中 國 物 理 學 報
第八卷第三期，235頁，1951年12月

關於狹義相對論內之速度變換式

東 星 北
（浙江大學物理系）

普通速度變換式太複雜且無羣性。若每個變換式沿速度方向作軸加以旋轉，其旋轉角度 δ 為 $\sin\delta=\dfrac{i\beta}{\sqrt{1-\beta^2}}$，則新的速度變換式即有羣性，因原點仍以原速度進行，故其物理意義仍不變。

若坐標與四元速度均以四元數代表之，則新坐標即為四元速度與四元坐標的乘積：

$$ict'+ix'+jy'+kz'=\left(\frac{1}{\sqrt{1-\beta^2}}+\frac{iv_x+jv_y+kv_z}{ic\sqrt{1-\beta^2}}\right)\left(ict+ix+jy+kz\right)$$

此處 $i\equiv\sqrt{-1}$, $\quad ij=-ji=-k \quad i^2=-1=j^2=k^2$ 等。

CHINESE JOURNAL OF PHYSICS
Vol. 8, No. 3, 1951, pp. 236–238

NOTE ON VELOCITY TRANSFORMATIONS IN SPECIAL RELATIVITY.

By HSIN P. SOH

Department of Physics, Chekiang University, Hangchow

(Received August 28, 1951)

ABSTRACT

Three dimensional velocity transformations in special relativity can form quaternion group if each is subject to a further rotation about the direction of the velocity through the same imaginary angle.

It may be observed that the usual form of Lorentz transformation in three dimensional velocity,

$$\begin{array}{c|cccc}
 & x & y & z & t \\
\hline
x' & 1-\frac{v_x^2}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & -\frac{v_x v_y}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & -\frac{v_x v_z}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & \frac{v_x}{\sqrt{1-\beta^2}} \\
y' & -\frac{v_x v_y}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & 1-\frac{v_y^2}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & -\frac{v_y v_z}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & \frac{v_y}{\sqrt{1-\beta^2}} \\
z' & -\frac{v_x v_z}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & -\frac{v_y v_z}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & 1-\frac{v_z^2}{v^2}\left(1-\frac{1}{\sqrt{1-\beta^2}}\right) & \frac{v_z}{\sqrt{1-\beta^2}} \\
t' & \frac{+v_x}{c^2\sqrt{1-\beta^2}} & \frac{v_y}{c^2\sqrt{1-\beta^2}} & \frac{v_z}{c^2\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}}
\end{array} \tag{1}$$

can be put, through a rotation of an imaginary angle δ with $\sin\delta = \frac{i\beta}{\sqrt{1-\beta^2}}$ about the direction of the velocity axis, into the following elegant form:

$$
\begin{array}{c|cccc}
 & x & y & z & t \\
\hline
x' & \dfrac{1}{\sqrt{1-\beta^2}} & \dfrac{-i\,v_z}{\sqrt{1-\beta^2}} & \dfrac{i\,v_y}{\sqrt{1-\beta^2}} & \dfrac{v_x}{\sqrt{1-\beta^2}} \\
y' & \dfrac{i\,v_z}{\sqrt{1-\beta^2}} & \dfrac{1}{\sqrt{1-\beta^2}} & \dfrac{-i\,v_x}{\sqrt{1-\beta^2}} & \dfrac{v_y}{\sqrt{1-\beta^2}} \\
z' & \dfrac{-i\,v_y}{\sqrt{1-\beta^2}} & \dfrac{i\,v_x}{\sqrt{1-\beta^2}} & \dfrac{1}{\sqrt{1-\beta^2}} & \dfrac{v_z}{\sqrt{1-\beta^2}} \\
t' & \dfrac{v_x}{c^2\sqrt{1-\beta^2}} & \dfrac{v_y}{c^2\sqrt{1-\beta^2}} & \dfrac{v_z}{c^2\sqrt{1-\beta^2}} & \dfrac{1}{\sqrt{1-\beta^2}}
\end{array}
\qquad (2)
$$

For, according to the formula given for space rotation about an axis, see p. 103 of 1936 edition of Madelung's Mathematischen Hilfsmittel, with now

$$
\cos\alpha_1 = \frac{v_x}{v},\ \cos\alpha_2 = \frac{v_y}{v},\ \cos\alpha_3 = \frac{v_z}{v},\ \cos\delta = \frac{1}{\sqrt{1-\beta^2}},\ \sin\delta = \frac{i\beta}{\sqrt{1-\beta^2}},
$$

we have

$$
\begin{array}{c|cccc}
 & x & y & z & t \\
\hline
x' & 1-\left(1-\dfrac{v_x^2}{v^2}\right) & \dfrac{v_x v_y}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & \dfrac{v_x v_z}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & 0 \\
 & \left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & +\dfrac{v_z}{v}\dfrac{i\beta}{\sqrt{1-\beta^2}} & -\dfrac{v_y}{v}\dfrac{i\beta}{\sqrt{1-\beta^2}} & \\
y' & \dfrac{v_x v_z}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & 1-\left(1-\dfrac{v_y^2}{v^2}\right) & \dfrac{v_y v_z}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & 0 \\
 & -\dfrac{v_z}{v}\dfrac{i\beta}{\sqrt{1-\beta^2}} & \left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & +\dfrac{v_x}{v}\dfrac{i\beta}{\sqrt{1-\beta^2}} & \\
z' & \dfrac{v_x v_z}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & \dfrac{v_y v_z}{v^2}\left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right) & 1-\left(1-\dfrac{v_z^2}{v^2}\right) & 0 \\
 & +\dfrac{v_y}{v^2}\dfrac{i\beta}{\sqrt{1-\beta^2}} & -\dfrac{v_x}{v}\dfrac{i\beta}{\sqrt{1-\beta^2}} & \left(1-\dfrac{1}{\sqrt{1-\beta^2}}\right). & \\
t' & 0 & 0 & 0 & 1
\end{array}
\qquad (3)
$$

The form (2) results when we apply (3) to (1).

As the ordinary forms (1), besides complication of the appearance of v^2 in the denominator, do not possess the group property, that is, successive applications of (1) do not lead to an expression of the same form, they have many drawbacks. The transformations (2), while differing from (1) simply by a reorientation of axes, which has no physical significance since the origin moves with the same velocity v_x, v_y, v_z do possess the group property with the combination rule

$$\boldsymbol{v}'' = \frac{\boldsymbol{v} + \boldsymbol{v}' + \frac{i}{c}[\boldsymbol{v}' \times \boldsymbol{v}]}{1 + \frac{\boldsymbol{v}' \cdot \boldsymbol{v}}{c^2}}, \tag{4}$$

inverse transformation $-v_x, -v_y, -v_z$ and non-connected operators

$$\begin{aligned} O_1 &= i\left(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}\right) + i\left(x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1}\right) \\ O_2 &= i\left(x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}\right) + i\left(x_2 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_2}\right) \\ O_3 &= i\left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) + i\left(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}\right) \end{aligned} \tag{5}$$

and possess no other algebraic invariant except $x_1^2+x_2^2+x_3^2+x_4^2$ where $x_4 = ict$. Hence (2) can be conveniently employed, especially as it can be represented by Quaternions,

$$\begin{aligned} &(ict' + \boldsymbol{i}x' + \boldsymbol{j}y' + \boldsymbol{k}z') \\ &= \left\{ \frac{1}{\sqrt{1-\beta^2}} + \frac{\boldsymbol{i}v_x + \boldsymbol{j}v_y + \boldsymbol{k}v_z}{ic\sqrt{1-\beta^2}} \right\} (ict + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) \end{aligned} \tag{6}$$

with $\boldsymbol{ij} = -\boldsymbol{ji} = -\boldsymbol{k}$, $\boldsymbol{i}^2 = -1$, etc., in problems where the group property of the 'velocity transformations' is required.