.\begin{aligned} C &= \frac{2^l}{(2l+1)!}\left(\frac{k}{\lambda}\right)^{l+\frac{1}{2}} \left|\,\Gamma(l+1-i\eta)\,\right| e^{-\eta\pi/2}, \quad D = \frac{1}{(2l+1)\,C}, \\ p &= (2l+1)\,C^2\,(e^{2\pi\eta}-1)\,\pi^{-1} \\ &= \frac{2^{2l}}{(2l)!\,(2l+1)!\,\lambda R}\left\{\left(\frac{1}{2\,\lambda R}\right)^2+\left(\frac{lk}{\lambda}\right)^2\right\}\cdots\left\{\left(\frac{1}{2\,\lambda R}\right)^2+\left(\frac{k}{\lambda}\right)^2\right\}, \\ Q &= p\left[2\gamma - \sum_{s=1}^{2l+1}\frac{1}{s} + \sum_{s=1}^{l}\frac{s}{s^2+\eta^2} + Re\,\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} + \log\frac{2k}{\lambda}\right] \\ &\quad + \frac{(-)^{l+1}}{2l!}\left(\frac{k}{\lambda}\right)^{2l+1} Im \sum_{s=0}^{2l}\frac{2^s\,(i\eta-l)_s}{s!\,(2l+1-s)}, \end{aligned}\right\} \tag{22}$$

[2] 約斯特，灰勒及布來特 (Yost, Wheeler 及 Breit), *Phys. Rev.* **49** (1936), 174. 應注意者即我們此處所用的符號和他們所用的之間有些不同 (差 k/λ 之冪次)。

$\gamma=0.5772\ldots=$ 歐衣拉氏常數，Re 及 Im 指實數及虛數部。(21) 式中的係數 A_j 及 a_j 等則由下列關係決定

$$\left.\begin{aligned}
&A_j=0\,(j<l+1),\ A_{l+1}=1,\ A_j=(j+l)^{-1}(j-l-1)^{-1}\times\\
&\qquad\left\{\frac{1}{\lambda R}A_{j-1}-\left(\frac{k}{\lambda}\right)^2 A_{j-2}\right\},\ (j>l+1);\\
&a_j=0\,(j<-l),\ a_{-l}=1,\ a_{l+1}=0,\ a_j=(j+l)^{-1}(j-l-1)^{-1}\times\\
&\qquad\left\{\frac{1}{\lambda R}a_{j-1}-\left(\frac{k}{\lambda}\right)^2 a_{j-2}-(2j-1)\,p\,A_j\right\},\ (j>-l,\ j\neq l+1).
\end{aligned}\right\}\tag{23}$$

由 (21) 容易算出積分 $R(s,\,t,\,x)$，$S(s,\,t,\,x)$ 及 $T(s,\,t,\,x)$. 在計算中所出現的積分除 $I(s,\,t,\,x)$ 外，還有它對 t 的一兩次導數，

$$I^{(n)}(s,t,x)=\frac{\partial^n}{\partial t^n}I(s,t,x)=\int_0^\infty(1-e^{-\lambda r})^s(\lambda r)^{t-1}(\log\lambda r)^n e^{-x\lambda r}\lambda\,dr\,. \tag{24}$$

變積分 I 重新可以寫成 (12) 式之狀，但其中的記號則代表下列意義：

$$y=C^2\cot\delta=\frac{2^{2l}(l^2+\eta^2)\cdots(1^2+\eta^2)}{(2l+1)!^2}\,\frac{2\pi\eta}{e^{2\pi\eta}-1}\left(\frac{k}{\lambda}\right)^{2l+1}\cot\delta\,, \tag{25}$$

$$\left.\begin{aligned}
&\alpha_{m,n}=\sum_{\nu=0}^{\infty}f_\nu\,a(\nu)\,,\\
&\beta_{-m,n}=\beta_{n,-m}=n\,\frac{\Gamma(m+n)\,\Gamma(2l+2)}{\Gamma(m+n+2l+2)}+\delta_{-m,n}\\
&\qquad+\frac{1}{2l+1}\sum_{\nu=0}^{\infty}\left\{h_\nu\,b(\nu)+f_\nu\,Q\,b(\nu+2l+1)+f_\nu\,p\,b'(\nu+2l+1)\right\},\\
&\gamma_{-m,n}=\frac{1}{(2l+1)^2}\left\{g_\nu\,c(\nu)+2\,h_\nu\,Q\,c(\nu+2l+1)+f_\nu\,Q^2\,c(\nu+4l+2)\right.\\
&\qquad\left.+2\,f_\nu\,Q\,p\,c'(\nu+4l+2)+f_\nu\,p^2\,c''(\nu+4l+2)\right\}.
\end{aligned}\right\}\begin{matrix}(26)\\ \\(m\geqslant 0)\\(n\geqslant 0)\end{matrix}$$

此處 $a(\nu)$, $b(\nu)$ 及 $c(\nu)$ 如 (15) 式所示，而 $b'(\nu)$, $c'(\nu)$ 及 $c''(\nu)$ 則可由 (15) 對 ν 微分得到。此外我們還用了下面的簡寫，

$$\left.\begin{aligned}
f_\nu &= \sum_{j=l+1}^{\infty} A_j\, A_{2l+2+\nu-j}\,, \\
g_\nu &= \sum_{j=-l}^{\infty} a_j\, a_{-2l+\nu-j}\,, \\
h_\nu &= \sum_{j=l+1}^{\infty} A_j\, a_{1+\nu-j}\,.
\end{aligned}\right\} \tag{27}$$

若 m 或 n 小於零，則 $\alpha_{m,n}$, $\beta_{-m,n}$, $\beta_{n,-m}$, 及 $\gamma_{-m,-n}$ 均$=0$.

從 (22), (23) 及 (27) 可以看出 p, f_ν, g_ν 及 h_ν 均爲 $(k/\lambda)^2$ 之冪級數；而 Q 則除包含 $(k/\lambda)^2$ 之冪外，尙含有對數項。 因此可以看出 y 之解亦爲一 $(k/\lambda)^2$ 之冪級數，此外則尙有對數項出現。

恆等式 (7) 又可寫成 (17) 式之狀，但其中

$$\left.\begin{aligned}
\beta_{m,\cos} &= \frac{1}{2l+1}\sum_{\nu=0}^{\infty}\sum_{\alpha} w_\alpha \Big\{ h_\nu\, I(0, \nu+1, m+\xi_\alpha) + f_\nu\, Q\, I(0, \nu+2l+2, m+\xi_\alpha) \\
&\quad + f_\nu\, p\, I'(0, \nu+2l+2, m+\xi_\alpha) \Big\} + 1 - (1+\varepsilon_{\cos})\,\delta_{m,0}\,, \\
\gamma_{-m,\cos} &= \frac{1}{2l+1} + \frac{1}{(2l+1)^2}\sum_{\nu=0}^{\infty}\sum_{\alpha} w_\alpha \Big\{ g_\nu\, I(2l+1, \nu-2l, m+\xi_\alpha) \\
&\quad + 2\, h_\nu\, Q\, I(2l+1, \nu+1, m+\xi_\alpha) + f_\nu\, Q^2\, I(2l+1, \nu+2l+2, m+\xi_\alpha) \\
&\quad + 2\, f_\nu\, Q\, p\, I'(2l+1, \nu+2l+2, m+\xi_\alpha) + f_\nu\, p^2\, I''(2l+1, \nu+2l+2, m+\xi) \Big\}, \\
\beta_{m,\sin} &= \sum_{\nu=0}^{\infty}\sum_{\alpha} w_\alpha\, f_\nu\, I(0, \nu+2l+2, m+\xi_\alpha)\,, \\
\gamma_{-m,\sin} &= \frac{1}{2l+1}\sum_{\nu=0}^{\infty}\sum_{\alpha} \Big\{ h_\nu\, I(2l+1, \nu+1, m+\xi_\alpha) \\
&\quad + f_\nu\, Q\, I(2l+1, \nu+2, m+\xi_\alpha) + f_\nu\, p\, I'(2l+1, \nu+2l+2, m+\xi_\alpha) \Big\} \\
&\quad + (1+\varepsilon_{\sin})\,\delta_{-m,0}\,.
\end{aligned}\right\} \begin{array}{l}(28)\\ (m \\ \geqslant 0)\end{array}$$

若 $m<0$, 則 β_m 及 γ_{-m} 均 $=0$.

從本節中之公式，若令 $\eta\rightarrow 0$，則我們可以重新得到第二節中所得到的在質子對中子散射底情形的公式。

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DETERMINATION OF THE PHASE FOR NUCLEON-NUCLEON SCATTERING BY HULTHÉN'S VARIATIONAL METHOD*

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ABSTRACT

A variational method for the determination of the phase, originally developed by Hulthén, is applied here to a two nucleon system, the nuclear potential being taken as a superposition of Yukawa potentials of different ranges. Hulthén's method is also generalized to include a Coulomb potential when necessary. Formulae are obtained in the form of power series which are useful for energies less than 40 Mev.

1. HULTHEN'S VARIATIONAL METHOD[1]

In this paragraph we shall describe briefly Hulthén's variational method for the determination of the phase, and add some remark which greatly simplifies the practical calculation.

The mathematical problem is to determine, for the equation

$$L\,u \equiv \left\{ \frac{d^2}{dr^2} - \frac{l\,(l+1)}{r^2} + k^2 - U \right\} u = 0\,, \tag{1}$$

where $U(r)$ falls off at infinity faster than r^{-1}, the phase δ which occurs in the asymptotic expansion of that solution u of (1) which vanishes at the origin. That is, if normaized to unit amplitude, we have $u=0$ at $r=0$, and $u \sim \sin(kr + \frac{1}{2}\, l\pi + \delta)$ as $r \longrightarrow \infty$.

Hulthén introduces the variational integral $I' = \int_0^\infty u' L u'\, dr$, where the trial function u' for u satisfies the same boundary condition at the origin and possesses similar asymptotic expansion as the exact solution u, the phase δ being now regarded as a parameter to be varied. In addition u' may contain

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1. Hulthén, *K. Fysiograf. Saellsk. Lund Foerhandl.* **14** (1944), No. 8 and No. 21.

other parameters to be varied, which we denote by c_ν, $\nu=\pm 1, \pm 2, \cdots$ Hulthén has shown that the parameters c_ν and δ are determined by the following conditions:

$$\frac{\partial I'}{\partial c_\nu} = 0, \nu = \pm 1, \pm 2, \cdots ; \text{ and } I' = 0 . \tag{2}$$

If $U(r)$ is a short-range potential which falls off rapidly for $\lambda r > 1$ Hulthén advises to use, for the trial function u', the following expression

$$\begin{aligned} u' = & (1 + c_1 e^{-\lambda r} + c_2 e^{-2\lambda r} + \cdots) F(r) \cos\delta \\ & + (1 + c_{-1} e^{-\lambda r} + \cdots) (1 - e^{-\lambda r})^{2l+1} G(r) \sin\delta , \end{aligned} \tag{3}$$

where

$$\left. \begin{aligned} F(r) &= (\tfrac{1}{2}\pi kr)^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr) \sim \sin(kr - \tfrac{1}{2} l\pi) , \\ G(r) &= (\tfrac{1}{2}\pi kr)^{\frac{1}{2}} (-)^l J_{-l-\frac{1}{2}}(kr) \sim \cos(kr - \tfrac{1}{2} l\pi) , \end{aligned} \right\} \tag{4}$$

are the asymptotic solutions of the equation (1) with the potential U omitted.

We note that the conditions (2) can be made to appear in a symmetrical form by writing C_ν/C_0 for c_ν, $\nu=\pm 1, \pm 2, \ldots$ and considering the variational integral $I = C_0^2 I'$. Then (2) becomes, as is easily verified,

$$\frac{\partial I}{\partial C_m} = 0, \quad m = 0, \ \pm 1, \ \pm 2, \cdots \tag{5}$$

If trial function of the form (3) is used, the variational integral I can be directly obtained from the trial function

$$u = C_0 u' = fF + Gg , \tag{6}$$

$$f = \sum_{n=0}^{\infty} C_n e^{-n\lambda r} \cos\delta, \quad g = \sum_{m=0}^{\infty} C_{-m} e^{-m\lambda r} (1 - e^{-\lambda r})^{2l+1} \sin\delta , \tag{7}$$

by integration, viz. $I = \int_0^\infty uLudr$. It is therefore of the form of a homogeneous quadratic form in the C_n, say

$$I = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m I_{mn} C_n , \tag{8}$$

so the condition (5) leads to the homogeneous linear equations

$$\sum_{n=-\infty}^{\infty} I_{mn} C_n = 0, \ m = 0, \ \pm 1, \ \pm 2, \cdots \tag{9}$$

Since not all C_n are zero, we must have

$$\det \left| I_{mn} \right| = 0 \tag{10}$$

which determines the parameter δ as an approximation to the phase. That the determination of the phase as above is ambiguous is of no practical disadvantage, as Hulthén already explained, because the accuracy of the variational method is in any case to be tested by examining how far the following identities are verified for the final trial function and the phase, viz.

$$\left.\begin{aligned} C_o \cos\delta &= \lim_{r \to o} \frac{u}{F} + \frac{1}{k} \int_o^\infty G\, U\, u\, dr , \\ C_o \sin\delta &= -\frac{1}{k} \int_o^\infty F\, U\, u\, dr . \end{aligned}\right\} \tag{11}$$

These are identities satisfied by the exact solution u of (1), as can be proved by eliminating Uu by means of (1) and then integrating by parts, using meanwhile the relation

$$G \frac{dF}{dr} - F \frac{dG}{dr} = \text{constant} = k . \tag{12}$$

Substituing (6) for u in Lu, we get for the variational integral

$$I = \int_o^\infty u \left\{ F \frac{d^2 f}{dr^2} + G \frac{d^2 g}{dr^2} + 2 \frac{dF}{dr} \frac{df}{dr} + 2 \frac{dG}{dr} \frac{dg}{dr} - U u \right\} dr . \tag{13}$$

Integrate once by parts the terms containing second derivatives, then substitute (6) for u and use (12). The variational integral becomes

$$\left.\begin{aligned} I &= -I_1 - I_2\,, \\ I_1 &= \int_o^\infty \left(f\,\frac{dg}{dr} - g\,\frac{df}{dr}\right) k\,dr\,, \end{aligned}\right\} \tag{14}$$

$$I_2 = \int_o^\infty \left[F^2 \left\{ \left(\frac{df}{dr}\right)^2 + U f^2 \right\} + G^2 \left\{ \left(\frac{dg}{dr}\right)^2 + U g^2 \right\} \right.$$
$$\left. + 2\,F\,G \left\{ \frac{df}{dr}\,\frac{dg}{dr} + U f g \right\} \right] dr\,.$$

The first part can be integrated by using $e^{-\lambda r}$ as a new variable of integration, thus we get

$$I_1 = k \sin\delta \cos\delta \left\{ C_o^2 + \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} 2n \frac{\Gamma(m+n)\,\Gamma(2l+2)}{\Gamma(m+u+2l+2)}\, C_{-m}\, C_n \right\}. \tag{15}$$

For the second part, we shall consider the case of a superposition of Yukawa potentials, say

$$U = \sum_\alpha U_\alpha \frac{e^{-\lambda_\alpha r}}{r} = \sum_\alpha \lambda w_\alpha \frac{e^{-\xi_\alpha \lambda r}}{r}\,, \tag{16}$$

where λ is so chosen that none of the ξ_α is smaller than unity, in order that the whole potentials falls off rapidly for $\lambda r > 1$ as assumed above. We then get, for the second part of the variational integral,

$$I_2 = \lambda \cos^2\delta \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} C_m\, C_n \left\{ mn\, R(0, 1, m+n) + \sum_\alpha w_\alpha R(0, 0, m+n+\xi_\alpha) \right\}$$
$$+ \lambda \sin^2\delta \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} C_{-m}\, C_{-n} \left\{ mn\, T(4l+2, 1, m+n) \right.$$
$$-(2l+1)(m+n)\, T(4l+1, 1, m+n+1) + (2l+1)^2\, T(4l, 1, m+n+2)$$
$$\left. + \sum_\alpha w_\alpha\, T(4l+2, 0, m+n+\xi_\alpha) \right\}$$
$$+ 2\,\lambda \cos\delta \sin\delta \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} C_{-m}\, C_n \left\{ mn\, S(2l+1, 1, m+n) \right.$$
$$\left. -(2l+1)\, n\, S(2l, 1, m+n+1) + \sum_\alpha w_\alpha\, S(2l+1, 0, m+n+\xi_\alpha) \right\} \tag{17}$$

with R, S and T defined as follows

$$\left.\begin{aligned} R(s,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s\,(\lambda r)^{t-1}\,e^{-x\lambda r}\,F^2\,\lambda\,dr\,,\\ S(s,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s\,(\lambda r)^{t-1}\,e^{-x\lambda r}\,F\,G\,\lambda\,dr\,,\\ T(s,t,x) &= \int_o^\infty (1-e^{-\lambda r})^s\,(\lambda r)^{t-1}\,e^{-x\lambda r}\,G^2\,\lambda\,dr\,. \end{aligned}\right\} \tag{18}$$

2. Case of proton-neutron scattering

To evaluate the above integrals with F and G given by Bessel's functions, we use the following series copied from § 5.41 of Watson's 'Bessel Functions'

$$J_\mu(z)\,J_\nu(z) = \sum_{\rho=o}^{\infty} \frac{(-)^\rho\, z^{\mu+\nu+2\rho}\,(\mu+\nu+\rho+1)_\rho}{2^{\mu+\nu+2\rho}\,\rho!\,\Gamma(\mu+\rho+1\Gamma(\nu+\rho+1)} \tag{19}$$

with the notation $(\mu)_\rho \equiv \mu(\mu+1)\cdots(\mu+\rho-1)$, and $(\mu)_0 \equiv 1$. We then get the following series

$$\left.\begin{aligned} R(s,t,x) &= \sum_{\rho=o}^{\infty} \frac{(-)^\rho\,(2l+\rho+2)_\rho\,\pi}{2^{2l+2\rho+2}\,\rho!\,\Gamma^2(l+\rho+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2l+2\rho+2} I(s,t+2l+2\rho+2,x),\\ S(s,t,x) &= \sum_{\rho=o}^{\infty} \frac{(-)^{\rho+l}\,(\rho+1)_\rho\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\frac{3}{2})\,\Gamma(-l+\rho+\frac{1}{2})}\left(\frac{k}{\lambda}\right)^{2\rho+1} I(s,t+2\rho+1,x),\\ T(s,t,x) &= \sum_{\rho=o}^{\infty} \frac{(-)^\rho\,(-2l+\rho)_\rho\,\pi}{2^{2\rho-2l}\,\rho!\,\Gamma^2(-l+\rho+\frac{1}{2})}\left(\frac{k}{\lambda}\right)^{-2l+2\rho} I(s,t-2l+2\rho,x)\,, \end{aligned}\right\} \tag{20}$$

all in terms of the integral (s being always an integer ≥ 0)

$$I(s,t,x) = \int_o^\infty (1-e^{-\lambda r})^s\,(\lambda r)^{t-1}\,e^{-x\lambda r}\,\lambda\,dr\,. \tag{21}$$

The integral converges for $s+t>0$, when it can be evaluated with the help of the theory of analytic functions, noting that for $t>0$ we have, by expanding the factor $(1-e^{-\lambda r})^s$ by binomial theorem,

$$I(s,t,x) = \sum_{j=o}^{\infty} (-)^j \binom{s}{j} (j+x)^{-t}\,\Gamma(t)\,. \tag{22}$$

This result is therefore valid, by the theory of analytic continuation, for all values of t so that the real part of $s+t>0$, except when t is a negative integer or zero. In the exceptional case, a limiting process leads from (22) to the expression

$$I(s,t,x)=\sum_{j=0}^{s}(-)^j\binom{s}{j}(j+x)\frac{(-)^{1-t}}{\Gamma(1-t)}\log(j+x),\ (t=\text{negative integer}),\tag{23}$$

if we note in this connection the identity

$$\sum_{j=0}^{s}(-)^j\binom{s}{j}(j+x)^{-t}=0,\quad (s+t>0,\ t=\text{negative integer}).\tag{24}$$

The series (20) converge for $(k/\lambda)^2<1$, and thus are useful for the scattering of a proton by a neutron at rest, provided the energy E of the proton is less than $2\hbar^2\lambda^2/M$ which is about 40 Mev.

Combining (15), (17) and (20) we get for the variational integral

$$-I=k\sin\delta\cos\delta\, y^{-1}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}C_m(\alpha_{mn}y^2+\beta_{mn}y+\gamma_{mn})C_n,\tag{25}$$

with the following abbreviation

$$y=\frac{\pi}{2^{2l+2}\Gamma^2(l+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2l+1}\cot\delta;\tag{26}$$

$$\left.\begin{aligned}
\alpha_{mn}&=\sum_{\rho=0}^{\infty}\frac{(-)^\rho(2l+\rho+2)_\rho\Gamma^2(l+\frac{3}{2})}{2^{2\rho}\rho!\,\Gamma^2(l+\rho+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}a(2\rho),\\
\beta_{-m,n}&=\beta_{n,-m}=n\frac{\Gamma(m+n)\Gamma(2l+2)}{\Gamma(m+n+2l+2)}+\delta_{-m,n}\\
&\quad+\sum_{\rho=0}^{\infty}\frac{(-)^{\rho+1}(\rho+1)_\rho\pi}{2^{2\rho+1}\rho!\,\Gamma(l+\rho+\frac{3}{2})\Gamma(-l+\rho+\frac{1}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}b(2\rho),\\
\gamma_{-m,-n}&=\sum_{\rho=0}^{\infty}\frac{(-)^\rho(-2l+\rho)_\rho\pi^2}{2^{2\rho+2}\rho!\,\Gamma^2(-l+\rho+\frac{1}{2})\Gamma^2(l+\frac{3}{2})}\left(\frac{k}{\lambda}\right)^{2\rho}c(2\rho),
\end{aligned}\right\}\ (m\geq 0,\ n\geq 0)\tag{27}$$

where

$$\left.\begin{aligned}
a(\nu)&=mn\,I(0,\nu+2l+3,m+n)+\sum_{\alpha}w_{\alpha}\,I(0,\nu+2l+2,m+n+\xi_{\alpha}),\\
b(\nu)&=mn\,I(2l+1,\nu+2,m+n)-(2l+1)\,n\,I(2l,\nu+2,m+n+1)\\
&\qquad+\sum_{\alpha}w_{\alpha}\,I(2l+1,\nu+1,m+n+\xi_{\alpha}),\\
c(\nu)&=mn\,I(4l+2,\nu-2l+1,m+n)-(2l+1)(m+n)\,I(4l+1,\nu-2l+1,m+n+1)\\
&\quad+(2l+1)^2\,I(4l,\nu-2l+1,m+n+2)+\sum_{\alpha}w_{\alpha}\,I(4l+2,\nu-2l,m+n+\xi_{\alpha}).
\end{aligned}\right\}\tag{28}$$

The α_{mn}, β_{mn}, γ_{mn} for m, n other than those indicated in (27) all vanish. The condition (10) for the determination of the phase δ is therefore of the form (we assume for example that only C_-, C_0, C_1 and C_2 are different from zero)

$$\begin{vmatrix}
\gamma_{-1,-1}, & \beta_{-1,0}\,y+\gamma_{-1,0}, & \beta_{-1,1}\,y, & \beta_{-1,2}\,y\\
\beta_{0,-1}\,y+\gamma_{0,-1}, & \alpha_{00}\,y^2+\beta_{00}\,y+\gamma_{00}, & \alpha_{01}\,y^2+\beta_{01}\,y, & \alpha_{02}\,y^2+\beta_{02}\,y\\
\beta_{1,-1}\,y, & \alpha_{10}\,y^2+\beta_{10}\,y, & \alpha_{11}\,y^2, & \alpha_{12}\,y^2\\
\beta_{2,-1}\,y, & \alpha_{20}\,y^2+\beta_{20}\,y, & \alpha_{21}\,y^2, & \alpha_{22}\,y^2
\end{vmatrix}=0 \tag{29}$$

or in view of the fact that $y\neq 0$,

$$\begin{vmatrix}
\gamma_{-1,-1}, & \beta_{-1,0}\,y+\gamma_{-1,0}, & \beta_{-1,1}, & \beta_{-1,2}\\
\beta_{0,-1}\,y+\gamma_{0,-1}, & \alpha_{00}\,y^2+\beta_{00}\,y+\gamma_{00}, & \alpha_{01}\,y+\beta_{01}, & \alpha_{02}\,y+\beta_{02}\\
\beta_{1,-1}, & \alpha_{10}\,y+\beta_{10}, & \alpha_{11}, & \alpha_{12}\\
\beta_{2,-1}, & \alpha_{20}\,y+\beta_{20}, & \alpha_{21}, & \alpha_{22}
\end{vmatrix}=0. \tag{30}$$

This represents a quadratic equation in y, with coefficients expressible as power series in $(k/\lambda)^2$. The solution for y is therefore also expressible as a power series in $(k/\lambda)^2$ for small values of (k/λ). To decide between the two solutions for y we have to test the identities (11). Because we have introduced only a finite number of parameters C_n, these identities will not be satisfied exactly. So we denote the ratio of the right-hand-side to the left-hand-side by $1+\varepsilon_{\cos}$ or by $1+\varepsilon_{\sin}$ for the first or the second identity of (11), the deviation $\varepsilon_{\cos}$ and $\varepsilon_{\sin}$ ought to be small. For our variational wave function (cf.(6) and (7)) the identities are of the following form

$$\sum_{m=-\infty}^{\infty} (\beta_m y + \gamma_m)\, C_m = 0 \tag{31}$$

with only the following non-vanishing coefficients

$$\beta_{m,\,\cos} = \sum_{\rho=0}^{\infty} \frac{(-)^{\rho+l}\,(\rho+1)_\rho\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\frac{3}{2})\,\Gamma(-l+\rho+\frac{1}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} \sum_\alpha w_\alpha\, I(0, 2\rho+1, m+\xi_\alpha)$$
$$+ 1 - (1+\varepsilon_{\cos})\,\delta_{m,0}\,, \qquad (m \geq 0)\,,$$

$$\gamma_{-m,\,\cos} = \sum_{\rho=0}^{\infty} \frac{(-)^{\rho}\,(-2l+\rho)_\rho\,\pi^2}{2^{2\rho+2}\,\rho!\,\Gamma^2(-l+\rho+\frac{1}{2})\,\Gamma^2(l+\frac{3}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} \sum_\alpha w_\alpha I(2l+1, 2\rho-2l, m+\xi_\alpha)$$
$$+ \frac{1}{2l+1}\,, \qquad (m \geq 0)\,,$$

$$\beta_{m,\,\sin} = \sum_{\rho=0}^{\infty} \frac{(-)^{\rho}\,(2l+\rho+2)_\rho\,\Gamma^2(l+\frac{3}{2})}{2^{2\rho}\,\rho!\,\Gamma^2(l+\rho+\frac{3}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} \sum_\alpha^{\infty} w_\alpha\, I(0, 2\rho+2l+2, m+\xi_\alpha)\,,$$
$$(m \geq 0)\,,$$

$$\gamma_{-m,\,\sin} = \sum_{\rho=0}^{\infty} \frac{(-)^{\rho+l}\,(\pi+1)_\rho\,\pi}{2^{2\rho+1}\,\rho!\,\Gamma(l+\rho+\frac{3}{2})\,\Gamma(-l+\rho+\frac{1}{2})} \left(\frac{k}{\lambda}\right)^{2\rho} \sum_\alpha w_\alpha I(2l+1, 2\rho+2l, m+\xi_\alpha)$$
$$+ (1+\varepsilon_{\sin})\,\delta_{m,0}\,, \qquad (m \geq 0)\,. \tag{32}$$

The parameters C_m can be eliminated from the identity (31) with the help of (9), resulting in a determinantal equation like (30), but with the second row replaced by γ_{-1}, $\beta_1 y + \gamma_0$, β_1 and β_2, which is a linear equation in y.

3. Case of proton-proton scattering

In the case of proton-proton scattering, the mathematical problem is to determine, for the equation

$$Lu \equiv \left\{ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 - \frac{1}{Rr} - U \right\} u = 0, \left(\frac{1}{R} = \frac{Me^2}{\hbar^2} = 2k\eta \right) \tag{33}$$

the additional phase-shift δ due to the short-range potential U. This occurs in the asymptotic expansion of that solution of (33) which vanish at the origin as indicated by

$$u \sim \sin\left(kr - \eta \log 2kr - \frac{1}{2} l\pi + \sigma + \delta\right),$$

where

$$\sigma = arg\,\Gamma\,(l + 1\, i\eta) \tag{34}$$

gives the Coulomb phase. The general consideration given in the first paragraph holds good throughout, provided that we take for F and G the exact solutions for the Coulomb potential (i.e. they satisfy (33) with U omitted), normalized to unit amplitude in the asymptotic expansions

$$\left.\begin{aligned} F &\sim \sin\left(kr - \eta \log 2kr - \frac{1}{2} l\pi + \sigma\right) \\ G &\sim \cos\left(kr - \eta \log 2kr - \frac{1}{2} l\pi + \sigma\right) \end{aligned}\right\} \tag{35}$$

Following Yost, Wheeler and Breit[2], we have the series expansion

$$\left.\begin{aligned} F(r) &= \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} C\,(\lambda r)^{l+1} \sum_{j=l+1}^{\infty} A_j\,(\lambda r)^{j-l-1}, \\ G(r) &= \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} D\,(\lambda r)^{-l} \sum_{j=-l}^{\infty} a_j\,(\lambda r)^{j+l} \\ &\quad + \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} D\,(Q + p \log \lambda r)\,(\lambda r)^{l+1} \sum_{j=l+1}^{\infty} A_j\,(\lambda r)^{j-l-1}. \end{aligned}\right\} \tag{36}$$

Here we have used, for abbreviation,

$$C = \frac{2^l}{(2l+1)!}\left(\frac{k}{\lambda}\right)^{l+\frac{1}{2}} |\,\Gamma\,(l+1-i\eta)\,|\,e^{-\eta\pi/2},\quad D = \frac{1}{(2l+1)\,C},$$

$$p = (2l+1)\,C^2\,(e^{2\pi\eta}-1)\,\pi^{-1}$$

$$= \frac{2^{2l}}{(2l)!\,(2l+1)!\,\lambda R}\left\{\left(\frac{1}{2\lambda R}\right)^2 + \left(\frac{lk}{\lambda}\right)^2\right\}\cdots\left\{\left(\frac{1}{2\lambda R}\right)^2 + \left(\frac{k}{\lambda}\right)^2\right\}$$

2. Yost, Wheeler and Breit, *Phys. Rev.* **49** (1936), 174. We note for comparison that our coefficients A and a differ from theirs by powers of k/λ. Also we have omitted the subscript l throughout.

$$Q = p\left[2\gamma - \sum_{s=1}^{2l+1}\frac{1}{s} + \sum_{s=1}^{l}\frac{s}{s^2+\eta^2} + Re\,\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)}\right] + p\log\frac{2k}{\lambda}$$

$$+\frac{(-)^{l+1}}{2l!}\left(\frac{k}{\lambda}\right)^{2l+1} Im\sum_{s=o}^{2l}\frac{2^s(i\eta-l)_s}{s!\,(2l+1-s)}, \tag{37}$$

where $\gamma = 0.5772\ldots$ is Euler's constant, while Re and Im denote respectively the real and imaginary parts of the expressions following these symbols. The A_j and the a_j are to be determined from the recurrence formulae

$$\left.\begin{aligned}
&A_j = 0\,(j < l+0), \quad A_{l+1} = 1,\\
&\qquad A_j = \frac{1}{(j+l)(j-l-1)}\left\{\frac{1}{\lambda R}A_{j-1} - \left(\frac{k}{\lambda}\right)^2 A_{j-2}\right\}, (j > l+1);\\
&a_j = 0\,(j < -l), \quad a_{-l} = 1, \quad a_{l+1} = 0,\\
&a_j = \frac{1}{(j+l)(j-l-1)}\left\{\frac{1}{\lambda R}a_{j-1} - \left(\frac{k}{\lambda}\right)^2 a_{j-2} - (2j-1)\,p\,A_j,\right\}\\
&\qquad\qquad\qquad\qquad (j > -l,\ j \neq l+1).
\end{aligned}\right\} \tag{38}$$

With (36) the integrals (18) can be expressed in terms of the integral (21) and its first and second derivatives with respect to t, namely

$$I^{(n)}(s,t,x) = \frac{\partial^n}{\partial t^n}I(s,t,x) = \int_0^\infty (1-e^{-\lambda r})^s(\lambda r)^{t-1}(\log\lambda r)^n e^{-x\lambda r}\lambda dr. \tag{39}$$

Thus we get for the variational integral again the expression (25), now with the following abbreviation

$$y = C^2\cot\delta = \frac{2^{2l}(l^2+\eta^2)\cdots(1^2+\eta^2)}{(2l+1)!^2}\,\frac{2\pi\eta}{e^{2\pi\eta}-1}\left(\frac{k}{\lambda}\right)^{2l+1}\cot\delta, \tag{40}$$

$$\left.\begin{aligned}
&\alpha_{m,n} = \sum_{\nu=o}^{\infty} f_\nu\, a(\nu),\\
&\beta_{-m,n} = \beta_{n,m} = n\frac{\Gamma(m+n)\,\Gamma(2l+2)}{\Gamma(m+n+2l+2)} + \delta_{-m,n}\\
&\qquad + \frac{1}{2l+1}\sum_{\nu=o}^{\infty}\left\{h_\nu b(\nu) + f_\nu Q\, b(\nu+2l+1) + f_\nu\, p\, b'(\nu+2l+1)\right\}\\
&\gamma_{-m,-n} = \frac{1}{(2l+1)^2}\sum_{\nu=o}^{\infty}\left\{g_\nu c(\nu) + 2h_\nu Q c(\nu+2l+1) + f_\nu Q^2 c(\nu+4l+2)\right.\\
&\qquad \left.+2f_\nu Q\,p\,c'(\nu+4l+2) + f_\nu\, p^2 c''(\nu+4l+2)\right\}.
\end{aligned}\right\} \tag{41}$$

$(m \geq 0,\ n \geq 0)$

Here $a(\nu)$, $b(\nu)$ and $c(\nu)$ are as given above, (28), and $b'(\nu)$, $c'(\nu)$ and $c''(\nu)$ are obtained from (28) by differentiation with respect to ν. Further we have introduced for abbreviation

$$\left.\begin{aligned} f_\nu &= \sum_{j=l+1}^{\infty} A_j A_{2l+2+\nu-j}, \\ g_\nu &= \sum_{j=-l}^{\infty} a_j a_{-2l+\nu-j}, \quad h_\nu = \sum_{j=l+1}^{\infty} A_j a_{1+\nu-j}. \end{aligned}\right\} \quad (\nu \geq 0) \qquad (42)$$

From the alternate expression for p we see that p is of the form of a power series in $(k/\lambda)^2$. Hence, by the recurrence formula (38), the A_j and the a_j, hence also f_ν, g_ν and h_ν are all power series in $(k/\lambda)^2$. The solution for y is thus a power series in $(k/\lambda)^2$, complicated further by the appearance of logarithm terms contained in Q.

The identities are again of the form (31), now with

$$\beta_{m,\cos} = \frac{1}{2l+1} \sum_{\nu=0}^{\infty} \sum_{\alpha} w_\alpha \Big\{ h_\nu I(0, \nu+1, m+\xi_\alpha) + f_\nu Q I(0, \nu+2l+2, m+\xi_\alpha)$$

$$+ f_\nu p I'(0, \nu+2l+2, m+\xi_\alpha) \Big\} + 1 - (1+\varepsilon_{\cos})\delta_{m,0}, \qquad (m \geq 0)$$

$$\gamma_{-m,\cos} = \frac{1}{2l+1} + \frac{1}{(2l+1)^2} \sum_{\nu=0}^{\infty} \sum_{\alpha} w_\alpha \Big\{ g_\nu I(2l+1, \nu-2l, m+\xi_\alpha)$$

$$+ 2h_\nu Q I(2l+1, \nu+1, m+\xi_\alpha) + f_\nu Q^2 I(2l+1, \nu+2l+2, m+\xi_\alpha)$$

$$+ 2f_\nu Q p I'(2l+1, \nu+2l+2, m+\xi_\alpha) + f_\nu p^2 I''(2l+1, \nu+2l+2, m+\xi_\alpha) \Big\},$$

$$(m \geq 0)$$

$$\beta_{m,\sin} = \sum_{\nu=0}^{\infty} \sum_{\alpha} w_\alpha f_\nu I(0, \nu+2l+2, m+\xi_\alpha), \qquad (m \geq 0)$$

$$\gamma_{-m,\sin} = \frac{1}{2l+1} \sum_{\nu=0}^{\infty} \sum_{\alpha} w_\alpha \Big\{ h_\nu I(2l+1, \nu+1, m+\xi_\alpha)$$

$$+ f_\nu Q I(2l+2, m+\xi_\alpha) + f_\nu p I'(2l+1, \nu+2l+2, m+\xi_\alpha) \Big\}$$

$$+ (1+\varepsilon_{\sin})\delta_{-m,0}, \qquad (m \geq 0). \qquad (43)$$

If we take the limit as η approaches zero in all the formulae considered in this paragraph, we arrive again at the same result as obtained above for the case of proton-neutron scattering.