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相空间中变质量力学系统的 Hojman 守恒量

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研究一般的无限小变换下相空间中变质量力学系统 Lie 对称性的 Hojman 守恒量. 给出了相空间中变质量力学系统 Lie 对称性的确定方程和 Hojman 守恒量定理,并举例说明结果的应用.

关键词：相空间，变质量系统，一般的无限小变换，Lie 对称性，Hojman 守恒量

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1. 引 言

力学系统守恒量的研究具有重要的理论意义和实际意义. 分析力学的方法仅能找到极少数的守恒量，如广义动量守恒、广义能量守恒等. 近代寻求守恒量的方法是对称性方法，主要有 Noether 对称性、Lie 对称性和 Mei 对称性[1—15]. Noether 对称性总可导致守恒量，而 Lie 对称性和 Mei 对称性一般没有这种性质. 由 Lie 对称性寻求守恒量往往要通过 Noether 对称性来找 Lie 对称性的 Noether 型守恒量. 1992 年 Hojman 给出了由 Lie 对称性找守恒量的一种直接方法,得到了一类新的守恒量[16],被称为 Hojman 守恒量. 近年来人们对 Hojman 守恒量定理做了一些推广[17—23]. 然而以往的研究仅限于时间不变的特殊无限小变换. 本文研究在一般的无限小变换下，相空间中变质量力学系统的 Hojman 守恒量，将 Hojman 定理的结果进一步推广，给出相空间中变质量力学系统的 Hojman 守恒量定理.

2. 系统的运动微分方程

研究 N 个质点组成的力学系统. 在时刻 t,第 i 个质点的质量为 $m_i(i=1,2,\cdots,N)$，在时刻 $t+\mathrm{d}t$，由 i 质点分离(或并入)的微粒质量为 $\mathrm{d}m_i$. 设 m_i 是时间和广义坐标的函数,即 $m_i=m_i(t,\boldsymbol{q})$,系统受到的约束是理想的完整约束,系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$确定，则位形空间中系统的运动微分方程为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+P_s \qquad (s=1,2,\cdots,n), \tag{1}$$

其中 L 为系统的 Lagrange 函数，Q_s 为非势广义力，P_s 为广义反推力,有

$$P_s=\dot{m}_i(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s}-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s}, \tag{2}$$

$\boldsymbol{u}_i$ 为分离或并入 m_i 的微粒相对 m_i 的速度.

在相空间中方程(1)可表示为

$$\begin{aligned}\dot{q}_s&=\frac{\partial H}{\partial p_s},\\ \dot{p}_s&=-\frac{\partial H}{\partial q_s}+\widetilde{Q}_s+\widetilde{P}_s,\end{aligned} \tag{3}$$

其中 $H=H(t,\boldsymbol{q},\boldsymbol{p})$为系统的 Hamiltonian 函数，

$$\widetilde{Q}_s=\widetilde{Q}_s(t,\boldsymbol{q},\boldsymbol{p})=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p})),$$

$$\widetilde{P}_s=\widetilde{P}_s(t,\boldsymbol{q},\boldsymbol{p})=P_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p})).$$

方程(3)可展开为显式

$$\begin{aligned}\dot{q}_s&=g_s(t,\boldsymbol{q},\boldsymbol{p}),\\ \dot{p}_s&=h_s(t,\boldsymbol{q},\boldsymbol{p}) \qquad (s=1,\cdots,n).\end{aligned} \tag{4}$$

3. 系统的 Lie 对称性确定方程

取一般的无限小变换

$$\begin{aligned}t^*&=t+\varepsilon\xi_0(t,\boldsymbol{q},\boldsymbol{p}),\\ q_s^*(t^*)&=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\boldsymbol{p}),\\ p_s^*(t^*)&=p_s(t)+\varepsilon\eta_s(t,\boldsymbol{q},\boldsymbol{p}),\end{aligned} \tag{5}$$

其中 ε 为无限小参数,ξ_0，ξ_s，η_s 为无限小单参数群变换的生成元. 在无限小变换(5)式下系统(4)的 Lie 对称性确定方程为

$$\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - g_s \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \xi_0 \frac{\partial g_s}{\partial t} + \xi_k \frac{\partial g_s}{\partial q_k} + \eta_k \frac{\partial g_s}{\partial p_k},$$
$$\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - h_s \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \xi_0 \frac{\partial h_s}{\partial t} + \xi_k \frac{\partial h_s}{\partial q_k} + \eta_k \frac{\partial h_s}{\partial p_k}$$
$$(s,k = 1,\cdots,n), \tag{6}$$

其中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_s \frac{\partial}{\partial q_s} + h_s \frac{\partial}{\partial p_s}. \tag{7}$$

如果无限小变换(5)式的生成元 ξ_0, ξ_s, η_s 满足(6)式,则变换是 Lie 对称的.

4. 系统的 Hojman 守恒量

定理 对变质量力学系统(4),如果无限小变换的生成元 ξ_0, ξ_s 和 η_s 满足(6)式和条件

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) = 0, \tag{8}$$

且存在某函数 $\mu = \mu(t, \boldsymbol{q}, \boldsymbol{p})$,使得

$$\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{9}$$

则系统存在如下的 Hojman 守恒量:

$$I = \frac{1}{\mu}\frac{\partial(\mu\xi_0)}{\partial t} + \frac{1}{\mu}\frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu}\frac{\partial(\mu\eta_s)}{\partial p_s} = \mathrm{const}. \tag{10}$$

证 我们有

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s}, \tag{11}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} = \frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \frac{\partial\xi_0}{\partial q_k}\frac{\partial g_k}{\partial t} - \frac{\partial\xi_0}{\partial p_k}\frac{\partial h_k}{\partial t}, \tag{12}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \frac{\partial\xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s}, \tag{13}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} = \frac{\partial}{\partial p_s}\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - \frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s} - \frac{\partial\eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}. \tag{14}$$

将(12)—(14)式代入(11)式,并利用(6)式,得

$$\begin{aligned}\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} ={}& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right)\\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \frac{\partial g_s}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\partial P_s}{\partial p_s}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \xi_0\frac{\partial^2 g_s}{\partial q_s\partial t}\\ &+ \xi_0\frac{\partial^2 h_s}{\partial p_s\partial t} + \frac{\partial^2 g_s}{\partial q_s\partial q_k}\xi_k + \frac{\partial^2 g_s}{\partial q_s\partial p_k}\eta_k\\ &+ \frac{\partial^2 h_s}{\partial p_s\partial q_k}\xi_k + \frac{\partial^2 h_s}{\partial p_s\partial p_k}\eta_k.\end{aligned} \tag{15}$$

将条件(9)对 t, q_s 和 $\dot{q}_s$ 求偏导数,并将其代入(15)式,可得

$$\begin{aligned}\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} ={}& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right)\\ &- \xi_0\frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu - \xi_k\frac{\partial}{\partial q_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\\ &- \eta_k\frac{\partial}{\partial p_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\\ ={}& \frac{\partial}{\partial q_k}\ln\mu\left(\frac{\bar{\mathrm{d}}\xi_k}{\mathrm{d}t} - \xi_0\frac{\partial g_k}{\partial t} - \xi_s\frac{\partial g_k}{\partial q_s} - \eta_s\frac{\partial g_k}{\partial p_s}\right)\\ &+ \frac{\partial}{\partial p_k}\ln\mu\left(\frac{\bar{\mathrm{d}}\eta_k}{\mathrm{d}t} - \xi_0\frac{\partial h_k}{\partial t} - \xi_s\frac{\partial h_k}{\partial q_s} - \eta_s\frac{\partial h_k}{\partial p_s}\right)\\ &+ \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\partial}{\partial t}\ln\mu + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu.\end{aligned} \tag{16}$$

考虑到(6)和(8)式,则有

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} = 0. \tag{17}$$

5. 讨 论

本文给出的 Hojman 守恒量定理具有更一般的意义. 如果 $\xi_0 = 0$, $p_s = \dot{q}_s$, $\eta_s = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s$,则本文的定理化为文献[20]的结果 如果不考虑方程(4)的具体含义,则当 $\xi_0 = 0$ 时,本文的定理化为文献[19]的结果,而当 $\xi_0 = 0$, $p_s = \dot{q}_s$, $\eta_s = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s$ 时,本文给出的定理化为文献[16]的结果 如果非势广义力和广义反推力为零,根据方程(3)和(4)可知 $\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} = 0$,再由(9)式可知 u 是常数,这时(10)式可化为

$$I = \frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial\eta_s}{\partial p_s} = \mathrm{const}. \tag{18}$$

6. 算 例

系统的 Lagrange 函数为

$$L = \frac{1}{2}m(t)(\dot{q}_1^2 + \dot{q}_2^2), \tag{19}$$

质量的变化规律为

$$m = m_0(1 - kt) \qquad (m_0, k \text{ 为常数}), \tag{20}$$

微粒分离的相对速度为

$$\boldsymbol{u} = -\frac{1}{2}\dot{\boldsymbol{r}}, \tag{21}$$

非势广义力为零,试研究系统在相空间中的 Hojman 守恒量.

我们有

$$H = p_s \dot{q}_s - L = \frac{1}{2m_0(1-kt)}(p_1^2 + p_2^2), \tag{22}$$

$$\tilde{P}_1 = -\frac{kp_1}{2(1-kt)},\quad \tilde{P}_2 = -\frac{kp_2}{2(1-kt)}. \tag{23}$$

由方程(3)可得

$$\dot{q}_1 = \frac{p_1}{m_0(1-kt)},\quad \dot{q}_2 = \frac{p_2}{m_0(1-kt)},\quad \dot{p}_1 = -\frac{kp_1}{2(1-kt)},\quad \dot{p}_2 = -\frac{kp_2}{2(1-kt)}. \tag{24}$$

确定方程(6)给出

$$\frac{\bar{d}\xi_1}{dt} - \frac{p_1}{m_0(1-kt)}\frac{\bar{d}\xi_0}{dt} = \xi_0 \frac{kp_1}{m_0(1-kt)^2} + \eta_1 \frac{1}{m_0(1-kt)}, \tag{25}$$

$$\frac{\bar{d}\xi_2}{dt} - \frac{p_2}{m_0(1-kt)}\frac{\bar{d}\xi_0}{dt} = \xi_0 \frac{kp_2}{m_0(1-kt)^2} + \eta_2 \frac{1}{m_0(1-kt)}, \tag{26}$$

$$\frac{\bar{d}\eta_1}{dt} + \frac{kp_1}{2(1-kt)}\frac{\bar{d}\xi_0}{dt} = -\xi_0 \frac{k^2 p_1}{2(1-kt)^2} - \eta_1 \frac{k}{2(1-kt)}, \tag{27}$$

$$\frac{\bar{d}\eta_2}{dt} + \frac{kp_2}{2(1-kt)}\frac{\bar{d}\xi_0}{dt} = -\xi_0 \frac{k^2 p_2}{2(1-kt)^2} - \eta_2 \frac{k}{2(1-kt)}. \tag{28}$$

方程(25)—(28)有如下解：

$$\xi_0 = (1-kt),\quad \xi_1 = \left(p_1 + \frac{1}{2}km_0 q_1\right)^2,\quad \xi_2 = 1,\quad \eta_1 = 0,\quad \eta_2 = 0. \tag{29}$$

显然 ξ_0 满足条件(8).

方程(9)给出

$$\frac{k}{(1-kt)} - \frac{\bar{d}}{dt}\ln\mu = 0, \tag{30}$$

于是有

$$\mu = \frac{1}{(1-kt)}. \tag{31}$$

由(10)式可得到 Hojman 守恒量

$$I = km_0\left(p_1 + \frac{1}{2}km_0 q_1\right) = \text{const}. \tag{32}$$

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The conserved quantity of Hojman for mechanical systems with variable mass in phase space

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Abstract

In this paper, we study the Hojman conserved quantity of Lie symmetry for mechanical systems with variable mass in phase space under a general infinitesimal transformation. The determining equations and Hojman conservation theorem of Lie symmetry of the system in phase space are obtained. An example is given to illustrate the application of the result.

Keywords: phase space, variable mass system, general infinitesimal transformation, Lie symmetry, Hojman conserved quantity

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