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# 两组新的广义的 Ito 方程组的多种行波解*

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利用推广的形变映射法,得到两组广义的 Ito 方程组的丰富的行波解,包括孤子解,三角函数解,椭圆函数解,幂函数解.

关键词:广义的 Ito 方程,非线性 Klein-Gordon 方程,行波解

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## 1. 引 言

在众多领域中,许多科学问题最终可用非线性演化方程来描述,因此寻找非线性演化方程的精确解,特别是孤子解,一直是物理学家和数学家感兴趣的课题.近年来,已发展了许多求解非线性模型精确解的方法.如反散射法[1],Bäcklund 变换和 Darboux 变换[2],Painlevé 截断展开法[3],齐次平衡法[4],双曲函数展开法[5],三角函数展开法[6],Jacobian 椭圆函数展开法[7—9],形变映射法[10—13]等.

形变映射法是求解非线性偏微分方程精确解的一种比较有效的方法之一.其主要思想是找到两组非线性方程之间的映射关系,利用一组方程的已知解得到另一组方程的精确解.而这种映射关系的获得一般凭经验作出猜测.我们对这种方法作了进一步的推广[14].基于齐次平衡的思想,得到了任一非线性偏微分方程或方程组与立方非线性 Klein-Gordon(NKG)方程之间一般的映射关系的方法.利用立方 NKG 方程丰富的精确解,可以得到非线性偏微分方程或方程组非常丰富的精确行波解.这种推广的形变映射法包含了双曲函数展开法,三角函数展开法,椭圆函数展开法.

最近,Tam 等人[15]提出了两组非线性耦合方程组

$$
\begin{aligned}
u_t &= v_x,\\
v_t &= -2v_{xxx} - 6(uv)_x - 12\omega\omega_x + 6p_x,\\
\omega_t &= \omega_{xxx} + 3u\omega_x,\\
p_t &= p_{xxx} + 3up_x;
\end{aligned}
\tag{1}
$$

和

$$
\begin{aligned}
u_t &= v_x,\\
v_t &= -2v_{xxx} - 6(uv)_x - 6(\omega p)_x,\\
\omega_t &= \omega_{xxx} + 3u\omega_x,\\
p_t &= p_{xxx} + 3up_x.
\end{aligned}
\tag{2}
$$

其实这是著名的 Ito 可积模型[16]的两种不同推广.我们分别把它们称做 GIto(1)和 GIto(2).

$$
\begin{aligned}
u_t &= v_x,\\
v_t &= -2v_{xxx} - 6(uv)_x.
\end{aligned}
\tag{3}
$$

利用 Hirota 双线性方法,Tam 等人找到了 $p=0$ 时方程组(1)的 3 孤子解,4 孤子解[15]和方程组(2)的 3 孤子解[17].随后,Karasu-Kalkanli 等人对其可积性进行了研究,发现它们都具有 Painlevé 意义下的可积性[18].然而,这两组方程的行波解特别是周期解还没有进行过研究.本文利用推广的形变映射法,得到 GIto(1)和 GIto(2)方程组丰富的精确解,包括孤子解,三角函数解,椭圆函数解,幂函数解.

## 2. GIto(1)方程组的行波解

为了得到方程组(1)的行波解,先对它作一变换:$u(x,t)=u(\xi)$,$v(x,t)=v(\xi)$,$\omega(x,t)=\omega(\xi)$,$p(x,t)=p(\xi)$,$\xi=k(x-ct)$,$k$ 为波数,$c$ 为波速.这样,方程(1)就成为一个常微分方程组.

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$$cu_\xi + v_\xi = 0,$$
$$cv_\xi - 2k^2 v_{\xi\xi\xi} - 6(uv)_\xi - 12\omega\omega_\xi + 6p_\xi = 0,$$
$$c\omega_\xi + k^2\omega_{\xi\xi\xi} + 3u\omega_\xi = 0,$$
$$cp_\xi + k^2 p_{\xi\xi\xi} + 3up_\xi = 0. \quad (4)$$

为了得到常微分方程组(4)的精确解,假设

$$u(\xi) = \sum_{i=0}^{M_1} A_i\phi^i, v(\xi) = \sum_{j=0}^{M_2} B_j\phi^j,$$
$$\omega(\xi) = \sum_{l=0}^{M_3} R_l\phi^l, p(\xi) = \sum_{n=0}^{M_4} S_n\phi^n. \quad (5)$$

其中 $A_i, B_j, R_l, S_n$ 为待定系数,$\phi(\xi)$是立方非线性 Klein-Gordon 方程的精确解,

$$\phi_{\xi\xi} = \lambda\phi + \mu\phi^3,$$
$$\phi_\xi^2 = c_0 + \lambda\phi^2 + \frac{\mu}{2}\phi^4, \quad (6)$$

$c_0, \lambda$ 和 $\mu$ 为常数.把(5),(6)两式代入方程组(4),利用齐次平衡的思想,即平衡非线性波动方程组(4)中的非线性项和最高导数项,可解得 $M_1 = M_2 = M_3 = 2, M_4 = 4$,于是,(5)式变为

$$u = A_0 + A_1\phi + A_2\phi^2,$$
$$v = B_0 + B_1\phi + B_2\phi^2,$$
$$\omega = R_0 + R_1\phi + R_2\phi^2,$$
$$p = S_0 + S_1\phi + S_2\phi^2 + S_3\phi^3 + S_4\phi^4. \quad (7)$$

将(6),(7)式代入(4)式,利用 Maple 软件,得到一组关于 $A_i, B_i, R_i (i = 0,1,2), S_j (j = 0,1,2,3,4)$的代数方程组,

$$cA_1 + B_1 = 0,$$
$$cA_2 + B_2 = 0,$$
$$S_4 - \mu k^2 B_2 - R_2^2 - B_2 A_2 = 0,$$
$$3B_2A_1 + 6R_1R_2 + 3B_1A_2 - 3S_3 + \mu k^2 B_1 = 0,$$
$$6B_0A_2 + 8\lambda k^2 B_2 + 12R_0R_2 + 6B_1A_1 - cB_2 + 6B_2A_0 + 6R_1^2 - 6S_2 = 0,$$
$$2\lambda k^2 B_1 - 6S_1 + 12R_0R_1 - cB_1 + 6B_0A_1 + 6B_1A_0 = 0,$$
$$2\mu k^2 R_2 + A_2R_2 = 0,$$
$$2A_1R_2 + \mu k^2 R_1 + A_2R_1 = 0,$$
$$2cR_2 + 6A_0R_2 + 8\lambda k^2 R_2 + 3A_1R_1 = 0,$$
$$cR_1 + \lambda k^2 R_1 + 3A_0R_1 = 0,$$
$$5\mu k^2 S_4 + A_2S_4 = 0,$$
$$2A_2S_3 + 4A_1S_4 + 10\mu k^2 S_3 = 0,$$
$$4cS_4 + 6A_2S_2 + 9A_1S_3 + 12\mu k^2 S_2 + 12A_0S_4 + 64\lambda k^2 S_4 = 0,$$
$$9\lambda k^2 S_3 + 2A_1S_2 + \mu k^2 S_1 + 3A_0S_3 + A_2S_1 + cS_3 = 0,$$
$$6A_0S_2 + 24c_0k^2S_4 + 8\lambda k^2 S_2 + 2cS_2 + 3A_1S_1 = 0,$$
$$\lambda k^2 S_1 + cS_1 + 3A_0S_1 + 6c_0k^2S_3 = 0. \quad (8)$$

利用 Maple,得到四组有意义的解

$$A_0 = -\frac{1}{3}(c + 4\lambda k^2), A_1 = 0, A_2 = -2\mu k^2,$$
$$B_0 = B_0, B_1 = 0, B_2 = 2\mu k^2 c,$$
$$R_0 = R_0, R_1 = 0, R_2 = \pm\mu k^2\sqrt{2c},$$
$$S_0 = S_0, S_1 = S_3 = S_4 = 0,$$
$$S_2 = -\mu k^2(2B_0 + c^2 \mp 2\sqrt{2c}R_0); \quad (9)$$

$$A_0 = -\frac{1}{3}(c + 4\lambda k^2), A_1 = 0, A_2 = -\mu k^2,$$
$$B_0 = \frac{1}{2\mu k^2}(2\mu\lambda k^4 c - \mu k^2 c^2 + 2R_1^2),$$
$$B_1 = 0, B_2 = \mu k^2 c,$$
$$R_0 = R_0, R_1 = R_1, R_2 = 0,$$
$$S_0 = S_0, S_1 = 2R_0R_1, S_2 = S_3 = S_4 = 0; \quad (10)$$

$$A_0 = -\frac{1}{3}(c + 10\lambda k^2) \pm k^2(4\lambda^2 - 6\mu c_0)^{\frac{1}{2}},$$
$$A_1 = 0, A_2 = -5\mu k^2,$$
$$B_0 = \frac{1}{6}c(10\lambda k^2 - 3c \pm 22k^2(4\lambda^2 - 6\mu c_0)^{\frac{1}{2}}),$$
$$B_1 = 0, B_2 = 5\mu k^2 c,$$
$$R_0 = R_0, R_1 = R_2 = 0,$$
$$S_0 = S_0, S_1 = S_3 = 0,$$
$$S_2 = -\frac{40}{3}\mu k^2 c(2\lambda k^2 + 4c \pm k^2(4\lambda^2 - 6\mu c_0)^{\frac{1}{2}}),$$
$$S_4 = -20\mu^2 k^4 c; \quad (11)$$

$$A_0 = A_0, A_1 = 0, A_2 = -\mu k^2,$$
$$B_0 = \frac{4}{3}\lambda k^2 c - \frac{1}{6}c^2 + cA_0, B_1 = 0, B_2 = \mu k^2 c,$$
$$R_0 = R_0, R_1 = R_2 = 0,$$
$$S_0 = S_0, S_1 = S_2 = S_3 = S_4 = 0. \quad (12)$$

其中 $c > 0, k, \mu, \lambda, c_0$ 为任意常数.

因此,我们可以得到方程组(1)的两组行波解

$$u = -\frac{1}{3}(c + 4\lambda k^2) - 2\mu k^2\phi^2,$$
$$v = B_0 + 2\mu k^2 c\phi^2,$$
$$\omega = R_0 \pm \mu k^2\sqrt{2c}\phi^2,$$
$$p = S_0 - \mu k^2(2B_0 + c^2 \mp 2\sqrt{2c}R_0)\phi^2; \quad (13)$$

和

$$u=-\frac{1}{3}(c+4\lambda k^2)-\mu k^2\phi^2,$$
$$v=\frac{1}{2\mu k^2}(2\mu\lambda k^4c-\mu k^2c^2+2R_1^2)+\mu k^2c\phi^2,$$
$$\omega=R_0+R_1\phi,$$
$$p=S_0+2R_0R_1\phi. \quad (14)$$

耦合方程组(即方程组(1)中少第三个方程)的一组行波解

$$u=-\frac{1}{3}(c+10\lambda k^2)\pm k^2\sqrt{4\lambda^2-6\mu c_0}-5\mu k^2\phi^2,$$
$$v=\frac{c}{6}(10\lambda k^2-3c\pm 22k^2\sqrt{4\lambda^2-6\mu c_0})+5\mu k^2c\phi^2,$$
$$p=S_0-\frac{40}{3}\mu k^2c(2\lambda k^2+4c\pm k^2\sqrt{4\lambda^2-6\mu c_0})\phi^2-20\mu^2k^4c\phi^4; \quad (15)$$

Ito 方程组(3)的一组行波解

$$u=A_0-\mu k^2\phi^2,$$
$$v=\frac{4}{3}\lambda k^2c-\frac{1}{6}c^2+cA_0+\mu k^2c\phi^2. \quad (16)$$

$\phi$ 为立方 NKG 方程的精确解. 文献[10]已给出了立方 NKG 方程的许多精确解,利用映射关系(13)和(14),可以得到 GIto(1)方程组的许多显式的精确的行波解,包括孤子解,三角函数解,椭圆函数解,幂函数解.

## 2.1. 孤子解

A. 当 $\lambda=1,\mu=-2,c_0=0$ 时,$\phi=\mathrm{sech}\xi$

$$u_1=-\frac{1}{3}(c+4k^2)+4k^2\mathrm{sech}^2\xi,$$
$$v_1=B_0-4k^2c\,\mathrm{sech}^2\xi,$$
$$\omega_1=R_0\mp 2k^2\sqrt{2c}\,\mathrm{sech}^2\xi,$$
$$p_1=S_0+2k^2(2B_0+c^2\mp 2\sqrt{2c}R_0)\mathrm{sech}^2\xi; \quad (17)$$
$$u_2=-\frac{1}{3}(c+4k^2)+2k^2\mathrm{sech}^2\xi,$$
$$v_2=-\frac{1}{2k^2}(-2k^4c+k^2c^2+R_1^2)-2k^2c\,\mathrm{sech}^2\xi,$$
$$\omega_2=R_0+R_1\mathrm{sech}\xi,$$
$$p_2=S_0+2R_0R_1\mathrm{sech}\xi. \quad (18)$$

B. 当 $\lambda=-2,\mu=2,c_0=1$ 时,$\phi=\tanh\xi$

$$u_3=-\frac{1}{3}(c-8k^2)-4k^2\tanh^2\xi,$$
$$v_3=B_0+4k^2c\tanh^2\xi,$$
$$\omega_3=R_0\pm 2k^2\sqrt{2c}\tanh^2\xi,$$
$$p_3=S_0-2k^2(2B_0+c^2\mp 2\sqrt{2c}R_0)\tanh^2\xi; \quad (19)$$
$$u_4=-\frac{1}{3}(c-8k^2)-2k^2\tanh^2\xi,$$
$$v_4=\frac{1}{2k^2}(-4k^4c-k^2c^2+R_1^2)+2k^2c\tanh^2\xi,$$
$$\omega_4=R_0+R_1\tanh\xi,$$
$$p_4=S_0+2R_0R_1\tanh\xi. \quad (20)$$

## 2.2. 三角函数解

C. 当 $\lambda=2,\mu=2,c_0=1$ 时,$\phi=\tan\xi$

$$u_5=-\frac{1}{3}(c+8k^2)-4k^2\tan^2\xi,$$
$$v_5=B_0+4k^2c\tan^2\xi,$$
$$\omega_5=R_0\pm 2k^2\sqrt{2c}\tan^2\xi,$$
$$p_5=S_0-2k^2(2B_0+c^2\mp 2\sqrt{2c}R_0)\tan^2\xi; \quad (21)$$
$$u_6=-\frac{1}{3}(c+8k^2)-2k^2\tan^2\xi,$$
$$v_6=\frac{1}{2k^2}(4k^4c-k^2c^2+R_1^2)+2k^2c\tan^2\xi,$$
$$\omega_6=R_0+R_1\tan\xi,$$
$$p_6=S_0+2R_0R_1\tan\xi. \quad (22)$$

D. 当 $\lambda=-1,\mu=2,c_0=0$ 时,$\phi=\sec\xi$

$$u_7=-\frac{1}{3}(c-4k^2)-4k^2\sec^2\xi,$$
$$v_7=B_0+4k^2c\sec^2\xi,$$
$$\omega_7=R_0\pm 2k^2\sqrt{2c}\sec^2\xi,$$
$$p_7=S_0-2k^2(2B_0+c^2\mp 2\sqrt{2c}R_0)\sec^2\xi; \quad (23)$$
$$u_8=-\frac{1}{3}(c-4k^2)-2k^2\sec^2\xi,$$
$$v_8=\frac{1}{2k^2}(-2k^4c-k^2c^2+R_1^2)+2k^2c\sec^2\xi,$$
$$\omega_8=R_0+R_1\sec\xi,$$
$$p_8=S_0+2R_0R_1\sec\xi. \quad (24)$$

## 2.3. 椭圆函数解

E. 当 $\lambda=-(1+K^2),\mu=2K^2,c_0=1$ 时,$\phi=\mathrm{sn}\xi$

$$u_9=-\frac{1}{3}(c-4(1+K^2)k^2)-4K^2k^2\mathrm{sn}^2\xi,$$
$$v_9=B_0+4K^2k^2c\,\mathrm{sn}^2\xi,$$
$$\omega_9=R_0\pm 2K^2k^2\sqrt{2c}\,\mathrm{sn}^2\xi,$$
$$p_9=S_0-2K^2k^2(2B_0+c^2\mp 2\sqrt{2c}R_0)\mathrm{sn}^2\xi; \quad (25)$$

$$u_{10}=-\frac{1}{3}\left(c-4\left(1+K^2\right)k^2\right)-2K^2k^2\mathrm{sn}^2\xi,$$

$$v_{10}=\frac{1}{2K^2k^2}\left(-2K^2\left(1+K^2\right)k^4c-K^2k^2c^2+R_1^2\right)+2K^2k^2c\,\mathrm{sn}^2\xi,$$

$$\omega_{10}=R_0+R_1\mathrm{sn}\xi,$$

$$p_{10}=S_0+2R_0R_1\mathrm{sn}\xi. \quad (26)$$

F. 当 $\lambda=2K^2-1$，$\mu=-2K^2$，$c_0=1-K^2$ 时，$\phi=\mathrm{cn}\xi$

$$u_{11}=-\frac{1}{3}\left(c+4\left(2K^2-1\right)k^2\right)+4K^2k^2\mathrm{cn}^2\xi,$$

$$v_{11}=B_0-4K^2k^2c\,\mathrm{cn}^2\xi,$$

$$\omega_{11}=R_0\mp2K^2k^2\sqrt{2c}\,\mathrm{cn}^2\xi,$$

$$p_{11}=S_0+2K^2k^2\left(2B_0+c^2\mp2\sqrt{2c}R_0\right)\mathrm{cn}^2\xi; \quad (27)$$

$$u_{12}=-\frac{1}{3}\left(c+4\left(2K^2-1\right)k^2\right)+2K^2k^2\mathrm{cn}^2\xi,$$

$$v_{12}=-\frac{1}{2K^2k^2}\left(-2K^2\left(2K^2-1\right)k^4c+K^2k^2c^2+R_1^2\right)-2K^2k^2c\,\mathrm{cn}^2\xi,$$

$$\omega_{12}=R_0+R_1\mathrm{cn}\xi,$$

$$p_{12}=S_0+2R_0R_1\mathrm{cn}\xi. \quad (28)$$

G. 当 $\lambda=2-K^2$，$\mu=-2$，$c_0=K^2-1$ 时，$\phi=\mathrm{dn}\xi$

$$u_{13}=-\frac{1}{3}\left(c+4\left(2-K^2\right)k^2\right)+4k^2\mathrm{dn}^2\xi,$$

$$v_{13}=B_0-4k^2c\,\mathrm{dn}^2\xi,$$

$$\omega_{13}=R_0\mp2k^2\sqrt{2c}\,\mathrm{dn}^2\xi,$$

$$p_{13}=S_0+2k^2\left(2B_0+c^2\mp2\sqrt{2c}R_0\right)\mathrm{dn}^2\xi; \quad (29)$$

$$u_{14}=-\frac{1}{3}\left(c+4\left(2-K^2\right)k^2\right)+2k^2\mathrm{dn}^2\xi,$$

$$v_{14}=-\frac{1}{2k^2}\left(-2\left(2-K^2\right)k^4c+k^2c^2+R_1^2\right)-2k^2c\,\mathrm{dn}^2\xi,$$

$$\omega_{14}=R_0+R_1\mathrm{dn}\xi,$$

$$p_{14}=S_0+2R_0R_1\mathrm{dn}\xi. \quad (30)$$

其中 $\mathrm{sn}\xi$，$\mathrm{cn}\xi$ 和 $\mathrm{dn}\xi$ 是模为 $K$ 的 Jacobi 椭圆函数.

H. 当 $\lambda=2-K^2$，$\mu=2\left(1-K^2\right)$，$c_0=1$ 时，$\phi=\mathrm{tn}\xi$

$$u_{15}=-\frac{1}{3}\left(c+4\left(2-K^2\right)k^2\right)-4\left(1-K^2\right)k^2\mathrm{tn}^2\xi,$$

$$v_{15}=B_0+4\left(1-K^2\right)k^2c\,\mathrm{tn}^2\xi,$$

$$\omega_{15}=R_0\mp2\left(1-K^2\right)k^2\sqrt{2c}\,\mathrm{tn}^2\xi,$$

$$p_{15}=S_0-2\left(1-K^2\right)k^2\left(2B_0+c^2\mp2\sqrt{2c}R_0\right)\mathrm{tn}^2\xi; \quad (31)$$

$$u_{16}=-\frac{1}{3}\left(c+4\left(2-K^2\right)k^2\right)-2\left(1-K^2\right)k^2\mathrm{tn}^2\xi,$$

$$v_{16}=\frac{1}{2\left(1-K^2\right)k^2}\left(2\left(1-K^2\right)\left(2-K^2\right)k^4c-\left(1-K^2\right)k^2c^2+R_1^2\right)+2\left(1-K^2\right)k^2c\,\mathrm{tn}^2\xi,$$

$$\omega_{16}=R_0+R_1\mathrm{tn}\xi,$$

$$p_{16}=S_0+2R_0R_1\mathrm{tn}\xi. \quad (32)$$

I. 当 $\lambda=\left(1+K^2\right)/2$，$\mu=-\left(1-K^2\right)/2$，$c_0=-\left(1-K^2\right)/4$ 时，$\phi=\mathrm{dn}\xi/\left(1+K\mathrm{sn}\xi\right)$

$$u_{17}=-\frac{1}{3}\left(c+2\left(1+K^2\right)k^2\right)+\left(1-K^2\right)k^2\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2},$$

$$v_{17}=B_0-\left(1-K^2\right)k^2c\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2},$$

$$\omega_{17}=R_0\mp\frac{\left(1-K^2\right)}{2}k^2\sqrt{2c}\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2},$$

$$p_{17}=S_0+\frac{\left(1-K^2\right)}{2}k^2\left(2B_0+c^2\mp2\sqrt{2c}R_0\right)\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2}; \quad (33)$$

$$u_{18}=-\frac{1}{3}\left(c+2\left(1+K^2\right)k^2\right)+\frac{\left(1-K^2\right)}{2}k^2\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2},$$

$$v_{18}=-\frac{1}{\left(1-K^2\right)k^2}\left(-\frac{\left(1-K^4\right)}{2}k^4c+\frac{\left(1-K^2\right)}{2}k^2c^2+R_1^2\right)-\frac{\left(1-K^2\right)}{2}k^2c\frac{\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)^2},$$

$$\omega_{18}=R_0+R_1\frac{\mathrm{dn}\xi}{1+K\mathrm{sn}\xi},$$

$$p_{18}=S_0+2R_0R_1\frac{\mathrm{dn}\xi}{1+K\mathrm{sn}\xi}. \quad (34)$$

J. 当 $\lambda=\left(1-6K+K^2\right)/4$，$\mu=2K$，$c_0=-\left(1-K^2\right)/4$ 时，$\phi=\sqrt{1-K}\,\mathrm{dn}\xi/\sqrt{2K\left(1+K\mathrm{sn}\xi\right)\left(1+\mathrm{sn}\xi\right)}$

$$u_{19}=-\frac{1}{3}\left(c-\left(1-6K+K^2\right)k^2\right)-2k^2\frac{\left(1-K\right)\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)\left(1+\mathrm{sn}\xi\right)},$$

$$v_{19}=B_0+2k^2c\frac{\left(1-K\right)\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)\left(1+\mathrm{sn}\xi\right)}$$

$$\omega_{19}=R_0\pm k^2\sqrt{2c}\frac{\left(1-K\right)\mathrm{dn}^2\xi}{\left(1+K\mathrm{sn}\xi\right)\left(1+\mathrm{sn}\xi\right)},$$

$$p_{19}=S_0-k^2\Big(2B_0+c^2\mp 2\sqrt{2c}R_0\Big)\frac{(1-K)\mathrm{dn}^2\xi}{(1+K\mathrm{sn}\xi)(1+\mathrm{sn}\xi)};\quad(35)$$

$$u_{20}=-\frac{1}{3}\big(c-(1-6K+K^2)k^2\big)-k^2\frac{(1-K)\mathrm{dn}^2\xi}{(1+K\mathrm{sn}\xi)(1+\mathrm{sn}\xi)},$$

$$v_{20}=\frac{1}{4Kk^2}\big(-K(1-6K+K^2)k^4c-2Kk^2c^2+2R_1^2\big)+k^2c\frac{(1-K)\mathrm{dn}^2\xi}{(1+K\mathrm{sn}\xi)(1+\mathrm{sn}\xi)},$$

$$\omega_{20}=R_0+R_1\frac{\sqrt{1-K}\,\mathrm{dn}\xi}{\sqrt{2K(1+K\mathrm{sn}\xi)(1+\mathrm{sn}\xi)}},$$

$$p_{20}=S_0+2R_0R_1\frac{\sqrt{1-K}\,\mathrm{dn}\xi}{\sqrt{2K(1+K\mathrm{sn}\xi)(1+\mathrm{sn}\xi)}}.\quad(36)$$

K. 当 $\lambda=1-6K'+K'^2$，$\mu=\mp 2(1\mp K')^2$，$c_0=K'$ 时，$\phi=2\sqrt{K'}\,\mathrm{sn}\xi\mathrm{cn}\xi/(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)$

$$u_{21}=-\frac{1}{3}\big(c+4(1+6K'+K'^2)k^2\big)\pm 16k'(1\mp K')^2k^2\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2},$$

$$v_{21}=B_0\mp 16K'(1\mp K')^2k^2c\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2}$$

$$\omega_{21}=R_0\pm 8K'(1\mp K')^2k^2\sqrt{2c}\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2},$$

$$p_{21}=S_0\pm 2(1\mp K')^2k^2\Big(2B_0+c^2\mp 2\sqrt{2c}R_0\Big)\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2};\quad(37)$$

$$u_{22}=-\frac{1}{3}\big(c+(1\mp 6K'+K'^2)k^2\big)\pm 8K'(1\mp K')^2k^2\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2},$$

$$v_{22}=\mp\frac{1}{2(1\mp K')^2k^2}\big(\mp 2(1\mp K')^2(1\mp 6K'+K'^2)k^4c\pm 2(1\mp K')^2k^2c^2+R_1^2\big)\mp 2(1\mp K')^2k^2c\frac{\mathrm{sn}^2\xi\mathrm{cn}^2\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)^2},$$

$$\omega_{22}=R_0+2\sqrt{K'}R_1\frac{\mathrm{sn}\xi\mathrm{cn}\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)},$$

$$p_{22}=S_0+4\sqrt{K'}R_0R_1\frac{\mathrm{sn}\xi\mathrm{cn}\xi}{(\mathrm{cn}^2\xi\pm K'\mathrm{sn}^2\xi)}.\quad(38)$$

其中 $K'=\sqrt{1-K^2}$ 为椭圆函数的余模.

L. 当 $\lambda=-2(1+K')^2$，$\mu=2(1-K')^2$，$c_0=(1+K')^2$ 时，$\phi=(\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi)/(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)$

$$u_{23}=-\frac{1}{3}\big(c-8(1+K')^2k^2\big)-4(1-K')^2k^2\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2,$$

$$v_{23}=B_0+4(1-K')^2k^2c\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2,$$

$$\omega_{23}=R_0\pm 2(1-K')^2k^2\sqrt{2c}\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2,$$

$$p_{23}=S_0-2(1-K')^2k^2\Big(2B_0+c^2\mp 2\sqrt{2c}R_0\Big)\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2;\quad(39)$$

$$u_{24}=-\frac{1}{3}(c-8(1+K')^2)k^2)-2(1-K')^2k^2\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2,$$

$$v_{24}=\frac{1}{2(1-K')^2k^2}\big(-4(1-K'^2)^2k^4c-(1-K')^2k^2c^2+R_1^2\big)+2(1-K')^2k^2c\left(\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}\right)^2,$$

$$\omega_{24}=R_0+R_1\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}^2,$$

$$p_{24}=S_0+2R_0R_1\frac{\mathrm{cn}^2\xi-K'\mathrm{sn}^2\xi}{(\mathrm{cn}^2\xi+K'\mathrm{sn}^2\xi)}^2.\quad(40)$$

M. 当 $\lambda=-2(1+K'^2)$，$\mu=2(1+K')^2$，$c_0=(1-K')^2$ 时，$\phi=(K'-\mathrm{dn}^2\xi)/(K'+\mathrm{dn}^2\xi)$

$$u_{25}=-\frac{1}{3}\big(c-8(1+K'^2)k^2\big)-4(1+K')^2k^2\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2,$$

$$v_{25}=B_0+4(1+K')^2k^2c\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2,$$

$$\omega_{25}=R_0\pm 2(1+K')^2k^2\sqrt{2c}\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2,$$

$$p_{26}=S_0-2(1+K')^2k^2\Big(2B_0+c^2\mp 2\sqrt{2c}R_0\Big)\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2;\quad(41)$$

$$u_{26}=-\frac{1}{3}\big(c-8(1+K'^2)k^2\big)-2(1+K')^2k^2\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2,$$

$$v_{26}=\frac{1}{2(1+K')^2k^2}\big(-4(1+K')^2(1+K'^2)k^4c-(1+K')^2k^2c^2+R_1^2\big)+2(1+K')^2k^2c\left(\frac{K'-\mathrm{dn}^2\xi}{(K'+\mathrm{dn}^2\xi)}\right)^2,$$

$$\omega_{26} = R_0 + R_1 \frac{K' - \mathrm{dn}^2\xi}{(K' + \mathrm{dn}^2\xi)},$$

$$p_{26} = S_0 + 2R_0R_1 \frac{K' - \mathrm{dn}^2\xi}{(K' + \mathrm{dn}^2\xi)}. \tag{42}$$

利用映射关系(13),(14)和文献[10]中的表1,还可以得到一些其他类型的椭圆函数解.

### 2.4. 幂函数解

当 $\lambda = 0, \mu = 2, c_0 = 0, \phi = 1/\xi$. 可以得到GIto(1)的幂函数解

$$u_{27} = -\frac{c}{3} - \frac{4k^2}{\xi^2},$$

$$v_{27} = B_0 + \frac{4k^2c}{\xi^2},$$

$$\omega_{27} = R_0 \pm \frac{2k^2\sqrt{2c}}{\xi^2},$$

$$p_{27} = S_0 - 2k^2(2B_0 + c^2 \mp 2\sqrt{2c}R_0)\frac{1}{\xi^2}; \tag{43}$$

$$u_{28} = -\frac{c}{3} - \frac{2k^2}{\xi^2},$$

$$v_{28} = \frac{1}{2k^2}(-k^2c^2 + R_1^2) + \frac{2k^2c}{\xi^2},$$

$$\omega_{28} = R_0 + \frac{R_1}{\xi},$$

$$p_{28} = S_0 + \frac{2R_0R_1}{\xi}. \tag{44}$$

## 3. GIto(2)方程组的行波解

采用类似的方法和步骤,可以得到方程组(2)的两组行波解

$$u = -\frac{1}{3}(c + \lambda k^2) - \mu k^2\phi^2,$$

$$v = B_0 + \mu k^2 c\phi^2,$$

$$\omega = \frac{\mu k^2(-2\lambda k^2 c + c^2 + 2B_0)S_0}{2S_1}\left(\frac{S_0}{S_1} - \phi\right),$$

$$p = S_0 + S_1\phi; \tag{45}$$

$$u = -\frac{1}{3}(c + \lambda k^2) - 2\mu k^2\phi^2,$$

$$v = B_0 + 2\mu k^2 c\phi^2,$$

$$\omega = \frac{\mu k^2(-2\mu k^2 cS_0 + c^2S_2 + 2B_0S_2)}{S_2^2} + \frac{2\mu^2k^4c}{S_2}\phi^2,$$

$$p = S_0 + 2S_0\phi^2. \tag{46}$$

和耦合方程组(即方程组(2)中少第四个方程)的一组行波解

$$u = -\frac{1}{3}(c + \lambda k^2) - \mu k^2\phi^2,$$

$$v = \lambda k^2 c - \frac{c^2}{2} + \mu k^2 c\phi^2,$$

$$\omega = R_0 + R_1\phi. \tag{47}$$

$\phi$ 为立方NKG方程的精确解. 利用映射关系(45)和(46)和文献[10]中的表1,同样可以得到GIto(2)方程组的许多显式的精确的行波解,包括孤子解,三角函数解,椭圆函数解,幂函数解,本文不再一一列出.

## 4. 结 论

本文利用推广的形变映射法,得到了GIto(1)和GIto(2)两组非线性演化方程组非常丰富的精确的行波解,同时还得到了两组耦合Ito方程组(即方程组(1)中少第三个方程和方程组(2)中少第四个方程)和Ito方程组(3)的行波解. 这些行波解包括孤子解,三角函数解,椭圆函数解和幂函数解. 因此这种推广的形变映射法已经包含了双曲函数展开法,三角函数展开法和椭圆函数展开法. 该方法还可以广泛应用于其他一些非线性偏微分方程或方程组,是求解线性偏微分方程孤子解和周期波解的一种既简单又有效的方法.

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# Multiple travelling wave solutions of two newly generalized Ito systems*

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Abstract

By use of the extended mapping deformation method, many explicit and exact travelling wave solutions of the two newly generalized Ito systems are obtained which contain solitary wave solutions, trigonometric function solutions, Jacobian elliptic function solutions and rational solutions.

**Keywords**: generalized Ito system, nonlinear cubic Klein-Gordon equation, exact sodution
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