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BBM 方程和修正的 BBM 方程新的精确孤立波解*

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采用一种双曲函数假设和一类新的辅助常微分方程相结合的方法给出 BBM 方程和修正的 BBM 方程新的精确孤立波解.这种方法也可用于寻找其他非线性发展方程新的孤立波解.

关键词:辅助方程,双曲函数假设,孤立波解

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1. 引 言

Benjamin-Bona-Mahoney (BBM)方程是弱非线性色散介质中长波单向传播的重要模型,且 BBM 方程和修正的 BBM (mBBM)方程是不可积方程.BBM 方程连同一个条件方程才能用反散射方法求解.文献[1]以 Lamé 方程为基础利用 Jacobi 椭圆函数展开法构造了 BBM 方程多级准确解.文献[2]结合直接方法、假设方法并根据 BBM 方程和 mBBM 方程解的一般形式构造了这两个方程的钟状孤立波解和奇异行波解,正割与余割型三角函数波解,而文献[3]用待定系数法构造了钟状型和扭状型孤立波解.文献[4]用直接方法得到了高阶广义 BBM 方程的孤立波解,文献[5]用非线性发展方程的解表示为两个待定函数的线性形式的方法构造了二维色散长波方程组的精确孤立波解,文献[6]利用 sinh-Gordon 方程展开法构造了非线性耦合标量场方程的双周期解.本文由文献[5,6]得到启示,在双曲正切函数法[7]、齐次平衡法[8]和辅助方程法[9]的基础上不需要条件方程而利用一个双曲函数型假设和一类辅助方程相结合,借助计算机代数系统 Mathematica 构造了 BBM 方程、mBBM 方程新的精确孤立波解.用这种方法构造非线性发展方程(组)的精确解具有普遍意义.

2. 函数假设和辅助方程

假设给定的非线性发展方程

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{1}$$

具有行波解 $u(x,t) = u(\xi), \xi = x + \omega t$,并将其代入方程(1)后得到如下常微分方程:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \cdots) = 0. \tag{2}$$

我们用双曲正弦函数和双曲余弦函数组合而成的复合函数来假设方程(2)的解为

$$u(\xi) = g_0 + \sum_{i=1}^{n} (g_i \sinh z(\xi) + f_i \cosh z(\xi))^i, \tag{3}$$

其中 $g_0, g_i, f_i (i = 1, 2, \cdots, n)$为待定常数,$n$ 为由领头项分析法[10]确定的自然数.下面用两种辅助方程构造 BBM 和 mBBM 方程的精确解.

第一个方程为

$$\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} = a + b\sinh z(\xi). \tag{4}$$

我们用 $\sinh z(\xi) = \frac{1}{2}[\mathrm{e}^{z(\xi)} - \mathrm{e}^{-z(\xi)}]$和变量分离法容易得到

$$\mathrm{e}^{z(\xi)} = -\frac{1}{b}\left(a + \sqrt{a^2 + b^2}\coth\frac{1}{2}\sqrt{a^2 + b^2}\,\xi\right)$$

从而得到

$$\sinh z(\xi) = \frac{b^2 - \left(a + \sqrt{a^2 + b^2}\coth\frac{\sqrt{a^2 + b^2}}{2}\xi\right)^2}{2b\left(a + \sqrt{a^2 + b^2}\coth\frac{\sqrt{a^2 + b^2}}{2}\xi\right)},$$

$$\cosh z(\xi) = -\frac{b^2 + \left(a + \sqrt{a^2 + b^2}\coth\frac{\sqrt{a^2 + b^2}}{2}\xi\right)^2}{2b\left(a + \sqrt{a^2 + b^2}\coth\frac{\sqrt{a^2 + b^2}}{2}\xi\right)}, \tag{5}$$

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第二个方程为

$$\frac{dz(\xi)}{d\xi} = a + b\cosh z(\xi). \tag{6}$$

我们用 $\cosh z(\xi) = \frac{1}{2}[e^{z(\xi)} + e^{-z(\xi)}]$ 和变量分离法容易得到如下关系式:

$$e^{z(\xi)} = \frac{1}{b}\left[-a + \sqrt{b^2 - a^2}\tan\frac{1}{2}\sqrt{b^2 - a^2}\,\xi\right] \quad (b^2 > a^2),$$

$$e^{z(\xi)} = \frac{1}{b}\left[-a - \sqrt{a^2 - b^2}\coth\frac{1}{2}\sqrt{a^2 - b^2}\,\xi\right] \quad (b^2 < a^2),$$

$$e^{z(\xi)} = -1 - \frac{2}{b\xi} \quad (a = b \neq 0).$$

从而得到下面的关系式:

$$\sinh z(\xi) = \frac{-b^2 + \left(-a + \sqrt{-a^2 + b^2}\tan\frac{\sqrt{-a^2 + b^2}}{2}\xi\right)^2}{2b\left(-a + \sqrt{-a^2 + b^2}\tan\frac{\sqrt{-a^2 + b^2}}{2}\xi\right)} \quad (b^2 > a^2),$$

$$\cosh z(\xi) = \frac{b^2 + \left(-a + \sqrt{-a^2 + b^2}\tan\frac{\sqrt{-a^2 + b^2}}{2}\xi\right)^2}{2b\left(-a + \sqrt{-a^2 + b^2}\tan\frac{\sqrt{-a^2 + b^2}}{2}\xi\right)} \quad (b^2 > a^2); \tag{7}$$

$$\sinh z(\xi) = \frac{b^2 - \left(a + \sqrt{a^2 - b^2}\coth\frac{\sqrt{a^2 - b^2}}{2}\xi\right)^2}{2b\left(a + \sqrt{a^2 - b^2}\coth\frac{\sqrt{a^2 - b^2}}{2}\xi\right)} \quad (b^2 < a^2),$$

$$\cosh z(\xi) = \frac{-b^2 - \left(a + \sqrt{a^2 - b^2}\coth\frac{\sqrt{a^2 - b^2}}{2}\xi\right)^2}{2b\left(a + \sqrt{a^2 - b^2}\coth\frac{\sqrt{a^2 - b^2}}{2}\xi\right)} \quad (b^2 < a^2); \tag{8}$$

$$\sinh z(\xi) = -\frac{2(1 + b\xi)}{b\xi(2 + b\xi)} \quad (a = b)$$

$$\cosh z(\xi) = -\frac{b^2\xi^2 + 2b\xi + 2}{b\xi(2 + b\xi)} \quad (a = b). \tag{9}$$

将(3),(4)式代入(2)式或将(3),(6)式代入(2)式并令 $\sinh^i z(\xi)$,$\cosh z(\xi)\sinh^j z(\xi)$,$\sinh 2z(\xi)$ $(i = 0,1,\cdots,n+2;j = 0,1,\cdots,n+1)$的系数为零,得到一个以 $g_0,f_k,g_k(k = 1,2,\cdots,n)$,$a,b,\omega$ 为未知量的非线性代数方程组,利用符号计算系统 Mathematica 或 Maple 求出该方程组的解,再将求出的代数方程组的每一组解连同(5)或(7)—(9)式一起代回到(3)式后得到非线性发展方程(组)的精确孤立波解.

3.具体例子

3.1. BBM 方程[1,4]

对于方程

$$u_t + \alpha u_x + \beta u u_x - \gamma u_{xxt} = 0, \tag{10}$$

我们只考虑 $\gamma = 1$ 的情况,即

$$u_t + \alpha u_x + \beta u u_x - u_{xxt} = 0. \tag{11}$$

将 $u(x,t) = u(\xi)$,$\xi = x + \omega t$ 代入(11)式并对 ξ 积分一次后得到如下常微分方程:

$$(\omega + \alpha)u + \frac{\beta}{2}u^2 - \omega u'' = 0. \tag{12}$$

由领头项分析法得到平衡常数 $n = 2$.从而可设方程(12)的解为

$$u(\xi) = g_0 + \sum_{i=1}^{2}(g_i\sinh z(\xi) + f_i\cosh z(\xi))^i, \tag{13}$$

其中

$$\cosh 2z(\xi) = 1 + 2\sinh^2 z(\xi),$$
$$\cosh^2 z(\xi) = 1 + \sinh^2 z(\xi),$$
$$\cosh^3 z(\xi) = \cosh z(\xi)[1 + \sinh^2 z(\xi)],$$
$$\cosh^4 z(\xi) = [1 + \sinh^2 z(\xi)]^2.$$

将(13),(4)式代入(12)式并令 $\sinh^i z(\xi)$,$\sinh 2z(\xi)$,$\cosh z(\xi)\sinh^j z(\xi)$ $(i = 0,1,2,3,4;j = 0,1,2,3)$的系

数为零后得到一个非线性代数方程组

$$
\begin{aligned}
&(\alpha+\omega)g_0+\frac{1}{2}\beta g_0^2-ab\omega g_1+\frac{1}{2}\beta f_1^2-2a^2\omega g_2^2\\
&+\left(\alpha+\omega-2a^2\omega+\beta g_0+\frac{1}{2}\beta f_2^2\right)f_2^2=0,\\
&(\alpha+\omega-a^2\omega-b^2\omega+\beta g_0)g_1\\
&+(-6ab\omega g_2+2\beta f_1f_2)g_2\\
&+(-6ab\omega+\beta g_1)f_2^2=0,\\
&\left(-3ab\omega+\frac{1}{2}\beta g_1\right)g_1+\frac{1}{2}\beta f_1^2\\
&+(\alpha+\omega-4a^2\omega-4b^2\omega+\beta g_0)g_2^2\\
&+(\alpha+\omega-4a^2\omega-4b^2\omega+\beta g_0\\
&+3\beta g_2^2+\beta f_2^2)f_2^2=0,\\
&-2b^2\omega g_1+(-10ab\omega+\beta g_1)g_2^2\\
&+2\beta f_1g_2f_2+(-10ab\omega+\beta g_1)f_2^2=0,\\
&(-12b^2\omega+\beta g_2^2)g_2^2+(-12b^2\omega\\
&+6\beta g_2^2+\beta f_2^2)f_2^2=0,\\
&(\alpha+\omega-a^2\omega+\beta g_0)f_1\\
&+(-2ab\omega g_2+\beta f_1f_2)f_2=0,\\
&(-3ab\omega+\beta g_1)f_1=0,\\
&(-2b^2\omega+\beta g_2^2)f_1+(-20ab\omega\\
&+2\beta g_1)g_2f_2+\beta f_1f_2^2=0,\\
&g_2f_2(-6b^2\omega+\beta g_2^2+\beta f_2^2)=0,\\
&(\alpha+\omega-4a^2\omega-b^2\omega+\beta g_0+\beta f_2^2)g_2f_2=0.
\end{aligned}
$$

用符号计算系统 Mathematica 求出该方程组的如下解：

$$
\begin{aligned}
&g_0=\frac{3b^2\alpha}{\beta-\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=\pm\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&f_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&\omega=\frac{\alpha}{b^2-1}\quad(\alpha\beta>0,|b|>1);
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&g_0=\frac{3b^2\alpha}{\beta-\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&f_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&\omega=\frac{\alpha}{b^2-1}\quad(\alpha\beta>0,|b|>1);
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&g_0=\frac{b^2\alpha}{\beta-\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=\pm\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&f_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\
&\omega=\frac{\alpha-2b^2\alpha}{b^2-1}\quad(\alpha\beta>0,|b|>1);
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&g_0=\frac{b^2\alpha}{\beta+\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=\mp\frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},\\
&f_2=\mp\frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},\\
&\omega=-\frac{\alpha}{1+b^2}\quad(\alpha\beta<0);
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&g_0=\frac{b^2\alpha}{\beta+\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=\pm\frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},\\
&f_2=\mp\frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},\\
&\omega=-\frac{\alpha}{1+b^2}\quad(\alpha\beta<0);
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&g_0=\frac{12b^2\alpha}{\beta-4\beta b^2},\\
&g_1=f_1=0,\quad a=0,\\
&g_2=0,\\
&f_2=\mp\frac{2\sqrt{3\alpha}b}{\sqrt{\beta(-1+4b^2)}},\\
&\omega=\frac{\alpha}{-1+4b^2}\quad\left(\alpha\beta>0,|b|>\frac{1}{2}\right);
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&g_0=\frac{4b^2\alpha}{\beta+4b^2\beta},\\
&g_1=f_1=a=g_2=0,\\
&\omega=-\frac{\alpha}{1+4b^2},\\
&f_2=\mp\frac{\mathrm{i}2\sqrt{3\alpha}b}{\sqrt{\beta+4b^2\beta}}\quad(\alpha\beta<0);
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&g_0=g_1=f_1=f_2=0,\quad a=0,\\
&g_2=\mp\frac{2\sqrt{3\alpha}b}{\sqrt{\beta(-1+4b^2)}},
\end{aligned}
$$

$$\omega=\frac{\alpha}{-1+4b^2}\quad\left(\alpha\beta>0,|b|>\frac{1}{2}\right);\tag{21}$$

$$\begin{aligned}&g_0=-\frac{8b^2\alpha}{\beta+4\beta b^2},\\&g_1=f_1=0,\quad a=0,\\&g_2=\mp\frac{\mathrm{i}2\sqrt{3\alpha}b}{\sqrt{\beta(1+4b^2)}},\\&f_2=0,\\&\omega=-\frac{\alpha}{1+4b^2}\quad(\alpha\beta<0).\end{aligned}\tag{22}$$

把(5)式分别与(14)—(22)式一起代入(13)式后得到 BBM 方程($\gamma=1$)的如下孤立波解：

$$u^{\pm}(x,t)=\frac{3b^2\alpha\operatorname{csch}^2\left(\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)\right)}{-\beta+\beta b^2}\quad(|b|>1),\tag{23}$$

$$u^{\pm}(x,t)=\frac{3b^2\alpha\operatorname{sech}^2\left(\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)\right)}{\beta-\beta b^2}\quad(|b|>1),\tag{24}$$

$$u^{\pm}(x,t)=-\frac{b^2\alpha\left[2+\cosh\left(\pm b\left(x-\frac{\alpha}{1+b^2}t\right)\right)\right]\operatorname{csch}^2\left(\pm\frac{b}{2}\left(x-\frac{\alpha}{1+b^2}t\right)\right)}{(1+b^2)\beta},\tag{25}$$

$$u^{\pm}(x,t)=\frac{b^2\alpha}{(1+b^2)\beta}\left[-2+\cosh\left(\pm b\left(x+\frac{\alpha-2b^2\alpha}{-1+b^2}t\right)\right)\right]\operatorname{sech}^2\left(\pm\frac{b}{2}\left(x+\frac{\alpha-2b^2\alpha}{-1+b^2}t\right)\right)\quad(|b|>1),\tag{26}$$

$$u^{\pm}(x,t)=-\frac{b^2\alpha\left[-2+\cosh\left(\pm b\left(x-\frac{\alpha}{1+b^2}t\right)\right)\right]\operatorname{sech}^2\left(\pm\frac{b}{2}\left(x-\frac{\alpha}{1+b^2}t\right)\right)}{(1+b^2)\beta},\tag{27}$$

$$u^{\pm}(x,t)=\frac{3b^2\alpha\operatorname{csch}^2\left(\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right)\operatorname{sech}^2\left(\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right)}{-\beta+4\beta b^2}\quad\left(|b|>\frac{1}{2}\right),\tag{28}$$

$$u^{\pm}(x,t)=-\frac{b^2\alpha\left[2+\cosh2b\left(x-\frac{\alpha}{1+4b^2}t\right)\right]\operatorname{csch}^2\left(\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right)\operatorname{sech}^2\left(\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right)}{\beta+4\beta b^2},\tag{29}$$

$$u^{\pm}(x,t)=\frac{3b^2\alpha}{-\beta+4b^2\beta}\operatorname{csch}^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]\operatorname{sech}^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]\quad\left(|b|>\frac{1}{2}\right),\tag{30}$$

$$u^{\pm}(x,t)=-\frac{b^2\alpha}{\beta+4b^2\beta}\left[8+3\operatorname{csch}^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right]\operatorname{sech}^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right]\right].\tag{31}$$

将(13)和(6)式代入(12)式并用上述方法得到一个非线性代数方程组(限于篇幅这里不列出)并用符号计算系统 Mathematica 求出该方程组的如下解：

$$\begin{aligned}&g_0=\frac{b^2\alpha}{-\beta+\beta b^2},\\&g_1=f_1=0,\quad a=0\\&g_2=\pm\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\&f_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\&\omega=\frac{\alpha}{-1+b^2}\quad(\alpha\beta>0,|b|>1);\end{aligned}\tag{32}$$

$$\begin{aligned}&g_0=\frac{b^2\alpha}{-\beta+\beta b^2},\\&g_1=f_1=0,\quad a=0\\&g_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\&f_2=\mp\frac{\sqrt{3\alpha}b}{\sqrt{\beta(-1+b^2)}},\\&\omega=\frac{\alpha}{-1+b^2}\quad(\alpha\beta>0,|b|>1);\end{aligned}\tag{33}$$

$$\begin{aligned}&g_0=\frac{-3b^2\alpha}{\beta(1+b^2)},\\&g_1=f_1=0,\quad a=0\\&g_2=\pm\frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},\end{aligned}$$

$$f_2 = \mp \frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},$$

$$\omega = -\frac{\alpha}{1+b^2} \quad (\alpha\beta < 0); \tag{34}$$

$$g_0 = -\frac{3b^2\alpha}{\beta(1+b^2)},$$

$$g_1 = f_1 = 0, \quad a = 0$$

$$g_2 = \mp \frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}}, \tag{35}$$

$$f_2 = \mp \frac{\mathrm{i}\sqrt{3\alpha}b}{\sqrt{\beta(1+b^2)}},$$

$$\omega = -\frac{\alpha}{1+b^2} \quad (\alpha\beta < 0);$$

$$g_0 = g_1 = f_1 = g_2 = a = 0,$$

$$f_2 = \mp \frac{\mathrm{i}2\sqrt{3\alpha}b}{\sqrt{\beta+4b^2\beta}}, \tag{36}$$

$$\omega = -\frac{\alpha}{1+4b^2} \quad (\alpha\beta < 0);$$

$$g_0 = \frac{8b^2\alpha}{\beta-\beta 4b^2},$$

$$g_1 = f_1 = 0, \quad a = 0, \quad g_2 = 0,$$

$$f_2 = \mp \frac{2\sqrt{3\alpha}b}{\sqrt{\beta(-1+4b^2)}},$$

$$\omega = \frac{\alpha}{-1+4b^2} \quad \left(\alpha\beta > 0, |b| > \frac{1}{2}\right); \tag{37}$$

$$g_0 = \frac{4b^2\alpha}{-\beta+4\beta b^2},$$

$$g_1 = f_1 = 0, \quad a = 0,$$

$$g_2 = \mp \frac{\mathrm{i}2\sqrt{3\alpha}b}{\sqrt{\beta(-1+4b^2)}}, \tag{38}$$

$$f_2 = 0,$$

$$\omega = \frac{\alpha}{-1+4b^2} \quad \left(\alpha\beta < 0, |b| > \frac{1}{2}\right);$$

$$g_0 = -\frac{12b^2\alpha}{\beta+4\beta b^2},$$

$$g_1 = f_1 = 0, \quad a = 0, \quad f_2 = 0,$$

$$g_2 = \mp \frac{\mathrm{i}2\sqrt{3\alpha}b}{\sqrt{\beta+4b^2\beta}}, \tag{39}$$

$$\omega = -\frac{\alpha}{1+4b^2} \quad (\alpha\beta < 0).$$

把(7)式分别与(32)—(39)式一起代入(13)式后得到 BBM 方程($\gamma = 1$)的如下三角函数形式解：

$$u^{\pm}(x,t) = -\frac{3b^2\alpha\csc^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+b^2}t\right)\right]}{(1+b^2)\beta}, \tag{40}$$

$$u^{\pm}(x,t) = -\frac{3b^2\alpha\sec^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+b^2}t\right)\right]}{(1+b^2)\beta}, \tag{41}$$

$$u^{\pm}(x,t) = \frac{b^2\alpha\left[2+\cos\left(\pm b\left(x+\frac{\alpha}{-1+b^2}t\right)\right)\right]\csc^2\frac{\pm b}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)}{\beta(-1+b^2)} \quad (|b| > 1), \tag{42}$$

$$u^{\pm}(x,t) = -\frac{b^2\alpha\left[-2+\cos\left(\pm b\left(x+\frac{\alpha}{-1+b^2}t\right)\right)\right]\sec^2\frac{\pm b}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)}{\beta(-1+b^2)} \quad (|b| > 1), \tag{43}$$

$$u^{\pm}(x,t) = -\frac{3b^2\alpha\csc^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right]\sec^2\left[\pm\frac{b}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right]}{\beta+4\beta b^2}, \tag{44}$$

$$u^{\pm}(x,t) = \frac{b^2\alpha\left\{-8+3\csc^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]\sec^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]\right\}}{(-1+4b^2)\beta} \quad \left(|b| > \frac{1}{2}\right), \tag{45}$$

$$u^{\pm}(x,t) = \frac{b^2\alpha\left[2+\cos\left(\pm 2b\left(x+\frac{\alpha}{-1+4b^2}t\right)\right)\right]\sec^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]\csc^2\left[\pm\frac{b}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]}{(-1+4b^2)\beta} \quad \left(|b| > \frac{1}{2}\right). \tag{46}$$

把(8)式分别与(32)—(39)式一起代入(13)式后得到 BBM 方程($\gamma = 1$)的如下孤立波解：

$$u_1(x,t) = -\frac{b^2\alpha}{\beta(-1+b^2)}\left[-2+\cosh\sqrt{-b^2}\left(x+\frac{\alpha}{-1+b^2}t\right)\right]\operatorname{sech}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)$$
$$(-1+b^2\neq 0), \tag{47}$$

$$u_2(x,t) = -\frac{b^2\alpha}{\beta(-1+b^2)}\left[2+\cosh\sqrt{-b^2}\left(x+\frac{\alpha}{-1+b^2}t\right)\right]\operatorname{csch}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+b^2}t\right)$$
$$(-1+b^2\neq 0), \tag{48}$$

$$u_3(x,t) = -\frac{3b^2\alpha}{(1+b^2)\beta}\operatorname{sech}^2\frac{\sqrt{-b^2}}{2}\left(x-\frac{\alpha}{1+b^2}t\right), \tag{49}$$

$$u_4(x,t) = \frac{3b^2\alpha}{(1+b^2)\beta}\operatorname{csch}^2\frac{\sqrt{-b^2}}{2}\left(x-\frac{\alpha}{1+b^2}t\right), \tag{50}$$

$$u_5(x,t) = \frac{3b^2\alpha}{\beta+4b^2\beta}\operatorname{csch}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\operatorname{sech}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right) \quad (-1+4b^2\neq 0), \tag{51}$$

$$u_6(x,t) = \frac{-b^2\alpha}{(-1+4b^2)\beta}\left[8+3\operatorname{csch}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\operatorname{sech}^2\frac{\sqrt{-b^2}}{2}\left(x+\frac{\alpha}{-1+4b^2}t\right)\right]$$
$$(-1+4b^2\neq 0), \tag{52}$$

$$u_7(x,t) = \frac{-b^2\alpha}{(-1+4b^2)\beta}\left[2+\cosh 2\sqrt{-b^2}\left(x-\frac{\alpha}{1+4b^2}t\right)\right]$$
$$\times\operatorname{sech}^2\frac{\sqrt{-b^2}}{2}\left(x-\frac{\alpha}{1+4b^2}t\right)\operatorname{csch}^2\frac{\sqrt{-b^2}}{2}\left(x-\frac{\alpha}{1+4b^2}t\right) \quad \left(b\neq\pm\frac{1}{2}\right). \tag{53}$$

(47)—(53)式中 b 为纯虚数.

3.2. mBBM 方程[2]

对于方程

$$u_t+\beta u^2u_x-\gamma u_{xxt}=0, \tag{54}$$

将 $u(x,t)=u(\xi),\xi=x+\omega t$ 代入并对 ξ 积分一次后得到如下方程：

$$\omega u+\frac{\beta}{3}u^3-\gamma\omega u''=0. \tag{55}$$

由领头项分析法得到平衡常数 $n=1$. 从而可设方程(55)的解为

$$u(\xi)=g_0+g_1\sinh z(\xi)+f_1\cosh z(\xi), \tag{56}$$

其中 g_0,g_1,f_1 为待定常数. 这里

$$\cosh 2z(\xi)=1+2\sinh^2z(\xi),$$
$$\cosh^2z(\xi)=1+\sinh^2z(\xi),$$
$$\cosh^3z(\xi)=\cosh z(\xi)[1+\sinh^2z(\xi)].$$

将(56)和(4)式代入(55)式并令 $\sinh^iz(\xi),\cosh z(\xi)\times\sinh^jz(\xi),\sinh 2z(\xi)$ $(i=0,1,2,3;j=0,1,2)$ 的系数为零后得到一个非线性代数方程组

$$3\omega g_0+\beta g_0^3-3ab\gamma\omega g_1+3\beta g_0f_1^2=0,$$
$$g_1(-\omega+a^2\gamma\omega+b^2\gamma\omega-\beta g_0^2-\beta f_1^2)=0,$$
$$(-3ab\gamma\omega g_1+\beta g_0g_1^2+\beta g_0f_1^2)=0,$$
$$g_1(-6b^2\gamma\omega+\beta g_1^2+3\beta f_1^2)=0,$$
$$f_1(3\omega-3a^2\gamma\omega+3\beta g_0^2+\beta f_1^2)=0,$$
$$f_1ab\gamma\omega=0,$$
$$f_1(-6\gamma b^2\omega+3\beta g_1^2+\beta f_1^2)=0,$$
$$\beta g_0g_1f_1=0.$$

用符号计算系统 Mathematica 求出该方程组的如下解：

$$a=0,\quad g_0=f_1=0,$$
$$g_1=-\frac{\sqrt{6\omega}}{\sqrt{\beta}}, \tag{57}$$
$$b=\mp\frac{1}{\sqrt{\gamma}}\quad(\gamma>0);$$

$$a=0,\quad g_0=f_1=0,$$
$$g_1=\frac{\sqrt{6\omega}}{\sqrt{\beta}}, \tag{58}$$
$$b=\mp\frac{1}{\sqrt{\gamma}}\quad(\gamma>0);$$

$$a=0,\quad g_0=0,$$
$$g_1=-\frac{\mathrm{i}\sqrt{3\omega}}{\sqrt{\beta}},$$
$$b=-\frac{\mathrm{i}\sqrt{2}}{\sqrt{\gamma}},$$

$$f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0);\tag{59}$$

$$\begin{aligned}&a=0,\quad g_0=0,\\&g_1=-\frac{i\sqrt{3\omega}}{\sqrt{\beta}},\\&b=\frac{i\sqrt{2}}{\sqrt{\gamma}},\\&f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0);\end{aligned}\tag{60}$$

$$\begin{aligned}&a=0,\quad g_0=0,\\&g_1=\frac{i\sqrt{3\omega}}{\sqrt{\beta}},\\&b=-\frac{i\sqrt{2}}{\sqrt{\gamma}},\\&f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0);\end{aligned}\tag{61}$$

$$\begin{aligned}&a=0,\quad g_0=0,\\&g_1=\frac{i\sqrt{3\omega}}{\sqrt{\beta}},\\&b=\frac{i\sqrt{2}}{\sqrt{\gamma}},\\&f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0);\end{aligned}\tag{62}$$

$$\begin{aligned}&b=-\frac{i}{\sqrt{2\gamma}},\\&a=g_1=g_0=0,\\&f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0);\end{aligned}\tag{63}$$

$$\begin{aligned}&b=\frac{i}{\sqrt{2\gamma}},\\&a=g_1=g_0=0,\\&f_1=\mp\frac{i\sqrt{3\omega}}{\sqrt{\beta}}\quad(\beta\omega<0,\gamma<0).\end{aligned}\tag{64}$$

把(5)式分别与(57)—(64)式一起代入(56)式后得到 mBBM 方程的如下孤立波解：

$$u^{\mp}(x,t)=\mp i\sqrt{\frac{3\omega}{\beta}}\coth\frac{1}{\sqrt{-2\gamma}}(x+\omega t)\quad(\beta\omega<0,\gamma<0),\tag{65}$$

$$u^{\mp}(x,t)=\mp i\sqrt{\frac{3\omega}{\beta}}\tanh\frac{1}{\sqrt{-2\gamma}}(x+\omega t)\quad(\beta\omega<0,\gamma<0),\tag{66}$$

$$\begin{aligned}u^{\pm}(x,t)=&\pm\frac{i\sqrt{3\omega}}{2\sqrt{\beta}}\cosh\frac{1}{\sqrt{-2\gamma}}(x+\omega t)\\&\times\operatorname{csch}\frac{1}{2\sqrt{-2\gamma}}(x+\omega t)\\&\times\operatorname{sech}\frac{1}{2\sqrt{-2\gamma}}(x+\omega t)\\&(\beta\omega<0,\gamma<0),\end{aligned}\tag{67}$$

$$\begin{aligned}u^{\mp}(x,t)=&\mp\sqrt{\frac{3\omega}{2\beta}}\operatorname{csch}\frac{1}{2\sqrt{\gamma}}(x+\omega t)\\&\times\operatorname{sech}\frac{1}{2\sqrt{\gamma}}(x+\omega t)\\&(\beta\omega>0,\gamma>0).\end{aligned}\tag{68}$$

将(56)和(6)式代入(55)式并用上述方法得到一个非线性代数方程组

$$\begin{aligned}&-6ab\gamma\omega f_1+3\omega g_0+3\beta f_1^2g_0+\beta g_0^3=0,\\&(-\omega+a^2\gamma\omega+2b^2\gamma\omega-\beta f_1^2-\beta g_0^2)g_1=0,\\&(-3ab\gamma\omega f_1+\beta f_1^2g_0+\beta g_0g_1^2)=0,\\&g_1(-6b^2\gamma\omega+3\beta f_1^2+\beta g_1^2)=0,\\&f_1(3\omega-3a^2\gamma\omega-3b^2\gamma\omega+\beta f_1^2+3\beta g_0^2)=0,\\&(-\omega+a^2\gamma\omega+2b^2\gamma\omega-\beta f_1^2-\beta g_0^2)g_1=0,\\&ab\gamma\omega g_1=0,\\&f_1(-6b^2\gamma\omega+\beta f_1^2+3\beta g_1^2)=0,\\&\beta f_1g_0g_1=0.\end{aligned}$$

用符号计算系统 Mathematica 求出该方程组的如下解：

$$\begin{aligned}&a=0,\quad g_0=0,\\&f_1=\mp\frac{i\sqrt{6\omega}}{\sqrt{\beta}},\\&g_1=0,\\&b=-\frac{i}{\sqrt{\gamma}}\quad(\gamma<0,\beta\omega<0);\end{aligned}\tag{69}$$

$$\begin{aligned}&a=0,\quad g_0=0,\\&f_1=\mp\frac{i\sqrt{6\omega}}{\sqrt{\beta}},\\&g_1=0,\\&b=\frac{i}{\sqrt{\gamma}}\quad(\gamma<0,\beta\omega<0);\end{aligned}\tag{70}$$

$$\begin{aligned}&g_1=\pm\sqrt{\frac{3\omega}{\beta}},\\&a=g_0=0,\\&b=-\frac{\sqrt{2}}{\sqrt{\gamma}},\\&f_1=\mp\sqrt{\frac{3\omega}{\beta}}\quad(\gamma>0,\beta\omega>0);\end{aligned}\tag{71}$$

$$
\begin{aligned}
&g_1 = \pm\sqrt{\frac{3\omega}{\beta}},\\
&a = g_0 = 0,\\
&b = \frac{\sqrt{2}}{\sqrt{\gamma}},\\
&f_1 = \mp\sqrt{\frac{3\omega}{\beta}} \qquad (\gamma > 0, \beta\omega > 0);
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
&g_1 = -\sqrt{\frac{3\omega}{\beta}},\\
&a = g_0 = 0,\\
&b = \mp\frac{\sqrt{2}}{\sqrt{\gamma}},\\
&f_1 = \mp\sqrt{\frac{3\omega}{\beta}} \qquad (\gamma > 0, \beta\omega > 0);
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
&g_1 = \sqrt{\frac{3\omega}{\beta}},\\
&a = g_0 = 0,\\
&b = \mp\frac{\sqrt{2}}{\sqrt{\gamma}},\\
&f_1 = \mp\sqrt{\frac{3\omega}{\beta}} \qquad (\gamma > 0, \beta\omega > 0).
\end{aligned} \tag{74}
$$

把(7)式分别与(69)—(74)式一起代入(56)式后得到 mBBM 方程的如下孤立波解:

$$
\begin{aligned}
u^{\pm}(x,t) = {} & \mathrm{i}\sqrt{\frac{3\omega}{2\beta}}\csc\frac{1}{2\sqrt{-\gamma}}(x+\omega t)\\
&\times \sec\frac{1}{2\sqrt{-\gamma}}(x+\omega t)\\
&(\gamma < 0, \beta\omega < 0),
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
u^{\pm}(x,t) = {} & \pm\sqrt{\frac{3\omega}{\beta}}\cot\frac{1}{\sqrt{2\gamma}}(x+\omega t)\\
&(\gamma > 0, \beta\omega > 0),
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
u(x,t) = {} & \sqrt{\frac{3\omega}{\beta}}\tan\frac{1}{\sqrt{2\gamma}}(x+\omega t)\\
&(\gamma > 0, \beta\omega > 0).
\end{aligned} \tag{77}
$$

4. 结　　论

我们在双曲正切函数法、齐次平衡法的基础上结合一种辅助方程与假设法,借助计算机代数系统 Mathematica 得到了文献[1,2]未能给出的 BBM 方程和 mBBM 方程的型如(25)—(31),(42)—(46),(47)—(53),(67)—(68),(75)新的精确孤立波解.本文给出的方法也能构造许多非线性发展方程(组)新的精确解.

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New exact solitary wave solutions to the BBM and mBBM equations*

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Abstract

In this paper, several new exact solitary wave solutions to the BBM and the mBBM equations are constructed explicitly by combined use of a hyperbolic function assumption and a new auxiliary ordinary differential equation. This method also can be used to find new solitary wave solutions to other nonlinear evolution equations.

Keywords: auxiliary equation, hyperbolic function assumption, solitary wave solutions
PACC: 0340K, 0290

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