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關於二組坐標有相對加速度時之變換

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在特種情況下，二組有相對加速度之坐標的變換係數前已求得之。即當 S 坐標對于 S' 坐標在 x 方向有相對加速度 Γ 時，其變換係數爲

$$T(\Gamma)=\begin{vmatrix} 1-\frac{x}{c^2}\Gamma & -\frac{y}{c^2}\Gamma & -\frac{z\Gamma}{c^2} & \frac{r}{c}\Gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{x^2}{c^3 r}\Gamma & -\frac{x\,y}{c^3 r}\Gamma & -\frac{x\,z}{c^3 r}\Gamma & 1+\frac{x}{c^2}\Gamma \end{vmatrix} \qquad (1)$$

此處 (x, y, z, t) 與 (x', y', z', t') 爲一點 P 對于 S 及 S' 坐標組的坐標，c 爲光速。並已證明：如欲由此求得一電荷的電磁場强度與由馬士威方程式所計算而得者符合，則上列變換係數爲唯一的解答。同時亦已證明此一組變換係數爲可以積分的，即 x', y', z' 及 t' 俱可用 x, y, z 及 t 表示之。

現在把 (1) 式中的 Γ 當作一向量看，則因對稱關係可求得二組坐標的相對加速度爲任意方向時的變換係數。再設 S' 與另一組坐標 S'' 有速度 $\boldsymbol{v}$，則 S 對于 S'' 有速度 $\boldsymbol{v}$ 及加速度 γ，γ 之値可由特殊相對論求之。再應用勞倫斯變換可求得 S 與 S'' 的變換係數如下

$$
T(\beta,\gamma)=\left|\begin{array}{ll}
1+\delta\dfrac{v_{x''}v_{x''}}{v^2}-\dfrac{A_{x''}x}{c^2}, & \delta\dfrac{v_{x''}v_{y''}}{v^2}-\dfrac{A_{x''}y}{c^2}, \\
\qquad\delta\dfrac{v_{x''}v_{z''}}{v^2}-\dfrac{A_{x''}z}{c^2}, & \dfrac{v_{x''}}{\sqrt{1-\beta^2}}+\dfrac{r\,A_{x''}}{c} \\
\delta\dfrac{v_{y''}v_{x''}}{v^2}-\dfrac{A_{y''}x}{c^2}, & 1+\delta\dfrac{v_{y''}v_{y''}}{v^2}-\dfrac{A_{y''}y}{c^2}, \\
\qquad\delta\dfrac{v_{y''}v_{z''}}{v^2}-\dfrac{A_{y''}z}{c^2}, & \dfrac{v_{y''}}{\sqrt{1-\beta^2}}+\dfrac{r\,A_{y''}}{c} \\
\delta\dfrac{v_{z''}v_{x''}}{v^2}-\dfrac{A_{z''}x}{c^2}, & \delta\dfrac{v_{z''}v_{y''}}{v^2}-\dfrac{A_{z''}y}{c^2}, \\
\qquad 1+\delta\dfrac{v_{z''}v_{z''}}{v^2}-\dfrac{A_{z''}z}{c^2}, & \dfrac{v_{z''}}{\sqrt{1-\beta^2}}+\dfrac{r\,A_{z''}}{c} \\
\dfrac{v_{x''}}{c^2\sqrt{1-\beta^2}}-\dfrac{A_{r''}x}{c^3}, & \dfrac{v_{z''}}{c^2\sqrt{1-\beta^2}}-\dfrac{A_{r''}y}{c^3}, \\
\qquad\dfrac{v_{z''}}{c^2\sqrt{1-\beta^2}}-\dfrac{A_{r''}z}{c^3}, & \dfrac{1}{\sqrt{1-\beta^2}}+\dfrac{r\,A_{r''}}{c^2}
\end{array}\right|
$$

此處 $\boldsymbol{r}''=\boldsymbol{r}+\dfrac{\boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{r})}{v^2}\left(\dfrac{1}{\sqrt{1-\beta^2}}-1\right)+\dfrac{r\,\boldsymbol{v}}{c\sqrt{1-\beta^2}}$, $r''=\dfrac{r+\boldsymbol{\beta}\cdot\boldsymbol{r}}{\sqrt{1-\beta^2}}$,

$$\delta=\left(\frac{1}{\sqrt{1-\beta^2}}-1\right),\qquad \boldsymbol{\beta}=\frac{\boldsymbol{v}}{c},\qquad \boldsymbol{A}=\left\{\gamma+\frac{\boldsymbol{v}\,\gamma_{r''}}{c\,(1-\beta_{r''})}\right\}\frac{1}{1-\beta^2}.$$

而 $v_{x''}$, $v_{y''}$, $A_{x''}$, $\beta_{r''}$ 則各爲 $\boldsymbol{v}$, $\boldsymbol{v}$, $\boldsymbol{A}$, $\boldsymbol{\beta}$ 在 x'', y'', x'', r'' 方向的分量，餘仿此。

今設有一電荷固定於 S 座標的原點，則在該坐標上所測得的電磁場強度爲

$$\boldsymbol{E}=\frac{e\,\boldsymbol{r}}{r^3},\quad \boldsymbol{H}=0.$$

在 S'' 座標上所測得的强度 $\boldsymbol{E}''$, $\boldsymbol{H}''$ 則可假定其爲張量應用張量變換式

$$F^{\mu\nu''}=\frac{\partial x''_\mu}{\partial x_\alpha}\frac{\partial x''_\nu}{\partial x_\beta}F^{\alpha\beta}$$

求得之。 其結果與由馬士威方程計算而得者完全符合。

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ON THE TRANSFORMATION CONNECTING TWO SYSTEMS IN RELATIVE ACCELERATION

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ABSTRACT

In previous papers,[1] we have obtained, for a special case, the transformation coefficients connecting two systems in relative acceleration and have shown that the transformation is the only possible one that gives electromagnetic fields the observed values and is integrable. In the present paper, we will give the general co-ordinate transformation connecting two systems in arbitrary motion and its application to obtain the general electromagnetic fields due to a charged particle in arbitrary motion.

We consider a fixed system S', in instantaneous coincidence with the system S which is moving with an acceleration Γ (but zero velocity) along x-direction. Let (x,y,z,t) and (x',y',z',t') be the co-ordinates of a certain point P with respect to the S- and S'-system respectively. r and r' are defined by

$$r'^2 = x'^2 + y'^2 + z'^2 \text{ and } r^2 = x^2 + y^2 + z^2 .$$

The transformation coefficients, which we have obtained, are

$$T(\Gamma) = \begin{vmatrix} 1 - \frac{x}{c^2}\Gamma & -\frac{y}{c^2}\Gamma & -\frac{z}{c^2}\Gamma & \frac{r}{c}\Gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{x^2}{c^3 r}\Gamma & -\frac{xy}{c^3 r}\Gamma & -\frac{xz}{c^3 r}\Gamma & 1 + \frac{x\Gamma}{c^2} \end{vmatrix}, \tag{1}$$

1. Hsin-Pei Soh and Mu-hsien Wang, *Science Record* **1** (1945), 431.
Hsin-Pei Soh, Mu-hsien Wang and Su-Chin Kiang, *Nature* **157** (1946), 809.
Mu-hsien Wang, *Phil. Mag.* **39** (1948), 84.

where c is the velocity of light.

By considering Γ as a vector in (1), we may obtain a general connection between the systems S and S' by symmetry consideration, S being moving with acceleration Γ relative to S'.

$$T(\Gamma)=\begin{vmatrix} 1-\dfrac{\Gamma_x x}{c^2} & -\dfrac{\Gamma_x y}{c^2} & -\dfrac{\Gamma_x z}{c^2} & \dfrac{r}{c}\Gamma_x \\ -\dfrac{\Gamma_y x}{c^2} & 1-\dfrac{\Gamma_y y}{c^2} & -\dfrac{\Gamma_y z}{c^2} & \dfrac{r}{c}\Gamma_y \\ -\dfrac{\Gamma_z x}{c^2} & -\dfrac{\Gamma_z y}{c^2} & 1-\dfrac{\Gamma_z z}{c^2} & \dfrac{r}{c}\Gamma_z \\ -\dfrac{\Gamma_r x}{c^3} & -\dfrac{\Gamma_r y}{c^3} & -\dfrac{\Gamma_r z}{c^3} & 1+\dfrac{\Gamma_r r}{c^2} \end{vmatrix} \tag{2}$$

We see that (2)'s from a group with determinant equal to unity.

Applying a finite general Lorentz transformation the form usually given

$$L(\beta)=\begin{vmatrix} 1+\dfrac{v_x^2}{v^2}\delta & \delta\dfrac{v_x v_y}{v^2} & \delta\dfrac{v_x v_z}{v^2} & \dfrac{v_x}{\sqrt{1-\beta^2}} \\ \dfrac{v_y v_x}{v^2}\delta & 1+\dfrac{v_y v_y}{v^2}\delta & \delta\dfrac{v_y v_z}{v^2} & \dfrac{v_y}{\sqrt{1-\beta^2}} \\ \dfrac{v_z v_x}{v^2}\delta & \delta\dfrac{v_z v_y}{v^2} & 1+\dfrac{v_z v_z}{v^2}\delta & \dfrac{v_z}{\sqrt{1-\beta^2}} \\ \dfrac{v_x}{c^2\sqrt{1-\beta^2}} & \dfrac{v_y}{c^2\sqrt{1-\beta^2}} & \dfrac{v_y}{c^2\sqrt{1-\beta^2}} & \dfrac{1}{\sqrt{1-\beta^2}} \end{vmatrix} \quad \begin{array}{l}\text{with } \beta=v/c,\\ \delta=(1-\beta^2)^{-1/2}-1\end{array} \tag{3}$$

to (2), we have

$$dX''=L(\beta)\,dX'=L(\beta)\,T(\Gamma)\,dX=T(\beta,\gamma)\,dX \tag{4}$$

a transformation connecting the system S and S'', with coefficients in terms of $\boldsymbol{v}$ and γ. Since S'' and S' are in uniform motion, the connection between Γ, the acceleration of S observed by S' and γ, the acceleration of S observed by S'' is well known from the Special Relativity i.e.,

$$\gamma = (1-\beta^2)\,(\sqrt{1-\beta^2}-1)\,\frac{(\boldsymbol{\Gamma}\cdot\boldsymbol{v})}{v^2}\,\boldsymbol{v} + (1-\beta^2)\,\boldsymbol{\Gamma}. \tag{5}$$

Substituting this value of γ in the results of (4) we obtain the general transformation connecting the systems S'', now at rest, and S, now moving with velocity $\boldsymbol{v}$ and acceleration γ, as

$$T(\beta,\gamma) = \left|\begin{array}{cccc}
1+\delta\frac{v_{x''}\,v_{x''}}{v^2}-\frac{A_{x''}\,x}{c^2}, & \delta\frac{v_{x''}\,v_{y''}}{v^2}-\frac{A_{x''}\,y}{c^2}, & \delta\frac{v_{x''}\,v_{z''}}{v^2}-\frac{A_{x''}\,z}{c^2}, & \frac{v_{x''}}{\sqrt{1-\beta^2}}+\frac{r\,A_{x''}}{c} \\
\delta\frac{v_{y''}\,v_{x''}}{v^2}-\frac{A_{y''}\,x}{c^2}, & 1+\delta\frac{v_{y''}\,v_{y''}}{v^2}-\frac{A_{y''}\,y}{c^2}, & \delta\frac{v_{y''}\,v_{z''}}{v^2}-\frac{A_{y''}\,z}{c^2}, & \frac{v_{y''}}{\sqrt{1-\beta^2}}+\frac{r\,A_{y''}}{c} \\
\delta\frac{v_{z''}\,v_{x''}}{v^2}-\frac{A_{z''}\,x}{c^2}, & \delta\frac{v_{z''}\,v_{y''}}{v^2}-\frac{A_{z''}\,y}{c^2}, & 1+\delta\frac{v_{z''}\,v_{z''}}{v^2}-\frac{A_{z''}\,z}{c^2}, & \frac{v_{z''}}{\sqrt{1-\beta^2}}+\frac{r\,A_{z''}}{c} \\
\frac{v_{x''}}{c^2\sqrt{1-\beta^2}}-\frac{A_{r''}\,x}{c^3}, & \frac{v_{y''}}{c^2\sqrt{1-\beta^2}}-\frac{A_{r''}\,y}{c^3}, & \frac{v_{z''}}{c^2\sqrt{1-\beta^2}}-\frac{A_{r''}\,z}{c^3}, & \frac{1}{\sqrt{1-\beta^2}}+\frac{r\,A_{r''}}{c^2}
\end{array}\right| \tag{6}$$

where
$$\boldsymbol{r}'' = \boldsymbol{r} + \frac{\boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{r})}{v^2}\left(\frac{1}{\sqrt{1-\beta^2}}\right) + \frac{r\,\boldsymbol{v}}{c\sqrt{1-\beta^2}},$$

$$r'' = \frac{r+\boldsymbol{\beta}\cdot\boldsymbol{r}}{\sqrt{1-\beta^2}}, \quad \delta = \left(\frac{1}{\sqrt{1-\beta^2}}-1\right);$$

and
$$\boldsymbol{A} = \left\{\gamma + \frac{\boldsymbol{v}\,\gamma_{r''}}{c\,(1-\beta_{r''})}\right\}\frac{1}{1-\beta^2}, \quad A_{r''} = \frac{(\boldsymbol{A}\cdot\boldsymbol{r}'')}{r''};$$

and $v_{x''}$, $v_{y''}$, $A_{x''}$, $\beta_{r''}$ are the components of $\boldsymbol{v}$, $\boldsymbol{v}$, $\boldsymbol{A}$, $\boldsymbol{\beta}=\boldsymbol{v}/c$ in the directions of x'', y'', x'', r'' respectively, etc.

Now, a charge "e" is considered as kept fixed at the origin O of the S-system. The electromagnetic fields due to the charge relative to S are pure

Coulomb's fields. The problem, to be solved, is to find the fields due to the same charge as observed in the S''-system. We assume the electromagnetic field F as a tensor, this property is also obvious for cases of uniform motion in the theory of Special Relativity, and write it as

$$F^{\alpha\beta} = \begin{vmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{vmatrix} . \tag{7}$$

The Field F'', as observed in S''-system is then given by

$$F^{\mu\nu''} = \frac{\partial x''_\mu}{\partial x_\alpha} \frac{\partial x''_\nu}{\partial x_\beta} F^{\alpha\beta} \tag{8}$$

and can now be calculated by using the coefficients given by (6). After a tedious algebraic) computation, the results are

$$\boldsymbol{E}'' = \frac{e(1-\beta^2)\,\overrightarrow{O_1P}}{r''^3(1-\beta_{r''})^3} + \frac{e\gamma_{r''}\,\overrightarrow{O_1P}}{c^2 r''^2(1-\beta_{r''})^3} - \frac{e\gamma}{c^2 r''(1-\beta_{r''})^2},$$

$$\boldsymbol{H}'' = \frac{e(1-\beta^2)\,[\boldsymbol{v}\times\boldsymbol{r}'']}{c\,r''^3(1-\beta_{r''})^3} + \frac{e\gamma_{r''}\,[\boldsymbol{v}\times\boldsymbol{r}'']}{c^3 r''^2(1-\beta_{r''})^3} + \frac{e\,[\gamma\times\boldsymbol{r}'']}{c^2 r''^2(1-\beta_{r''})^2}; \tag{9}$$

with $$\overrightarrow{O_1P} \equiv \boldsymbol{r}'' - \frac{r''}{c}\boldsymbol{v}, \quad \gamma_{r''} = \frac{\gamma\cdot\boldsymbol{r}''}{r''} \text{ and } \beta_{r''} = \frac{\beta\cdot\boldsymbol{r}''}{r''}, \tag{10}$$

in *perfect* agreement with the classical Maxwell's results.

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